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Gauge invariant perturbations of static spatially compact LRS II spacetimes

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Abstract

We present a framework to describe completely general first-order perturbations of static, spatially compact, and locally rotationally symmetric class II spacetimes within the theory of general relativity. The perturbation variables are by construction covariant and identification gauge invariant and encompass the geometry and the thermodynamics of the fluid sources. The new equations are then applied to the study of isotropic, adiabatic perturbations. We discuss how the choice of frame in which perturbations are described can significantly simplify the mathematical analysis of the problem and show that it is possible to change frames directly from the linear level equations. We find explicitly that the case of isotropic, adiabatic perturbations can be reduced to a singular Sturm–Liouville eigenvalue problem, and lower bounds for the values of the eigenfrequencies can be derived. These results lay the theoretical groundwork to analytically describe linear, isotropic, and adiabatic perturbations of static, spherically symmetric spacetimes.

Keywords: general relativity, perturbation theory, stellar compact objects

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1. Introduction

In the last few years, we have witnessed a true renaissance of relativistic astrophysics. The detection of gravitational waves [1] and the first images of the shadows of supermassive black holes [2] have brought a new wealth of data to the research community. It has, therefore, become paramount to devise new tools that allow for a clear interpretation of the latest data and deepen our understanding of relativistic astrophysical systems.

Due to the fundamental nonlinear nature of the field equations of the theory of general relativity (GR), finding exact solutions for these systems, even assuming highly symmetric setups with simple source fields, is a formidable task. Indeed, although several exact solutions have been found in vacuum or in the presence of matter, many of the latter type of solutions are, in general, not suitable to accurately model physically meaningful setups, especially in the so-called strong field regime, where relativistic gravitational effects play a pivotal role in the dynamics of the matter fields.

To circumvent the limitations in the applicability of idealized solutions, various perturbative schemes were developed to linearize the field equations and study perturbations of exact solutions. Indeed, the study of perturbations of black holes has been an area of intense development in the past few decades. Combining the advent of numerical relativity techniques with the analytical results from linear perturbation theory for vacuum solutions ultimately allowed for identifying patterns in the data and detecting gravitational waves. On the other hand, in the case of compact stars, several open problems remain in developing analytic tools to study perturbations of this type of objects. Especially in the strong field regime, we have so far relied almost exclusively on numerical methods to evolve the full nonlinear equations of GR. These methods allow us to understand oscillations of compact stellar objects in great generality, but carry the inevitable limitation of purely numerical approaches. Hence, there is a need for analytic methods that could complement the numerics in a synergic way.

The problem of linear perturbations of massive astrophysical objects is markedly different from black hole perturbations. Black holes are described by vacuum and electrovacuum solutions of the Einstein field equations. In contrast, compact stellar objects are described by solutions of GR with matter fluid sources characterized by perfect fluids with complex equations of state, fluids with non-trivial anisotropic pressure terms, or non-perfect multifluid models. Understandably, the evolution of the perturbations of these solutions is strongly tied to the properties of the matter fluid where, in general, anisotropies and momentum flows may be generated, even at a linear level, which then act as sources of shear and vorticity.

In 1964, by encoding the spacetime perturbations directly in the choice of gauge, Chandrasekhar derived for the first time in [3, 4] an equation to describe isotropic, adiabatic perturbations of static self-gravitating perfect fluids with a barotropic equation of state. In the 1990s, in [5, 6], structure equations were derived for nonradial perturbations of non-rotating perfect fluids. In [7–10], various extensions were proposed to study perturbations of slowly rotating stars. However, understanding the perturbative properties of relativistic stars using those frameworks has remained challenging because of fundamental mathematical limitations and the underlying methods used by those approaches. For instance, the original radial pulsation equation by Chandrasekhar for isotropic, adiabatic perturbations relies on the introduction of auxiliary trial functions, making it impossible to assert the stability of the background solution unequivocally. Moreover, it was later noticed that the choice of gauge in [5] leads to a higher-order system of equations when compared to the choice of another gauge because of an extraneous degree of freedom, adding unnecessary complication to the equations [11]. Several works have tried to improve these results in the following years, but all versions of those equations remain gauge-dependent.

In geometric gravity theories, the identification gauge problem in perturbation theory arises because, even if the choice of perturbation variables is physically reasonable, those might depend on the mapping between the equilibrium and the perturbed manifold. Hence, their values and rates of change are ambiguous. Therefore, to develop a rigorous perturbative framework in geometric theories of gravity, the perturbation variables have to be methodically chosen [12, 13].

In the case of vanishing or slow rotation, a locally rotationally symmetric (LRS) metric can be used to successfully describe the geometry of the spacetime for the interior of compact stellar objects. This class of spacetimes was first identified and classified according to their extra symmetries by Ellis [14] and Stewart J and Ellis [15] at the end of the 1960s and is characterized by a local rotational symmetry at every event. Many spacetimes of interest in astrophysics and cosmology belong to the LRS class. Examples are the Bianchi cosmologies [16], the Lemaître–Tolman–Bondi spacetime [17–19], the Oppenheimer–Snyder spacetime [20], and the vacuum Schwarzschild spacetime.

Recently, a new formalism has been proposed that is especially suitable to deal with LRS spacetimes: the 1+1+2 covariant approach [21–23]. This formalism, analogous to the better-known covariant 1+3 formalism extensively used in cosmology [24–26], is based on the procedure of covariant spacetime threading. It can also be considered a semi-tetradic approach to the description of spacetimes, which makes full use of the symmetries of the spacetime. The covariant approaches have two important properties: (i) they allow the maintenance of covariance at all stages of the calculations, and (ii) they enable the description of spacetimes in terms of well-defined physical quantities.

Analyzing LRS spacetimes with the 1+1+2 formalism can reveal many important aspects of these spacetimes and their physical processes. For instance, in [27, 28] it was found a covariant formulation of the Tolman–Oppenheimer–Volkoff equation for perfect fluids and fluids with non-trivial anisotropic pressure terms, allowing for the derivation of new solutions. In [29], using this formalism, it was proved a generalization of Derrick's theorem, showing that the conclusion of the original theorem holds independently of the geometric properties of the spacetime. In [30], the 1+1+2 formalism was applied to the Einstein-Cartan theory to derive the general structure equations for static, isotropic spacetimes, which were then used to find the first known regular solutions suitable to model the interior of massive astrophysical objects. However, the power of covariant formalisms is truly revealed in the context of perturbation theory. Indeed, these approaches are the cornerstone for constructing a covariant, gauge-invariant theory of perturbations, which can be employed in many contexts. In particular, the 1+1+2 formalism has been used to describe tensor perturbations of Schwarzschild black holes [21], to describe complex interaction between gravitational and electromagnetic degrees of freedom [31], or to study cosmological perturbations [32, 33].

The scope of this work is to construct a completely general covariant and gauge-invariant perturbation theory of non-vacuum, static, spatially compact LRS II spacetimes. Moreover, as a first application, we aim to describe the dynamics of adiabatic isotropic perturbations. Using the covariant nature of the equations, we will be able to describe the evolution of the perturbation equations from the point of view of an observer locally comoving with the fluid and one which is static with respect to an observer at spatial infinity. For both cases, we propose a method to find a family of exact solutions in the form of a power series for a wide variety of background solutions. We will also prove that the perturbation equations with appropriate boundary conditions constitute a singular Sturm–Liouville eigenvalue problem with a limit-point-non-oscillating endpoint, which, to our knowledge, has not been rigorously proven before. Using this property, we will be able to establish lower bounds for the absolute value of the fundamental eigenfrequency in terms of quantities of the background equilibrium solution.

The paper is organized as follows: section 2 summarizes the 1+1+2 formalism for a generic spacetime. In section 3, we derive the linearized covariant gauge-invariant perturbation equations for a static, spatially compact LRSII background, and we perform a harmonic decomposition, in particular, isolating even and odd perturbations. In section 4, we focus on adiabatic and isotropic perturbations, writing the equations in a comoving frame and a static frame, and for each frame, we properly define the boundary value problem. Here, we show that the perturbation equations in the static frame form a Sturm–Liouville eigenvalue problem and derive lower bounds for the values of the fundamental eigenvalue. Last, we summarize and draw some conclusions in section 6. The paper also contains six appendices. In appendix A we introduce the elementary definitions of the 1+1+2 potentials, and in appendix B we present the Einstein field equations in the language of the 1+1+2 formalism; in appendices C and D we display the linearized field equations for the two sets of gauge invariant quantities adopted in the body of the text. In appendix E, we present the definitions and properties of the eigenfunctions of the covariant Laplace–Beltrami operator on 2-hypersurfaces. In appendix F, we discuss the effects on the equation of state of a generalized Lorentz boost between two frames.

Throughout the article, we will work in the geometrized unit system where $8\pi G = c = 1$, and consider the metric signature $(-+++)$.

2. The 1+1+2 decomposition

To construct a general set of gauge-invariant, covariant equations for the perturbations of locally rotationally symmetric class II spacetimes, from hereon LRS II [15], we will adopt the language of the 1+1+2 covariant formalism [21–23]. In this section, we will then introduce the basic quantities and conventions used throughout the article.

2.1. Projectors and the Levi–Civita volume form

Consider a Lorentzian manifold of dimension 4, $(\mathcal{M}, g)$, where g represents the metric tensor, admitting in some open neighborhood the existence of a congruence of timelike curves with tangent vector field u. We will assume that the congruence to be affinely parameterized and $u_\alpha u^\alpha = -1$. Without loss of generality, we can locally foliate the manifold in 3-surfaces, V, orthogonal at each point to the curves of the congruence, such that all tensor quantities are defined by their behavior along the direction of u and in V. This procedure is usually called 1+3 spacetime decomposition. Such decomposition of the spacetime manifold relies on the existence of a pointwise projector to the cotangent space of V, which can be naturally defined as

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \,, \tag{1}$$

where $g_{\alpha\beta}$ represents the components of the metric tensor in some local coordinate system, with the following properties

$$\begin{aligned} h_{\alpha\beta} &= h_{\beta\alpha}\,, & h_{\alpha\beta}h^{\beta\gamma} &= h_\alpha{}^\gamma\,, \\ h_{\alpha\beta}u^\alpha &= 0\,, & h_\alpha{}^\alpha &= 3\,. \end{aligned} \tag{2}$$

The 1+1+2 decomposition builds from the 1+3 decomposition by defining a congruence of spacelike curves with tangent vector field e such that any tensor quantity defined in the submanifold V is defined by its behavior along e and the 2-surfaces W, orthogonal to both u and e at each point. We shall refer to each surface W as 'sheet'. We will consider that the spacelike congruence is affinely parameterized and $e_\alpha e^\alpha = 1$. We can then define a projector onto W by

$$N_{\alpha\beta} = h_{\alpha\beta} - e_\alpha e_\beta \,, \tag{3}$$

verifying

$$\begin{aligned} N_{\alpha\beta} &= N_{\beta\alpha}\,, & N_{\alpha\beta}N^{\beta\gamma} &= N_\alpha{}^\gamma\,, \\ N_{\alpha\beta}u^\alpha &= N_{\alpha\beta}e^\alpha = 0\,, & N_\alpha{}^\alpha &= 2\,. \end{aligned} \tag{4}$$

It is useful to introduce the following tensors derived from the covariant Levi–Civita tensor $\varepsilon_{\alpha\beta\gamma\sigma}$,

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma} &= \varepsilon_{\alpha\beta\gamma\sigma}u^\sigma\,, \\ \varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma}e^\gamma\,, \end{aligned} \tag{5}$$

with the following properties

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma} &= \varepsilon_{[\alpha\beta\gamma]}\,, & \varepsilon_{\alpha\beta} &= \varepsilon_{[\alpha\beta]}\,, \\ \varepsilon_{\alpha\beta\gamma}u^\gamma &= 0\,, & \varepsilon_{\alpha\beta}u^\alpha &= \varepsilon_{\alpha\beta}e^\alpha = 0\,, \\ \varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\sigma} &= 6h^\mu{}_{[\alpha}h^\nu{}_\beta h_{\gamma]}{}^\sigma & \varepsilon_{\alpha\beta}\varepsilon^{\mu\nu} &= N^\mu{}_\alpha N^\nu{}_\beta - N^\mu{}_\beta N^\nu{}_\alpha \\ \varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\gamma} &= h_\alpha{}^\mu h_\beta{}^\nu - h_\beta{}^\mu h_\alpha{}^\nu\,, & \varepsilon_\alpha{}^\gamma\varepsilon_{\beta\gamma} &= N_{\alpha\beta}\,, \\ \varepsilon_{\alpha\mu\nu}\varepsilon^{\beta\mu\nu} &= 2h_\alpha{}^\beta\,, & \varepsilon_{\alpha\beta\gamma} &= e_\alpha\varepsilon_{\beta\gamma} - e_\beta\varepsilon_{\alpha\gamma} + e_\gamma\varepsilon_{\alpha\beta}\,, \end{aligned} \tag{6}$$

where in the right-hand side of the relation for $\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\sigma}$ the anti-symmetrization is to be considered on all, and only, the lower indices. We will adopt the convention to indicate the symmetric and anti-symmetric part of a tensor using parentheses and brackets, such that for a 2-tensor χ

$$\chi_{(\alpha\beta)} = \frac{1}{2}\left(\chi_{\alpha\beta} + \chi_{\beta\alpha}\right)\,, \qquad \chi_{[\alpha\beta]} = \frac{1}{2}\left(\chi_{\alpha\beta} - \chi_{\beta\alpha}\right)\,. \tag{7}$$

2.2. Covariant derivatives of u and e

Using the definitions of the projector operators onto the surfaces V and W, and the definitions in appendix A we can fully characterize the covariant derivatives of the tangent vector fields u and e in terms of the 1+1+2 kinematical quantities, such that

$$\begin{aligned} \nabla_\alpha u_\beta = {} & N_{\alpha\beta}\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) + \Sigma_{\alpha\beta} + \varepsilon_{\alpha\beta}\Omega + \left(\frac{1}{3}\theta + \Sigma\right)e_\alpha e_\beta \\ & + 2\Sigma_{(\alpha}e_{\beta)} - \varepsilon_{\alpha\mu}\Omega^\mu e_\beta + e_\alpha\varepsilon_{\beta\mu}\Omega^\mu - u_\alpha\left(\mathcal{A}e_\beta + \mathcal{A}_\beta\right)\,, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \nabla_\alpha e_\beta = {} & \frac{1}{2}N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi + e_\alpha a_\beta - u_\alpha\alpha_\beta - \mathcal{A}u_\alpha u_\beta \\ & + \left(\frac{1}{3}\theta + \Sigma\right)e_\alpha u_\beta + \left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu\right)u_\beta\,. \end{aligned} \tag{9}$$

For notational convenience, given a tensor quantity χ, throughout the article, we will use the compact notation

$$\begin{aligned} \dot{\chi}_{\alpha\ldots\beta}{}^{\gamma\ldots\delta} := u^\mu\nabla_\mu\chi_{\alpha\ldots\beta}{}^{\gamma\ldots\delta}\,, \quad \widehat{\chi}_{\alpha\ldots\beta}{}^{\gamma\ldots\delta} &:= e^\mu D_\mu\chi_{\alpha\ldots\beta}{}^{\gamma\ldots\delta} \\ &:= e^\mu h_\alpha{}^\nu\ldots h_\beta{}^\rho h_\sigma{}^\gamma\ldots h_\tau{}^\delta\nabla_\mu\chi_{\nu\ldots\rho}{}^{\sigma\ldots\tau}\,, \end{aligned} \tag{10}$$

to respectively represent the covariant derivative along the integral curves of the vector field u, and the covariant derivative along the integral curves of the vector field e, fully projected onto V.

2.3. Weyl and stress-energy tensors

For the Levi–Civita connection, the Riemann tensor can be defined by the Ricci identity valid for an arbitrary 1-form χ:

$$R_{\alpha\beta\delta}{}^{\rho}\chi_\rho = \left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\chi_\delta\,. \tag{11}$$

In the case of a manifold of dimension 4, the components of the Riemann curvature tensor, $R_{\alpha\beta\gamma\delta}$, can be written as the following sum

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + R_{\alpha[\gamma}\, g_{\delta]\beta} - R_{\beta[\gamma}\, g_{\delta]\alpha} - \frac{1}{3} R\, g_{\alpha[\gamma}\, g_{\delta]\beta}\,, \tag{12}$$

where $C_{\alpha\beta\gamma\delta}$ represent the components of the Weyl tensor, $R_{\alpha\beta} := R_{\alpha\mu\beta}{}^{\mu}$ the components of the Ricci tensor and R the Ricci scalar. The Weyl tensor plays a pivotal role in relativistic gravity, describing the tidal forces and the properties of gravitational waves. The Weyl tensor itself is fully characterized by the 'electric' and 'magnetic' parts, such that

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = & -\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_\alpha E_{\beta[\gamma}\, u_{\delta]} + 2u_\beta E_{\alpha[\gamma}\, u_{\delta]} - 2\varepsilon_{\alpha\beta\mu}H^{\mu}{}_{[\gamma}\, u_{\delta]} \\ & - 2\varepsilon_{\mu\gamma\delta}H^{\mu}{}_{[\alpha}\, u_{\beta]}\,. \end{aligned} \tag{13}$$

In the 1+1+2 spacetime decomposition formalism, the components of the Weyl tensor are decomposed as

$$\begin{aligned} E_{\alpha\beta} &= \mathcal{E}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{E}_\alpha e_\beta + e_\alpha\mathcal{E}_\beta + \mathcal{E}_{\alpha\beta}\,, \\ H_{\alpha\beta} &= \mathcal{H}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{H}_\alpha e_\beta + e_\alpha\mathcal{H}_\beta + \mathcal{H}_{\alpha\beta}\,. \end{aligned} \tag{14}$$

To write the Einstein field equation in the 1+1+2 framework, we have to also decompose the metric stress-energy tensor, with components $T_{\alpha\beta}$ in a local coordinate system, in terms of its pointwise projections onto u, e, and W, finding:

$$\begin{aligned} T_{\alpha\beta} = & \,\mu\, u_\alpha u_\beta + (p+\Pi)\, e_\alpha e_\beta + \left(p - \frac{1}{2}\Pi\right)N_{\alpha\beta} + 2Qe_{(\alpha}\, u_{\beta)} \\ & + 2Q_{(\alpha}\, u_{\beta)} + 2\Pi_{(\alpha}\, e_{\beta)} + \Pi_{\alpha\beta}\,. \end{aligned} \tag{15}$$

Moreover, we have the following equations

$$\begin{aligned} p_r &= e^{\mu}e^{\mu}T_{\mu\nu} = p + \Pi\,, \\ p_\perp &= \frac{1}{2}N^{\mu\nu}T_{\mu\nu} = p - \frac{1}{2}\Pi\,, \end{aligned} \tag{16}$$

relating, respectively, the 1+1+2 variables p and Π with the 'radial' and 'tangential' components of the pressure.

2.4. Commutation relations

Given a scalar field χ in $(\mathcal{M},g)$, using the 1+1+2 spacetime decomposition, we have the following useful commutation relations for its covariant derivatives:

$$\begin{aligned}
\widehat{\dot{\chi}} - \dot{\widehat{\chi}} &= \left(\frac{1}{3}\theta + \Sigma\right)\widehat{\chi} - \mathcal{A}\dot{\chi} + \left(\Sigma_\mu + \varepsilon_{\mu\nu}\Omega^\nu - \alpha_\mu\right)\delta^\mu\chi\,,\\
\delta_\alpha\dot{\chi} - N_\alpha{}^\mu\left(\delta_\mu\chi\right)^{\cdot} &= \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)\delta_\alpha\chi + \left(\Sigma_{\alpha\mu} + \varepsilon_{\alpha\mu}\Omega\right)\delta^\mu\chi\\
&\quad + \left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu + \alpha_\alpha\right)\widehat{\chi} - \mathcal{A}_\alpha\dot{\chi}\,,\\
\delta_\alpha\widehat{\chi} - N_\alpha{}^\mu\left(\widehat{\delta_\mu\chi}\right) &= \frac{1}{2}\phi\delta_\alpha\chi + \left(\zeta_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\xi\right)\delta_\mu\chi - 2\varepsilon_{\alpha\mu}\Omega^\mu\dot{\chi} + \widehat{\chi}a_\alpha\,,\\
\delta_\alpha\delta_\beta\chi - \delta_\beta\delta_\alpha\chi &= 2\varepsilon_{\alpha\beta}\Omega\dot{\chi} - 2\varepsilon_{\alpha\beta}\xi\widehat{\chi}.
\end{aligned} \tag{17}$$

In the case of the covariant derivatives of a 1-form field χ defined on the sheet, that is $\chi_\alpha = N_\alpha{}^\mu\chi_\mu$, the commutation relations take the form:

$$\begin{aligned}
N_\alpha{}^\mu\widehat{\dot{\chi}_\mu} - N_\alpha{}^\mu\left(\widehat{\chi}_\mu\right)^{\cdot} &= \left(\frac{1}{3}\theta + \Sigma\right)N_\alpha{}^\mu\widehat{\chi}_\mu - \mathcal{A}N_\alpha{}^\mu\dot{\chi}_\mu + \chi_\mu\left(\Sigma^\mu + \varepsilon^{\mu\nu}\Omega_\nu\right)\mathcal{A}_\alpha\\
&\quad + \left(\Sigma^\mu + \varepsilon^{\mu\nu}\Omega_\nu - \alpha^\mu\right)\left(\delta_\mu\chi_\alpha\right) + \varepsilon_\alpha{}^\mu\chi_\mu\mathcal{H}\,,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\delta_\alpha\dot{\chi}_\beta - N_\alpha{}^\mu N_\beta{}^\nu\left(\delta_\mu\chi_\nu\right)^{\cdot} &= \left(\Sigma_\alpha + \alpha_\alpha - \varepsilon_{\alpha\lambda}\Omega^\lambda\right)N_\beta{}^\mu\widehat{\chi}_\mu - \mathcal{A}_\alpha N_\beta{}^\mu\dot{\chi}_\mu - \frac{1}{2}N_{\alpha\beta}\mathcal{Q}^\mu\chi_\mu + \mathcal{H}_\alpha\varepsilon_\beta{}^\mu\chi_\mu\\
&\quad + \left[N_\alpha{}^\mu\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) + \Sigma_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\Omega\right]\left(\delta_\mu\chi_\beta\right) - \chi_\mu\left(\zeta_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\xi\right)\alpha_\beta\\
&\quad + \chi_\alpha\left[\mathcal{A}_\beta\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) + \frac{1}{2}\mathcal{Q}_\beta - \frac{1}{2}\phi\alpha_\beta\right] + \chi_\mu\left(\Sigma_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\Omega\right)\mathcal{A}_\beta\,,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\delta_\alpha\widehat{\chi}_\beta - N_\alpha{}^\mu N_\beta{}^\nu\left(\widehat{\delta_\mu\chi_\nu}\right) &= a_\alpha N_\beta{}^\mu\widehat{\chi}_\mu - 2\varepsilon_{\alpha\mu}\Omega^\mu N_\beta{}^\nu\dot{\chi}_\nu + \left(\frac{1}{2}N_\alpha{}^\mu\phi + \zeta_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\xi\right)\delta_\mu\chi_\beta\\
&\quad + \chi_\alpha\left[\left(\Sigma_\beta + \varepsilon_{\beta\mu}\Omega^\mu\right)\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) - \mathcal{E}_\beta - \frac{1}{2}\Pi_\beta - \frac{1}{2}\phi a_\beta\right]\\
&\quad + \chi_\mu\left[\left(\Sigma_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\Omega\right)\left(\Sigma_\beta + \varepsilon_{\beta\lambda}\Omega^\lambda\right) - \left(\zeta_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\xi\right)a_\beta\right]\\
&\quad + N_{\alpha\beta}\chi_\mu\left[\mathcal{E}^\mu + \frac{1}{2}\Pi^\mu - \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)\left(\Sigma^\mu + \varepsilon^{\mu\nu}\Omega_\nu\right)\right]\\
&\quad - \chi_\mu\left(\Sigma_{\alpha\beta} + \varepsilon_{\alpha\beta}\Omega\right)\left(\Sigma^\mu + \varepsilon^{\mu\nu}\Omega_\nu\right)\,,
\end{aligned} \tag{20}$$

$$\begin{aligned}
\delta_\alpha\delta_\beta\chi_\gamma - \delta_\beta\delta_\alpha\chi_\gamma &= 2\varepsilon_{\alpha\beta}N_\gamma{}^\mu\left(\dot{\chi}_\mu\Omega - \widehat{\chi}_\mu\xi\right) + 2\chi_{[\alpha}N_{\beta]\gamma}\left[\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)^2 - \frac{1}{4}\phi^2 + \mathcal{E} + \frac{1}{2}\Pi\right.\\
&\quad \left. - \frac{1}{3}\left(\mu + \Lambda\right)\right] + 2\left[\left(\Sigma_{[\alpha|}{}^\mu + \varepsilon_{[\alpha|}{}^\mu\Omega\right)\left(\Sigma_{|\beta]\gamma} + \varepsilon_{|\beta]\gamma}\Omega\right) - \left(\zeta_{[\alpha|}{}^\mu + \varepsilon_{[\alpha|}{}^\mu\xi\right)\right.\\
&\quad \left.\times\left(\zeta_{|\beta]\gamma} + \varepsilon_{|\beta]\gamma}\xi\right)\right]\chi_\mu + 2\left[\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)\left(\Sigma_{[\alpha|}{}^\mu + \varepsilon_{[\alpha|}{}^\mu\Omega\right) - \frac{1}{2}\phi\left(\zeta_{[\alpha|}{}^\mu\right.\right.\\
&\quad \left.\left. + \varepsilon_{[\alpha|}{}^\mu\xi\right) - \frac{1}{2}\Pi_{[\alpha|}{}^\mu\right]N_{|\beta]\gamma}\chi_\mu + 2\chi_{[\alpha|}\left[\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)\left(\Sigma_{|\beta]\gamma} + \varepsilon_{|\beta]\gamma}\Omega\right)\right.\\
&\quad \left. - \frac{1}{2}\phi\left(\zeta_{|\beta]\gamma} + \varepsilon_{|\beta]\gamma}\xi\right) - \frac{1}{2}\Pi_{|\beta]\gamma}\right].
\end{aligned} \tag{21}$$

2.5. The set of 1+1+2 equations

The 1+1+2 variables introduced in the previous subsections completely describe the geometry of the spacetime manifold and the properties of the matter fields that permeate it. Using the Ricci and Bianchi identities and the Einstein field equations for the theory of General Relativity,

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = T_{\alpha\beta} \,, \tag{22}$$

where Λ represents the cosmological constant, we can find a set of evolution, propagation, and constraint equations for the 1+1+2 variables. In appendix B we give the general set of these equations. Using the freedom of choice of frame to describe the setup and providing an equation of state for the matter fields, those equations form a closed system, describing the geometry of the manifold and the dynamics of the permeating matter fields.

3. Linearized equations for the perturbed spacetime

3.1. Background spacetime

Having properly introduced the 1+1+2 covariant formalism, we are now in a position to derive a set of covariant and gauge invariant equations to describe linear order perturbations of a spacetime assumed to be static, LRS II, and permeated by a general matter fluid. As is customary in relativistic perturbation theory, throughout the article, we will refer to the unperturbed spacetime as the 'background spacetime'.

For a proper choice of frame, in an LRS II spacetime, all covariantly defined vector and tensor quantities of the 1+1+2 decomposition can be made to vanish identically. If, in addition, the spacetime is static, we can align the u vector field with the timelike, hypersurface orthogonal Killing vector field, such that all dot-derivatives of the covariantly defined quantities are zero and the scalars $\{\theta, \Sigma, \Omega, \xi, \mathcal{H}, Q\}$ also vanish. Hence, a static LRS II background spacetime can be completely characterized by the quantities $\{\phi_0, \mathcal{A}_0, \mathcal{E}_0, \mu_0, p_0, \Pi_0, \Lambda\}$, which satisfy the following equations

$$\begin{aligned}
\widehat{\mathcal{A}}_0 &= \frac{1}{2}(\mu_0 + 3p_0) - \Lambda - \mathcal{A}_0(\mathcal{A}_0 + \phi_0) \,, \\
\widehat{\phi}_0 &= -\frac{1}{2}\phi_0^2 - \frac{2}{3}(\mu_0 + \Lambda) - \frac{1}{2}\Pi_0 - \mathcal{E}_0 \,, \\
\widehat{p}_0 + \widehat{\Pi}_0 &= -\left(\frac{3}{2}\phi_0 + \mathcal{A}_0\right)\Pi_0 - (\mu_0 + p_0)\mathcal{A}_0 \,, \\
\widehat{\mathcal{E}}_0 - \frac{1}{3}\widehat{\mu}_0 + \frac{1}{2}\widehat{\Pi}_0 &= -\frac{3}{2}\phi_0\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right) ,
\end{aligned} \tag{23}$$

and the constraint

$$\mu_0 + 3p_0 - 2\Lambda - 3\mathcal{A}_0\phi_0 = -\frac{3}{2}\Pi_0 + 3\mathcal{E}_0 \,, \tag{24}$$

where the subscript '0' from hereon will be used to refer to the quantities that characterize the equilibrium configuration. The system is closed by providing an equation of state that relates the pressure components, p_0 and Π_0, with the energy density, μ_0, or some functional dependencies for these matter variables are imposed.

3.2. Gauge invariant variables

Following the Stewart–Walker lemma [12], to write a set of equations for the linear perturbations that are identification gauge invariant, we will consider perturbation variables that either vanish in the background spacetime or are scalars with the same constant value in the background and the perturbed spacetimes (e.g. the cosmological constant). Since the background spacetime is assumed to be static LRS II, provided the choice of frame adopted in the previous subsection, all vector and tensor quantities can be used as identification gauge invariant perturbation quantities. Moreover, for such spacetimes, the scalars $\{\theta, \Sigma, \Omega, \xi, \mathcal{H}, Q\}$ are identically zero, hence are identification gauge invariant quantities. Then, we only have to find a set of variables that vanish in the background spacetime, and that can be used to characterize the linear perturbations of the remaining scalars, i.e. $\{\mathcal{A}, \phi, \mathcal{E}, \mu, p, \Pi\}$. Given the symmetries of the background spacetime, two natural sets of variables can be adopted. The first one is given by

$$\begin{aligned} &\mathbb{A}_\alpha := \delta_\alpha \mathcal{A}, \quad \mathbb{F}_\alpha := \delta_\alpha \phi, \quad \mathbb{E}_\alpha := \delta_\alpha \mathcal{E}, \\ &\mathfrak{m}_\alpha := \delta_\alpha \mu, \quad \mathfrak{p}_\alpha := \delta_\alpha p, \quad \mathbb{P}_a := \delta_\alpha \Pi, \end{aligned} \tag{25}$$

which represent the gradients on the sheets of the various scalar quantities that describe the background spacetime. The second one instead contains

$$\begin{aligned} &\mathsf{A} := \dot{\mathcal{A}}, \quad \mathsf{F} := \dot{\phi}, \quad \mathsf{E} := \dot{\mathcal{E}}, \\ &\mathsf{m} := \dot{\mu}, \quad \mathsf{p} := \dot{p}, \quad \mathcal{P} := \dot{\Pi}, \end{aligned} \tag{26}$$

which represent the dot-derivatives of the various scalar quantities that describe the spacetime. In a static LRS II background, both the δ-gradients in equation (25), which are vectors, and the quantities in equation (26), which are related to the proper-time variation, vanish. Therefore, the quantities of the sets above can be used as gauge invariant variables that characterize the linear perturbations of a static LRS II spacetime.

In principle, we can use either set among equations (25) and (26). Moreover, as shown in equation (D34), the above perturbation variables are not independent. Then, in general, we can also construct a set of variables containing elements of both equations (25) and (26), provided that they form a closed system. Note, however, that dependence does not imply equivalence. For example, the δ-gradients variables in equation (25) preserve the information regarding the degrees of freedom on the sheet, however, if we only perturb the background spacetime in directions parallel to the u and e vector fields, these variables by themselves are not suitable to fully describe the perturbed spacetime, as they will remain identically zero. Conversely, the variables in equation (26) can be used to fully characterize linear perturbations of $\{\mathcal{A}, \phi, \mathcal{E}, \mu, p, \Pi\}$, only losing information in the case of constant-in-time perturbations. Thus, this second set of variables can appear more convenient to analyze the dynamics of the perturbations. The drawback is that some of the linearized 1+1+2 equations for the variables (26) are second order in time (cf appendix D), introducing extra complexity to the problem. Then, in what follows, we will opt for using the δ-gradients in equation (25) only to describe vector and tensor perturbations, since those will be characterized by first-order equations. Conversely, we will use the variables in equation (26) to describe the scalar modes.

3.3. The linearization procedure

The procedure to find the set of gauge invariant structure equations for linear perturbations of static, LRS II spacetimes is relatively straightforward, although rather laborious. After choosing the gauge-invariant quantities from those in equation (25) or (26), we deduce the equations for these quantities by applying the projected derivative operators 'dot' and δ_α to the 1+1+2 equations in appendix B. Then, using the commutation relations in section 2.4, we express all gauge-dependent terms as a combination of gauge-independent quantities. Successively, the terms containing the products of two or more first-order quantities are discarded as higher order. The gauge invariant quantities that complete characterization of the perturbed space are determined by a set of linearized equations found directly from the 1+1+2 equations considering only zeroth and first order terms and without requiring taking their derivatives.

We present in appendices C and D the general set of gauge independent field equations, valid at linear level, for the perturbations for both the δ-gradients and the dot-derivatives variables, respectively. As explained above, we will opt to use a mix of these equations to describe general linear perturbations. However, the set of equations in each appendix mentioned above is completely general and can be used by themselves.

3.4. Harmonic decomposition

The sets of the linearized equations in appendices C and D form two systems of partial differential equations. The appearance of δ-derivatives makes the system particularly complicated to integrate, such that finding solutions, even in relatively simple setups, an intractable problem. Following [21–23], to transform this system into a system of ordinary differential equations (ODEs) valid at linear level, we can use the fact that the background spacetime is static and LRS II.

Using the eigenfunctions of the projected covariant Laplace–Beltrami operator, $\delta^2 \equiv \delta_\mu \delta^\mu$, locally defined on the sheets of the LRSII background spacetime, we can make a harmonic decomposition and write the various quantities that characterize the perturbed spacetime as a linear combination of these eigenfunctions. In appendix E, we list the definitions and various useful properties of the scalar, vector, and tensor harmonics of the δ^2 operator. Then, given some scalar, 1-tensor, or symmetric, traceless 2-tensor quantity defined on a sheet of the perturbed spacetime, say χ, χ_α or $\chi_{\alpha\beta}$, at linear level we can formally write these as the following infinite sum

$$
\begin{aligned}
\chi &= \Psi_\chi^{(k,S)} \mathcal{Q}^{(k)}, \\
\chi_\alpha &= \Psi_\chi^{(k,V)} \mathcal{Q}_\alpha^{(k)} + \overline{\Psi}_\chi^{(k,V)} \bar{\mathcal{Q}}_\alpha^{(k)}, \\
\chi_{\alpha\beta} &= \Psi_\chi^{(k,T)} \mathcal{Q}_{\alpha\beta}^{(k)} + \overline{\Psi}_\chi^{(k,T)} \bar{\mathcal{Q}}_{\alpha\beta}^{(k)},
\end{aligned}
\tag{27}
$$

where summation or integration in the Laplace–Beltrami eigenvalue k (the so-called 'modes') is assumed, depending on the geometry of the sheets. We will use the compact notation $\Psi_\chi^{(k,S)}$ to refer to the harmonic coefficients associated with the eigenfunctions $\mathcal{Q}^{(k)}$ of the scalar quantity χ, $\Psi_\chi^{(k,V)}$ and $\overline{\Psi}_\chi^{(k,V)}$ to refer to the coefficients associated with the eigenfunctions $\mathcal{Q}_\alpha^{(k)}$ or $\bar{\mathcal{Q}}_\alpha^{(k)}$ of the 1-tensor quantity χ_α, and $\Psi_\chi^{(k,T)}$ and $\overline{\Psi}_\chi^{(k,T)}$ to refer to the coefficients associated with the eigenfunctions $\mathcal{Q}_{\alpha\beta}^{(k)}$ or $\bar{\mathcal{Q}}_{\alpha\beta}^{(k)}$ of the 2-tensor quantity $\chi_{\alpha\beta}$.

Additionally, since the background spacetime is assumed to be static, we can Fourier transform the harmonic coefficients of equation (27), explicitly factorizing their time dependence, writing them as a linear combination of the eigenfunctions of the Laplace operator in $\mathbb{R}$ (or some subset of it) for the appropriate boundary conditions. That is, we can write all coefficients for linear perturbations: $\Psi_\chi^{(k,I)}$, where $I = \{S, V, T\}$, as linear combinations of the eigenfunctions $e^{i\upsilon\tau}$, where τ represents the proper time of an observer with 4-velocity u and υ represent the eigenvalues of the Laplace operator in that manifold. The scalar υ takes discrete or continuous values depending on the chosen boundary conditions. To make this idea precise, consider the time-harmonic functions $T^{(\upsilon)}$ with the following properties

$$\begin{aligned} \dot{T}^{(\upsilon)} &= \mathrm{i}\upsilon T^{(\upsilon)}\,, \\ \widehat{T}^{(\upsilon)} &= \delta_\alpha T^{(\upsilon)} = 0\,, \\ \dot{\upsilon} &= \delta_\alpha \upsilon = 0\,, \end{aligned} \tag{28}$$

where i represents the imaginary unit. These properties imply that $\widehat{\dot{T}}^{(\upsilon)} = -\mathcal{A}_0 \dot{T}^{(\upsilon)}$, from which we find

$$\widehat{\upsilon} = -\mathcal{A}_0 \upsilon\,. \tag{29}$$

Knowing the function $\mathcal{A}_0 = \mathcal{A}_0(x^\alpha)$ in the background, where $\{x^\alpha\}$ represent some local coordinate system, equation (29) allow us to relate the eigenfrequencies υ defined with respect to τ and the eigenfrequencies defined with respect to the x^0 time coordinate.

Gathering these results, formally, we can expand any scalar, 1-tensor or a symmetric, traceless 2-tensor characterizing first-order quantities as

$$\begin{aligned} \chi &= \Psi_\chi^{(\upsilon,k,S)} \mathcal{Q}^{(k)} T^{(\upsilon)}\,, \\ \chi_\alpha &= \left(\Psi_\chi^{(\upsilon,k,V)} \mathcal{Q}_\alpha^{(k)} + \overline{\Psi}_\chi^{(\upsilon,k,V)} \bar{\mathcal{Q}}_\alpha^{(k)}\right) T^{(\upsilon)}\,, \\ \chi_{\alpha\beta} &= \left(\Psi_\chi^{(\upsilon,k,T)} \mathcal{Q}_{\alpha\beta}^{(k)} + \overline{\Psi}_\chi^{(\upsilon,k,T)} \bar{\mathcal{Q}}_{\alpha\beta}^{(k)}\right) T^{(\upsilon)}\,, \end{aligned} \tag{30}$$

where either discrete sums or multiple integrals in υ and k are assumed, once again, depending on the boundary conditions of the problem.

Using the harmonic decomposition just described, we can turn the sets of partial differential equations in appendices C and D, in a set of ODEs. We remark, though, that since the different harmonics are not defined for all values of k, we have to distinguish between the modes where the vector and tensor harmonics are not identically zero and those where these harmonics are zero. Then, we have the following set of gauge invariant equations for the linear perturbations that characterize the perturbed spacetime.

3.5. *Even sector*

3.5.1. *k-modes where* $\left\{\mathcal{Q}_{\alpha\beta}^{(k)}, \bar{\mathcal{Q}}_{\alpha\beta}^{(k)}\right\} \neq 0$

3.5.1.1. *Equations for the kinematical quantities associated with the timelike congruence*

- Evolution and propagation equations for the harmonic coefficients of gradients of scalar quantities:

$$\begin{aligned} \frac{1}{r}\mathrm{i}\upsilon\Psi_{\theta}^{(\upsilon,k,S)} - \widehat{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} = {}& \widehat{\mathcal{A}}_0 \Psi_a^{(\upsilon,k,V)} - \frac{1}{2}\left(\Psi_{\mathfrak{m}}^{(\upsilon,k,V)} + 3\Psi_{\mathfrak{p}}^{(\upsilon,k,V)}\right) - \frac{k^2}{r^2}\Psi_{\mathcal{A}}^{(\upsilon,k,V)} \\ & + \mathcal{A}_0 \Psi_{\mathbb{F}}^{(\upsilon,k,V)} + \left(\frac{3}{2}\phi_0 + 2\mathcal{A}_0\right)\Psi_{\mathbb{A}}^{(\upsilon,k,V)}\,, \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{2}{3r}\mathrm{i}\upsilon\Psi_{\theta}^{(\upsilon,k,S)} - \frac{1}{r}\mathrm{i}\upsilon\Psi_{\Sigma}^{(\upsilon,k,S)} = {}& \phi_0 \Psi_{\mathbb{A}}^{(\upsilon,k,V)} + \mathcal{A}_0 \Psi_{\mathbb{F}}^{(\upsilon,k,V)} - \frac{k^2}{r^2}\Psi_{\mathcal{A}}^{(\upsilon,k,V)} \\ & - \frac{1}{3}\Psi_{\mathfrak{m}}^{(\upsilon,k,V)} - \Psi_{\mathfrak{p}}^{(\upsilon,k,V)} - \frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)} + \Psi_{\mathbb{E}}^{(\upsilon,k,V)}\,, \end{aligned} \tag{32}$$

$$\frac{3}{2}\widehat{\Psi}_{\Sigma}^{(\upsilon,k,S)} - \widehat{\Psi}_{\theta}^{(\upsilon,k,S)} = \frac{3k^2}{2r}\left(\Psi_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) - \frac{3}{2}\Psi_{Q}^{(\upsilon,k,S)} - \frac{9}{4}\phi_0\Psi_{\Sigma}^{(\upsilon,k,S)}; \tag{33}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$2\mathrm{i}\upsilon\overline{\Psi}_{\Omega}^{(\upsilon,k,V)} + \widehat{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} = \Psi_{\mathbb{A}}^{(\upsilon,k,V)} - \frac{1}{2}\phi_0\Psi_{\mathcal{A}}^{(\upsilon,k,V)} - \mathcal{A}_0\Psi_{a}^{(\upsilon,k,V)}, \tag{34}$$

$$\mathrm{i}\upsilon\left(\Psi_{\Sigma}^{(\upsilon,k,V)} + \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Psi_{\mathcal{A}}^{(\upsilon,k,V)} + \Psi_{\mathbb{A}}^{(\upsilon,k,V)} - \Psi_{\mathcal{E}}^{(\upsilon,k,V)} + \frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}, \tag{35}$$

$$\begin{aligned}\widehat{\Psi}_{\Sigma}^{(\upsilon,k,V)} + \widehat{\overline{\Psi}}_{\Omega}^{(\upsilon,k,V)} &= \frac{1}{2r}\Psi_{\Sigma}^{(\upsilon,k,S)} + \frac{2}{3r}\Psi_{\theta}^{(\upsilon,k,S)} - \frac{3}{2}\phi_0\Psi_{\Sigma}^{(\upsilon,k,V)} - \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\overline{\Psi}_{\Omega}^{(\upsilon,k,V)} \\ &\quad - \frac{2-k^2}{2r}\Psi_{\Sigma}^{(\upsilon,k,T)} - \Psi_{Q}^{(\upsilon,k,V)};\end{aligned} \tag{36}$$

- Evolution and propagation equations for the harmonic coefficients of tensor quantities:

$$\mathrm{i}\upsilon\Psi_{\Sigma}^{(\upsilon,k,T)} = \frac{1}{r}\Psi_{\mathcal{A}}^{(\upsilon,k,V)} + \mathcal{A}_0\Psi_{\zeta}^{(\upsilon,k,T)} - \Psi_{\mathcal{E}}^{(\upsilon,k,T)} + \frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,T)}, \tag{37}$$

$$\widehat{\Psi}_{\Sigma}^{(\upsilon,k,T)} = \frac{1}{r}\left(\Psi_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) - \frac{1}{2}\phi_0\Psi_{\Sigma}^{(\upsilon,k,T)} + \overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,T)}; \tag{38}$$

- Constraint equations for the harmonic coefficients:

$$\begin{aligned}&\phi_0\left(\Psi_{\Sigma}^{(\upsilon,k,V)} + \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) + \frac{1}{r}\Psi_{\Sigma}^{(\upsilon,k,S)} - \frac{2}{3r}\Psi_{\theta}^{(\upsilon,k,S)} + \Psi_{Q}^{(\upsilon,k,V)} - 2\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)} \\ &\quad + \frac{2-k^2}{r}\Psi_{\Sigma}^{(\upsilon,k,T)} = 0.\end{aligned} \tag{39}$$

3.5.1.2. Equations for the kinematical quantities associated with the spacelike congruence

- Evolution and propagation equations for the harmonic coefficients of gradients of scalar quantities:

$$\begin{aligned}\mathrm{i}\upsilon\Psi_{\mathbb{F}}^{(\upsilon,k,V)} &= -\widehat{\phi}_0\left(\Psi_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\alpha}^{(\upsilon,k,V)} + \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) + \frac{1}{r}\Psi_{Q}^{(\upsilon,k,S)} \\ &\quad + \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\left(\frac{2}{3r}\Psi_{\theta}^{(\upsilon,k,S)} - \frac{1}{r}\Psi_{\Sigma}^{(\upsilon,k,S)}\right) - \frac{k^2}{r^2}\Psi_{\alpha}^{(\upsilon,k,V)},\end{aligned} \tag{40}$$

$$\widehat{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} = -\left(\widehat{\phi}_0 + \frac{k^2}{r^2}\right)\Psi_{a}^{(\upsilon,k,V)} - \frac{3}{2}\phi_0\Psi_{\mathbb{F}}^{(\upsilon,k,V)} - \frac{2}{3}\Psi_{\mathrm{m}}^{(\upsilon,k,V)} - \frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)} - \Psi_{\mathbb{E}}^{(\upsilon,k,V)}; \tag{41}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\begin{aligned}\widehat{\Psi}_{\alpha}^{(\upsilon,k,V)} - \mathrm{i}\upsilon\Psi_{a}^{(\upsilon,k,V)} &= \overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)} - \left(\mathcal{A}_0 + \frac{1}{2}\phi_0\right)\Psi_{\alpha}^{(\upsilon,k,V)} + \frac{1}{2}\Psi_{Q}^{(\upsilon,k,V)} \\ &\quad + \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\Psi_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right);\end{aligned} \tag{42}$$

- Evolution and propagation equations for the harmonic coefficients of tensor quantities:

$$\mathrm{i}\upsilon\Psi_{\zeta}^{(\upsilon,k,T)} = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Psi_{\Sigma}^{(\upsilon,k,T)} + \frac{1}{r}\Psi_{\alpha}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,T)}, \tag{43}$$

$$\widehat{\Psi}_{\zeta}^{(\upsilon,k,T)} = \frac{1}{r}\Psi_{a}^{(\upsilon,k,V)} - \phi_0\Psi_{\zeta}^{(\upsilon,k,T)} - \Psi_{\mathcal{E}}^{(\upsilon,k,T)} - \frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,T)}; \tag{44}$$

- Constraint equations for the harmonic coefficients:

$$\frac{2-k^2}{r}\Psi_{\zeta}^{(\upsilon,k,T)} - \Psi_{\mathbb{F}}^{(\upsilon,k,V)} - 2\Psi_{\mathcal{E}}^{(\upsilon,k,V)} - \Psi_{\Pi}^{(\upsilon,k,V)} = 0. \tag{45}$$

3.5.1.3. Equations for the Weyl tensor components and the matter variables

- Evolution and propagation equations for the harmonic coefficients of gradients of scalar quantities:

$$\begin{aligned}\frac{1}{r}\widehat{\Psi}_{Q}^{(\upsilon,k,S)} + \mathrm{i}\upsilon\Psi_{\mathrm{m}}^{(\upsilon,k,V)} = &-\widehat{\mu}_0\left(\overline{\Psi}_{\Omega}^{(\upsilon,k,V)} + \Psi_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\alpha}^{(\upsilon,k,V)}\right) - \frac{1}{r}\left(\mu_0 + p_0\right)\Psi_{\theta}^{(\upsilon,k,S)}\\ &- \frac{3}{2r}\Pi_0\Psi_{\Sigma}^{(\upsilon,k,S)} - \frac{1}{r}\left(\phi_0 + 2\mathcal{A}_0\right)\Psi_{Q}^{(\upsilon,k,S)} + \frac{k^2}{r^2}\Psi_{Q}^{(\upsilon,k,V)},\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{1}{3}\mathrm{i}\upsilon\Psi_{\mathrm{m}}^{(\upsilon,k,V)} - \frac{1}{2}\mathrm{i}\upsilon\Psi_{\mathbb{P}}^{(\upsilon,k,V)} - \mathrm{i}\upsilon\Psi_{\mathbb{E}}^{(\upsilon,k,V)} = &\left(\mu_0 + p_0 - \frac{1}{2}\Pi_0 - 3\mathcal{E}_0\right)\left(\frac{1}{2r}\Psi_{\Sigma}^{(\upsilon,k,S)} - \frac{1}{3r}\Psi_{\theta}^{(\upsilon,k,S)}\right)\\ &- \frac{3}{2}\phi_0\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\left(\overline{\Psi}_{\Omega}^{(\upsilon,k,V)} + \Psi_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\alpha}^{(\upsilon,k,V)}\right)\\ &+ \frac{k^2}{r^2}\left(\frac{1}{2}\Psi_{Q}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}\right) - \frac{1}{2r}\phi_0\Psi_{Q}^{(\upsilon,k,S)},\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{1}{r}\mathrm{i}\upsilon\Psi_{Q}^{(\upsilon,k,S)} + \widehat{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} + \widehat{\Psi}_{\mathrm{p}}^{(\upsilon,k,V)} = &\left(\mu_0 + p_0\right)\left(\mathcal{A}_0\Psi_{a}^{(\upsilon,k,V)} - \Psi_{\mathbb{A}}^{(\upsilon,k,V)}\right) - \mathcal{A}_0\left(\Psi_{\mathrm{m}}^{(\upsilon,k,V)} + \Psi_{\mathrm{p}}^{(\upsilon,k,V)}\right.\\ &\left.+\Psi_{\mathbb{P}}^{(\upsilon,k,V)}\right) - \frac{1}{2}\phi_0\left(\Psi_{\mathrm{p}}^{(\upsilon,k,V)} + 4\Psi_{\mathbb{P}}^{(\upsilon,k,V)}\right)\\ &+ \left(\frac{3}{2}\phi_0 + \mathcal{A}_0\right)\Pi_0\Psi_{a}^{(\upsilon,k,V)} - \Pi_0\left(\frac{3}{2}\Psi_{\mathbb{F}}^{(\upsilon,k,V)} + \Psi_{\mathbb{A}}^{(\upsilon,k,V)}\right)\\ &+ \frac{k^2}{r^2}\Psi_{\Pi}^{(\upsilon,k,V)},\end{aligned} \tag{48}$$

$$\begin{aligned}\widehat{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\widehat{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} - \frac{1}{3}\widehat{\Psi}_{\mathrm{m}}^{(\upsilon,k,V)} = &-2\phi_0\left(\Psi_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)} - \frac{1}{12}\Psi_{\mathrm{m}}^{(\upsilon,k,V)}\right)\\ &- \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\left(\Psi_{\mathbb{F}}^{(\upsilon,k,V)} - \phi_0\Psi_{a}^{(\upsilon,k,V)}\right)\\ &+ \frac{k^2}{r^2}\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)} + \frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right);\end{aligned} \tag{49}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\begin{aligned}\mathrm{i}\upsilon\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)} + \frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right) = &\left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\frac{1}{2}\Psi_{Q}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}\right) - \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\Psi_{\alpha}^{(\upsilon,k,V)}\\ &- \frac{1}{2}\left(\mu_0 + p_0 + \Pi_0\right)\left(\Psi_{\Sigma}^{(\upsilon,k,V)} + \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right)\\ &- \frac{1}{2r}\Psi_{Q}^{(\upsilon,k,S)} + \frac{2-k^2}{2r}\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,T)},\end{aligned} \tag{50}$$

$$
\begin{aligned}
\widehat{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\widehat{\Psi}_{\Pi}^{(\upsilon,k,V)}=&-\frac{3}{2}\phi_0\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Psi_{a}^{(\upsilon,k,V)}+\frac{1}{2}\Psi_{\mathbb{E}}^{(\upsilon,k,V)}\\
&+\frac{1}{3}\Psi_{\mathsf{m}}^{(\upsilon,k,V)}+\frac{1}{4}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}-\frac{2-k^2}{2r}\left(\Psi_{\mathcal{E}}^{(\upsilon,k,T)}+\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,T)}\right),
\end{aligned}\tag{51}
$$

$$
\begin{aligned}
\frac{1}{2}\widehat{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}-\mathrm{i}\upsilon\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}-\frac{1}{4}\widehat{\Psi}_{\Pi}^{(\upsilon,k,V)}=&\frac{3}{2}\mathcal{E}_0\Psi_{\mathcal{A}}^{(\upsilon,k,V)}+\frac{3}{4}\left(\Psi_{\mathbb{E}}^{(\upsilon,k,V)}-\frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}\right)-\frac{3}{4}\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\\
&\times\Psi_{a}^{(\upsilon,k,V)}-\frac{1}{4}\phi_0\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)}-\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right)-\mathcal{A}_0\Psi_{\mathcal{E}}^{(\upsilon,k,V)}\\
&+\frac{2-k^2}{4r}\left(\Psi_{\mathcal{E}}^{(\upsilon,k,T)}-\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,T)}\right),
\end{aligned}\tag{52}
$$

$$
\begin{aligned}
\widehat{\overline{\Psi}}_{\mathcal{H}}^{(\upsilon,k,V)}-\frac{1}{2}\widehat{\Psi}_{Q}^{(\upsilon,k,V)}=&\frac{2-k^2}{2r}\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,T)}-\frac{1}{2r}\Psi_{Q}^{(\upsilon,k,S)}-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Psi_{\Sigma}^{(\upsilon,k,V)}\\
&-\left(\mu_0+p_0+\frac{1}{4}\Pi_0-\frac{3}{2}\mathcal{E}_0\right)\overline{\Psi}_{\Omega}^{(\upsilon,k,V)}-\frac{3}{2}\phi_0\left(\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}-\frac{1}{6}\Psi_{Q}^{(\upsilon,k,V)}\right),
\end{aligned}\tag{53}
$$

$$
\begin{aligned}
\mathrm{i}\upsilon\Psi_{Q}^{(\upsilon,k,V)}+\widehat{\Psi}_{\Pi}^{(\upsilon,k,V)}=&\frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}-\Psi_{\mathsf{p}}^{(\upsilon,k,V)}-\frac{3}{2}\Pi_0\Psi_{a}^{(\upsilon,k,V)}-\frac{2-k^2}{2r}\Psi_{\Pi}^{(\upsilon,k,T)}\\
&-\left(\frac{3}{2}\phi_0+\mathcal{A}_0\right)\Psi_{\Pi}^{(\upsilon,k,V)}-\left(\mu_0+p_0-\frac{1}{2}\Pi_0\right)\Psi_{\mathcal{A}}^{(\upsilon,k,V)};
\end{aligned}\tag{54}
$$

- Evolution and propagation equations for the harmonic coefficients of tensor quantities:

$$
\begin{aligned}
\mathrm{i}\upsilon\left(\Psi_{\mathcal{E}}^{(\upsilon,k,T)}+\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,T)}\right)+\widehat{\overline{\Psi}}_{\mathcal{H}}^{(\upsilon,k,T)}=&-\frac{1}{r}\left(\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}+\frac{1}{2}\Psi_{Q}^{(\upsilon,k,V)}\right)-\left(\frac{1}{2}\phi_0+2\mathcal{A}_0\right)\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,T)}\\
&-\frac{1}{2}\left(\mu_0+p_0+3\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\Psi_{\Sigma}^{(\upsilon,k,T)},
\end{aligned}\tag{55}
$$

$$
\begin{aligned}
\frac{1}{2}\widehat{\Psi}_{\Pi}^{(\upsilon,k,T)}-\widehat{\Psi}_{\mathcal{E}}^{(\upsilon,k,T)}-\mathrm{i}\upsilon\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,T)}=&\frac{1}{r}\left(\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}-\Psi_{\mathcal{E}}^{(\upsilon,k,V)}\right)-\frac{1}{4}\phi_0\Psi_{\Pi}^{(\upsilon,k,T)}\\
&-\frac{3}{2}\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\Psi_{\zeta}^{(\upsilon,k,T)}+\left(\frac{1}{2}\phi_0+2\mathcal{A}_0\right)\Psi_{\mathcal{E}}^{(\upsilon,k,T)}.
\end{aligned}\tag{56}
$$

3.5.2. k-modes where $\left\{\mathcal{Q}_{\alpha}^{(k)},\bar{\mathcal{Q}}_{\alpha}^{(k)}\right\}\neq 0\wedge\left\{\mathcal{Q}_{\alpha\beta}^{(k)},\bar{\mathcal{Q}}_{\alpha\beta}^{(k)}\right\}=0$: $k^2=2$. For the $k^2=2$ modes, the tensor harmonics are not defined, but in general, the vector and scalar harmonics do not vanish. For those modes, we find the following equations for the harmonic coefficients.

3.5.2.1. Equations for the kinematical quantities associated with the timelike congruence

- Evolution and propagation equations for the harmonic coefficients of gradients of scalar quantities:

$$
\begin{aligned}
\frac{1}{r}\mathrm{i}\upsilon\Psi_{\Theta}^{(\upsilon,k,S)}-\widehat{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}=&\widehat{\mathcal{A}}_0\Psi_{a}^{(\upsilon,k,V)}-\frac{2}{r^2}\Psi_{\mathcal{A}}^{(\upsilon,k,V)}-\frac{1}{2}\left(\Psi_{\mathsf{m}}^{(\upsilon,k,V)}+3\Psi_{\mathsf{p}}^{(\upsilon,k,V)}\right)\\
&+\mathcal{A}_0\Psi_{\mathbb{F}}^{(\upsilon,k,V)}+\left(\frac{3}{2}\phi_0+2\mathcal{A}_0\right)\Psi_{\mathbb{A}}^{(\upsilon,k,V)},
\end{aligned}\tag{57}
$$

$$
\begin{aligned}
\frac{2}{3r}\mathrm{i}\upsilon\Psi_{\Theta}^{(\upsilon,k,S)}-\frac{1}{r}\mathrm{i}\upsilon\Psi_{\Sigma}^{(\upsilon,k,S)}=&\phi_0\Psi_{\mathbb{A}}^{(\upsilon,k,V)}+\mathcal{A}_0\Psi_{\mathbb{F}}^{(\upsilon,k,V)}-\frac{2}{r^2}\Psi_{\mathcal{A}}^{(\upsilon,k,V)}\\
&-\frac{1}{3}\Psi_{\mathsf{m}}^{(\upsilon,k,V)}-\Psi_{\mathsf{p}}^{(\upsilon,k,V)}-\frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}+\Psi_{\mathbb{E}}^{(\upsilon,k,V)},
\end{aligned}\tag{58}
$$

$$
\frac{3}{2}\widehat{\Psi}_{\Sigma}^{(\upsilon,k,S)}-\widehat{\Psi}_{\Theta}^{(\upsilon,k,S)}=\frac{3}{r}\left(\Psi_{\Sigma}^{(\upsilon,k,V)}-\overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right)-\frac{3}{2}\Psi_{Q}^{(\upsilon,k,S)}-\frac{9}{4}\phi_0\Psi_{\Sigma}^{(\upsilon,k,S)},\tag{59}
$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$2\mathrm{i}\upsilon\overline{\Psi}_{\Omega}^{(\upsilon,k,V)} + \widehat{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} = \Psi_{\mathbb{A}}^{(\upsilon,k,V)} - \frac{1}{2}\phi_0\Psi_{\mathcal{A}}^{(\upsilon,k,V)} - \mathcal{A}_0\Psi_{a}^{(\upsilon,k,V)}, \tag{60}$$

$$\mathrm{i}\upsilon\left(\Psi_{\Sigma}^{(\upsilon,k,V)} + \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Psi_{\mathcal{A}}^{(\upsilon,k,V)} + \Psi_{\mathbb{A}}^{(\upsilon,k,V)} - \Psi_{\mathcal{E}}^{(\upsilon,k,V)} + \frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}, \tag{61}$$

$$\widehat{\Psi}_{\Sigma}^{(\upsilon,k,V)} + \widehat{\overline{\Psi}}_{\Omega}^{(\upsilon,k,V)} = \frac{1}{2r}\Psi_{\Sigma}^{(\upsilon,k,S)} + \frac{2}{3r}\Psi_{\theta}^{(\upsilon,k,S)} - \frac{3}{2}\phi_0\Psi_{\Sigma}^{(\upsilon,k,V)} - \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\overline{\Psi}_{\Omega}^{(\upsilon,k,V)} - \Psi_{Q}^{(\upsilon,k,V)}; \tag{62}$$

- Constraint equations for the harmonic coefficients:

$$\phi_0\left(\Psi_{\Sigma}^{(\upsilon,k,V)} + \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) + \frac{1}{r}\Psi_{\Sigma}^{(\upsilon,k,S)} - \frac{2}{3r}\Psi_{\theta}^{(\upsilon,k,S)} + \Psi_{Q}^{(\upsilon,k,V)} - 2\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)} = 0. \tag{63}$$

3.5.2.2. Equations for the kinematical quantities associated with the spacelike congruence

- Evolution and propagation equations for the harmonic coefficients of gradients of scalar quantities:

$$i\upsilon\Psi_{\mathbb{F}}^{(\upsilon,k,V)} = -\widehat{\phi}_0\left(\Psi_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\alpha}^{(\upsilon,k,V)} + \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right) + \frac{1}{r}\Psi_{Q}^{(\upsilon,k,S)} + \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\left(\frac{2}{3r}\Psi_{\theta}^{(\upsilon,k,S)} - \frac{1}{r}\Psi_{\Sigma}^{(\upsilon,k,S)}\right) - \frac{2}{r^2}\Psi_{\alpha}^{(\upsilon,k,V)}, \tag{64}$$

$$\widehat{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} = -\left(\widehat{\phi}_0 + \frac{2}{r^2}\right)\Psi_{a}^{(\upsilon,k,V)} - \frac{3}{2}\phi_0\Psi_{\mathbb{F}}^{(\upsilon,k,V)} - \frac{2}{3}\Psi_{\mathrm{m}}^{(\upsilon,k,V)} - \frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)} - \Psi_{\mathbb{E}}^{(\upsilon,k,V)}; \tag{65}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\widehat{\Psi}_{\alpha}^{(\upsilon,k,V)} - \mathrm{i}\upsilon\Psi_{a}^{(\upsilon,k,V)} = \overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)} - \left(\mathcal{A}_0 + \frac{1}{2}\phi_0\right)\Psi_{\alpha}^{(\upsilon,k,V)} + \frac{1}{2}\Psi_{Q}^{(\upsilon,k,V)} + \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\Psi_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right); \tag{66}$$

- Constraint equations for the harmonic coefficients:

$$\Psi_{\mathbb{F}}^{(\upsilon,k,V)} + 2\Psi_{\mathcal{E}}^{(\upsilon,k,V)} + \Psi_{\Pi}^{(\upsilon,k,V)} = 0. \tag{67}$$

3.5.2.3. Equations for the Weyl tensor components and the matter variables

- Evolution and propagation equations for the harmonic coefficients of gradients of scalar quantities:

$$\frac{1}{r}\widehat{\Psi}_{Q}^{(\upsilon,k,S)} + \mathrm{i}\upsilon\Psi_{\mathrm{m}}^{(\upsilon,k,V)} = -\widehat{\mu}_0\left(\overline{\Psi}_{\Omega}^{(\upsilon,k,V)} + \Psi_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\alpha}^{(\upsilon,k,V)}\right) - \frac{1}{r}\left(\mu_0 + p_0\right)\Psi_{\theta}^{(\upsilon,k,S)} - \frac{3}{2r}\Pi_0\Psi_{\Sigma}^{(\upsilon,k,S)} - \frac{1}{r}\left(\phi_0 + 2\mathcal{A}_0\right)\Psi_{Q}^{(\upsilon,k,S)} + \frac{2}{r^2}\Psi_{Q}^{(\upsilon,k,V)}, \tag{68}$$

$$\frac{1}{3}\mathrm{i}\upsilon\Psi_{\mathfrak{m}}^{(\upsilon,k,V)}-\frac{1}{2}\mathrm{i}\upsilon\Psi_{\mathbb{P}}^{(\upsilon,k,V)}-\mathrm{i}\upsilon\Psi_{\mathbb{E}}^{(\upsilon,k,V)}=\left(\mu_0+p_0-\frac{1}{2}\Pi_0-3\mathcal{E}_0\right)\left(\frac{1}{2r}\Psi_{\Sigma}^{(\upsilon,k,S)}-\frac{1}{3r}\Psi_{\theta}^{(\upsilon,k,S)}\right)$$
$$-\frac{3}{2}\phi_0\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\left(\overline{\Psi}_{\Omega}^{(\upsilon,k,V)}+\Psi_{\Sigma}^{(\upsilon,k,V)}+\Psi_{\alpha}^{(\upsilon,k,V)}\right) \quad (69)$$
$$+\frac{2}{r^2}\left(\frac{1}{2}\Psi_{\mathcal{Q}}^{(\upsilon,k,V)}-\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}\right)-\frac{1}{2r}\phi_0\Psi_{\mathcal{Q}}^{(\upsilon,k,S)},$$

$$\frac{1}{r}\mathrm{i}\upsilon\Psi_{\mathcal{Q}}^{(\upsilon,k,S)}+\widehat{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}+\widehat{\Psi}_{\mathfrak{p}}^{(\upsilon,k,V)}=(\mu_0+p_0)\left(\mathcal{A}_0\Psi_{a}^{(\upsilon,k,V)}-\Psi_{\mathbb{A}}^{(\upsilon,k,V)}\right)-\mathcal{A}_0\left(\Psi_{\mathfrak{m}}^{(\upsilon,k,V)}+\Psi_{\mathfrak{p}}^{(\upsilon,k,V)}\right.$$
$$\left.+\Psi_{\mathbb{P}}^{(\upsilon,k,V)}\right)-\frac{1}{2}\phi_0\left(\Psi_{\mathfrak{p}}^{(\upsilon,k,V)}+4\Psi_{\mathbb{P}}^{(\upsilon,k,V)}\right)+\left(\frac{3}{2}\phi_0+\mathcal{A}_0\right) \quad (70)$$
$$\times\Pi_0\Psi_{a}^{(\upsilon,k,V)}-\Pi_0\left(\frac{3}{2}\Psi_{\mathbb{F}}^{(\upsilon,k,V)}+\Psi_{\mathbb{A}}^{(\upsilon,k,V)}\right)+\frac{2}{r^2}\Psi_{\Pi}^{(\upsilon,k,V)},$$

$$\widehat{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)}+\frac{1}{2}\widehat{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}-\frac{1}{3}\widehat{\Psi}_{\mathfrak{m}}^{(\upsilon,k,V)}=-2\phi_0\left(\Psi_{\mathbb{E}}^{(\upsilon,k,V)}+\frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}-\frac{1}{12}\Psi_{\mathfrak{m}}^{(\upsilon,k,V)}\right)$$
$$-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\left(\Psi_{\mathbb{F}}^{(\upsilon,k,V)}-\phi_0\Psi_{a}^{(\upsilon,k,V)}\right) \quad (71)$$
$$+\frac{2}{r^2}\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right);$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\mathrm{i}\upsilon\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right)=\left(\frac{1}{2}\phi_0-\mathcal{A}_0\right)\left(\frac{1}{2}\Psi_{\mathcal{Q}}^{(\upsilon,k,V)}-\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Psi_{\alpha}^{(\upsilon,k,V)}$$
$$-\frac{1}{2}(\mu_0+p_0+\Pi_0)\left(\Psi_{\Sigma}^{(\upsilon,k,V)}+\overline{\Psi}_{\Omega}^{(\upsilon,k,V)}\right)-\frac{1}{2r}\Psi_{\mathcal{Q}}^{(\upsilon,k,S)}, \quad (72)$$

$$\widehat{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\widehat{\Psi}_{\Pi}^{(\upsilon,k,V)}=-\frac{3}{2}\phi_0\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Psi_{a}^{(\upsilon,k,V)}$$
$$+\frac{1}{2}\Psi_{\mathbb{E}}^{(\upsilon,k,V)}+\frac{1}{3}\Psi_{\mathfrak{m}}^{(\upsilon,k,V)}+\frac{1}{4}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}, \quad (73)$$

$$\frac{1}{2}\widehat{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}-\mathrm{i}\upsilon\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}-\frac{1}{4}\widehat{\Psi}_{\Pi}^{(\upsilon,k,V)}=\frac{3}{2}\mathcal{E}_0\Psi_{\mathcal{A}}^{(\upsilon,k,V)}+\frac{3}{4}\left(\Psi_{\mathbb{E}}^{(\upsilon,k,V)}-\frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}\right)$$
$$-\frac{3}{4}\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\Psi_{a}^{(\upsilon,k,V)}-\frac{1}{4}\phi_0\left(\Psi_{\mathcal{E}}^{(\upsilon,k,V)}\right. \quad (74)$$
$$\left.-\frac{1}{2}\Psi_{\Pi}^{(\upsilon,k,V)}\right)-\mathcal{A}_0\Psi_{\mathcal{E}}^{(\upsilon,k,V)},$$

$$\widehat{\overline{\Psi}}_{\mathcal{H}}^{(\upsilon,k,V)}-\frac{1}{2}\widehat{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)}=-\frac{1}{2r}\Psi_{\mathcal{Q}}^{(\upsilon,k,S)}-\left(\mu_0+p_0+\frac{1}{4}\Pi_0-\frac{3}{2}\mathcal{E}_0\right)\overline{\Psi}_{\Omega}^{(\upsilon,k,V)}$$
$$-\frac{3}{2}\phi_0\left(\overline{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}-\frac{1}{6}\Psi_{\mathcal{Q}}^{(\upsilon,k,V)}\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Psi_{\Sigma}^{(\upsilon,k,V)}, \quad (75)$$

$$\mathrm{i}\upsilon\Psi_{\mathcal{Q}}^{(\upsilon,k,V)}+\widehat{\Psi}_{\Pi}^{(\upsilon,k,V)}=\frac{1}{2}\Psi_{\mathbb{P}}^{(\upsilon,k,V)}-\Psi_{\mathfrak{p}}^{(\upsilon,k,V)}-\left(\mu_0+p_0-\frac{1}{2}\Pi_0\right)\Psi_{\mathcal{A}}^{(\upsilon,k,V)}$$
$$-\frac{3}{2}\Pi_0\Psi_{a}^{(\upsilon,k,V)}-\left(\frac{3}{2}\phi_0+\mathcal{A}_0\right)\Psi_{\Pi}^{(\upsilon,k,V)}. \quad (76)$$

3.5.3. k-modes where $\mathcal{Q}^{(k)}\neq 0\wedge\left\{\mathcal{Q}_{\alpha}^{(k)},\bar{\mathcal{Q}}_{\alpha}^{(k)}\right\}=0\wedge\left\{\mathcal{Q}_{\alpha\beta}^{(k)},\bar{\mathcal{Q}}_{\alpha\beta}^{(k)}\right\}=0$*: $k=0$.* As mentioned above, for the *k*-modes where the vector and tensor harmonics are not defined, we cannot use the equations in appendix C, since those only relate to vector and tensor quantities, that is, those equations are trivially verified and do not contain information regarding linear perturbations that are only along the *u* and the *e* directions. Then, to describe the dynamics of the perturbations that are not along the directions on the sheets, we will use, instead, the relations in appendix D. The equations for the harmonic coefficients for the $k=0$ mode are

$$\begin{aligned}\widehat{\Psi}_{\mathsf{A}}^{(\upsilon,0,S)}+\upsilon^2\Psi_{\theta}^{(\upsilon,0,S)}=&\frac{1}{2}\left(\Psi_{\mathsf{m}}^{(\upsilon,0,S)}+3\Psi_{\mathsf{p}}^{(\upsilon,0,S)}\right)+\widehat{\mathcal{A}}_0\left(\frac{1}{3}\Psi_{\theta}^{(\upsilon,0,S)}+\Psi_{\Sigma}^{(\upsilon,0,S)}\right)\\&-\left(3\mathcal{A}_0+\phi_0\right)\Psi_{\mathsf{A}}^{(\upsilon,0,S)}-\mathcal{A}_0\Psi_{\mathsf{F}}^{(\upsilon,0,S)},\end{aligned}\tag{77}$$

$$\begin{aligned}\upsilon^2\left(\frac{2}{3}\Psi_{\theta}^{(\upsilon,0,S)}-\Psi_{\Sigma}^{(\upsilon,0,S)}\right)=&\frac{1}{3}\left(\Psi_{\mathsf{m}}^{(\upsilon,0,S)}+3\Psi_{\mathsf{p}}^{(\upsilon,0,S)}\right)+\frac{1}{2}\Psi_{\mathcal{P}}^{(\upsilon,0,S)}-\Psi_{\mathsf{E}}^{(\upsilon,0,S)}\\&-\mathcal{A}_0\Psi_{\mathsf{F}}^{(\upsilon,0,S)}-\phi_0\Psi_{\mathsf{A}}^{(\upsilon,0,S)},\end{aligned}\tag{78}$$

$$\frac{2}{3}\widehat{\Psi}_{\theta}^{(\upsilon,0,S)}-\widehat{\Psi}_{\Sigma}^{(\upsilon,0,S)}=\Psi_{\mathcal{Q}}^{(\upsilon,0,S)}+\frac{3}{2}\phi_0\Psi_{\Sigma}^{(\upsilon,0,S)},\tag{79}$$

$$\begin{aligned}-\upsilon^2\Psi_{\mathcal{Q}}^{(\upsilon,0,S)}+\widehat{\Psi}_{\mathsf{p}}^{(\upsilon,0,S)}+\widehat{\Psi}_{\mathcal{P}}^{(\upsilon,0,S)}=&\left(\widehat{p}_0+\widehat{\Pi}_0\right)\left(\frac{1}{3}\Psi_{\theta}^{(\upsilon,0,S)}+\Psi_{\Sigma}^{(\upsilon,0,S)}\right)-\left(\mu_0+p_0\right)\Psi_{\mathsf{A}}^{(\upsilon,0,S)}\\&-\Pi_0\left(\frac{3}{2}\Psi_{\mathsf{F}}^{(\upsilon,0,S)}+\Psi_{\mathsf{A}}^{(\upsilon,0,S)}\right)-\left(\frac{3}{2}\phi_0+2\mathcal{A}_0\right)\Psi_{\mathcal{P}}^{(\upsilon,0,S)}\\&-\mathcal{A}_0\left(\Psi_{\mathsf{m}}^{(\upsilon,0,S)}+2\Psi_{\mathsf{p}}^{(\upsilon,0,S)}\right),\end{aligned}\tag{80}$$

$$\Psi_{\mathsf{m}}^{(\upsilon,0,S)}+\widehat{\Psi}_{\mathcal{Q}}^{(\upsilon,0,S)}=-\left(\phi_0+2\mathcal{A}_0\right)\Psi_{\mathcal{Q}}^{(\upsilon,0,S)}-\frac{3}{2}\Pi_0\Psi_{\Sigma}^{(\upsilon,0,S)}-\left(\mu_0+p_0\right)\Psi_{\theta}^{(\upsilon,0,S)};\tag{81}$$

and the constraints

$$\Psi_{\mathsf{F}}^{(\upsilon,0,S)}=\Psi_{\mathcal{Q}}^{(\upsilon,0,S)}+\left(2\mathcal{A}_0-\phi_0\right)\left(\frac{1}{3}\Psi_{\theta}^{(\upsilon,0,S)}-\frac{1}{2}\Psi_{\Sigma}^{(\upsilon,0,S)}\right),\tag{82}$$

$$\begin{aligned}\Psi_{\mathsf{E}}^{(\upsilon,0,S)}+\frac{1}{2}\Psi_{\mathcal{P}}^{(\upsilon,0,S)}-\frac{1}{3}\Psi_{\mathsf{m}}^{(\upsilon,0,S)}=&\left(\mu_0+p_0-\frac{1}{2}\Pi_0-3\mathcal{E}_0\right)\left(\frac{1}{3}\Psi_{\theta}^{(\upsilon,0,S)}\right.\\&\left.-\frac{1}{2}\Psi_{\Sigma}^{(\upsilon,0,S)}\right)+\frac{1}{2}\phi_0\Psi_{\mathcal{Q}}^{(\upsilon,0,S)}.\end{aligned}\tag{83}$$

3.6. Odd sector

3.6.1. k-modes where $\left\{\mathcal{Q}_{\alpha\beta}^{(k)},\bar{\mathcal{Q}}_{\alpha\beta}^{(k)}\right\}\neq 0$

3.6.1.1. Equations for the kinematical quantities associated with the timelike congruence

- Evolution and propagation equations for the harmonic coefficients of scalars and gradients of scalar quantities:

$$\begin{aligned}\widehat{\overline{\Psi}}_{\mathbb{A}}^{(\upsilon,k,V)}=&-\widehat{\mathcal{A}}_0\overline{\Psi}_{a}^{(\upsilon,k,V)}-\left(\frac{3}{2}\phi_0+2\mathcal{A}_0\right)\overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}\\&-\mathcal{A}_0\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)}+\frac{1}{2}\left(\overline{\Psi}_{\mathbb{m}}^{(\upsilon,k,V)}+3\overline{\Psi}_{\mathbb{p}}^{(\upsilon,k,V)}\right),\end{aligned}\tag{84}$$

$$\mathrm{i}\upsilon\Psi_{\Omega}^{(\upsilon,k,S)}=\frac{k^2}{2r}\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)}+\mathcal{A}_0\Psi_{\xi}^{(\upsilon,k,S)},\tag{85}$$

$$\widehat{\Psi}_{\Omega}^{(\upsilon,k,S)}=\left(\mathcal{A}_0-\phi_0\right)\Psi_{\Omega}^{(\upsilon,k,S)}+\frac{k^2}{r}\Psi_{\Omega}^{(\upsilon,k,V)};\tag{86}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\mathrm{i}\upsilon\Psi_{\Omega}^{(\upsilon,k,V)} - \frac{1}{2}\widehat{\overline{\Psi}}_{\mathcal{A}}^{(\upsilon,k,V)} = \frac{1}{2}\left(\frac{1}{2}\phi_0\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} + \mathcal{A}_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}\right), \quad (87)$$

$$\mathrm{i}\upsilon\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right) = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)} + \frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}, \quad (88)$$

$$\begin{aligned}\widehat{\overline{\Psi}}_{\Sigma}^{(\upsilon,k,V)} - \widehat{\Psi}_{\Omega}^{(\upsilon,k,V)} &= -\frac{1}{r}\Psi_{\Omega}^{(\upsilon,k,S)} - \frac{3}{2}\phi_0\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{Q}^{(\upsilon,k,V)} \\ &\quad + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\Psi_{\Omega}^{(\upsilon,k,V)} + \frac{2-k^2}{2r}\overline{\Psi}_{\Sigma}^{(\upsilon,k,T)};\end{aligned} \quad (89)$$

- Evolution and propagation equations for the harmonic coefficients of tensor quantities:

$$\mathrm{i}\upsilon\overline{\Psi}_{\Sigma}^{(\upsilon,k,T)} = -\frac{1}{r}\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} + \mathcal{A}_0\overline{\Psi}_{\zeta}^{(\upsilon,k,T)} - \overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,T)} + \frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,T)}, \quad (90)$$

$$\widehat{\overline{\Psi}}_{\Sigma}^{(\upsilon,k,T)} = -\frac{1}{r}\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\Omega}^{(\upsilon,k,V)}\right) - \frac{1}{2}\phi_0\overline{\Psi}_{\Sigma}^{(\upsilon,k,T)} - \Psi_{\mathcal{H}}^{(\upsilon,k,T)}; \quad (91)$$

- Constraint equations for the harmonic coefficients:

$$\frac{k^2}{2r}\overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} + \hat{\mathcal{A}}_0\Psi_{\xi}^{(\upsilon,k,S)} = 0, \quad (92)$$

$$-\Psi_{\mathcal{H}}^{(\upsilon,k,S)} + (\phi_0 - 2\mathcal{A}_0)\Psi_{\Omega}^{(\upsilon,k,S)} + \frac{k^2}{r}\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right) = 0, \quad (93)$$

$$\phi_0\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right) + \overline{\Psi}_{Q}^{(\upsilon,k,V)} + 2\Psi_{\mathcal{H}}^{(\upsilon,k,V)} + \frac{2}{r}\Psi_{\Omega}^{(\upsilon,k,S)} - \frac{2-k^2}{r}\overline{\Psi}_{\Sigma}^{(\upsilon,k,T)} = 0, \quad (94)$$

$$\phi_0\overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} + \mathcal{A}_0\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} - \frac{1}{3}\overline{\Psi}_{\mathtt{m}}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathtt{p}}^{(\upsilon,k,V)} - \frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} = 0. \quad (95)$$

3.6.1.2. Equations for the kinematical quantities associated with the spacelike congruence

- Evolution and propagation equations for the harmonic coefficients of scalars and gradients of scalar quantities:

$$\mathrm{i}\upsilon\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} = -\widehat{\phi}_0\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} + \overline{\Psi}_{\alpha}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right), \quad (96)$$

$$\widehat{\overline{\Psi}}_{\mathbb{F}}^{(\upsilon,k,V)} = -\widehat{\phi}_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \frac{2}{3}\overline{\Psi}_{\mathtt{m}}^{(\upsilon,k,V)} - \frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} - \frac{3}{2}\phi_0\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)}, \quad (97)$$

$$\mathrm{i}\upsilon\Psi_{\xi}^{(\upsilon,k,S)} = \frac{1}{2}\Psi_{\mathcal{H}}^{(\upsilon,k,S)} + \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Psi_{\Omega}^{(\upsilon,k,S)} + \frac{k^2}{2r}\overline{\Psi}_{\alpha}^{(\upsilon,k,V)}, \quad (98)$$

$$\widehat{\Psi}_{\xi}^{(\upsilon,k,S)} = -\phi_0\Psi_{\xi}^{(\upsilon,k,S)} + \frac{k^2}{2r}\overline{\Psi}_{a}^{(\upsilon,k,V)}; \quad (99)$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\widehat{\overline{\Psi}}_{\alpha}^{(\upsilon,k,V)} - \mathrm{i}\upsilon\overline{\Psi}_{a}^{(\upsilon,k,V)} = -\Psi_{\mathcal{H}}^{(\upsilon,k,V)} - \left(\mathcal{A}_0 + \frac{1}{2}\phi_0\right)\overline{\Psi}_{\alpha}^{(\upsilon,k,V)} + \frac{1}{2}\overline{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)} + \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\Omega}^{(\upsilon,k,V)}\right); \tag{100}$$

- Evolution and propagation equations for the harmonic coefficients of tensor quantities:

$$\mathrm{i}\upsilon\overline{\Psi}_{\zeta}^{(\upsilon,k,T)} = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\overline{\Psi}_{\Sigma}^{(\upsilon,k,T)} - \frac{1}{r}\overline{\Psi}_{\alpha}^{(\upsilon,k,V)} - \Psi_{\mathcal{H}}^{(\upsilon,k,T)}, \tag{101}$$

$$\widehat{\overline{\Psi}}_{\zeta}^{(\upsilon,k,T)} = -\frac{1}{r}\overline{\Psi}_{a}^{(\upsilon,k,V)} - \phi_0\overline{\Psi}_{\zeta}^{(\upsilon,k,T)} - \overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,T)} - \frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,T)}; \tag{102}$$

- Constraint equations for the harmonic coefficients:

$$\frac{k^2}{2r}\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} = -\widehat{\phi}_0\Psi_{\xi}^{(\upsilon,k,S)}, \tag{103}$$

$$\frac{2}{r}\Psi_{\xi}^{(\upsilon,k,S)} - \frac{2-k^2}{r}\overline{\Psi}_{\zeta}^{(\upsilon,k,T)} - \overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} - 2\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)} - \overline{\Psi}_{\Pi}^{(\upsilon,k,V)} = 0. \tag{104}$$

3.6.1.3. Equations for the Weyl tensor components and the matter variables

- Evolution and propagation equations for the harmonic coefficients of scalars and gradients of scalar quantities:

$$\mathrm{i}\upsilon\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)} = \hat{\mu}_0\left(\Psi_{\Omega}^{(\upsilon,k,V)} - \overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\alpha}^{(\upsilon,k,V)}\right), \tag{105}$$

$$\mathrm{i}\upsilon\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\mathrm{i}\upsilon\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} = \left(\widehat{\mathcal{E}}_0 + \frac{1}{2}\widehat{\Pi}_0\right)\left(\Psi_{\Omega}^{(\upsilon,k,V)} - \overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\alpha}^{(\upsilon,k,V)}\right), \tag{106}$$

$$\begin{aligned}\widehat{\overline{\Psi}}_{\mathbb{P}}^{(\upsilon,k,V)} + \widehat{\overline{\Psi}}_{\mathsf{p}}^{(\upsilon,k,V)} &= (\mu_0 + p_0)\left(\mathcal{A}_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}\right) - \mathcal{A}_0\left(\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathsf{p}}^{(\upsilon,k,V)}\right.\\ &\quad\left. + \overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right) - \frac{1}{2}\phi_0\left(\overline{\Psi}_{\mathsf{p}}^{(\upsilon,k,V)} + 4\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right) + \left(\frac{3}{2}\phi_0 + \mathcal{A}_0\right)\\ &\quad\times\Pi_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \Pi_0\left(\frac{3}{2}\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}\right),\end{aligned} \tag{107}$$

$$\begin{aligned}\widehat{\overline{\Psi}}_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\widehat{\overline{\Psi}}_{\mathbb{P}}^{(\upsilon,k,V)} - \frac{1}{3}\widehat{\overline{\Psi}}_{\mathsf{m}}^{(\upsilon,k,V)} &= -2\phi_0\left(\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} - \frac{1}{12}\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)}\right)\\ &\quad - \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\left(\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} - \phi_0\overline{\Psi}_{a}^{(\upsilon,k,V)}\right),\end{aligned} \tag{108}$$

$$\mathrm{i}\upsilon\Psi_{\mathcal{H}}^{(\upsilon,k,S)} = \frac{k^2}{r}\left(\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}\right) + 3\left(\frac{1}{2}\Pi_0 - \mathcal{E}_0\right)\Psi_{\xi}^{(\upsilon,k,S)}, \tag{109}$$

$$\widehat{\Psi}_{\mathcal{H}}^{(\upsilon,k,S)} = \frac{k^2}{r}\left(\Psi_{\mathcal{H}}^{(\upsilon,k,V)} - \frac{1}{2}\overline{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)}\right) - \frac{3}{2}\phi_0\Psi_{\mathcal{H}}^{(\upsilon,k,S)} - \left(3\mathcal{E}_0 + \mu_0 + p_0 - \frac{1}{2}\Pi_0\right)\Psi_{\Omega}^{(\upsilon,k,S)}; \tag{110}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$
\begin{aligned}
\mathrm{i}\upsilon\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}\right)=&\left(\frac{1}{2}\phi_0-\mathcal{A}_0\right)\left(\frac{1}{2}\overline{\Psi}_{Q}^{(\upsilon,k,V)}+\Psi_{\mathcal{H}}^{(\upsilon,k,V)}\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\overline{\Psi}_{\alpha}^{(\upsilon,k,V)}\\
&-\frac{1}{2}\left(\mu_0+p_0+\Pi_0\right)\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)}-\Psi_{\Omega}^{(\upsilon,k,V)}\right)\\
&+\frac{1}{2r}\Psi_{\mathcal{H}}^{(\upsilon,k,S)}+\frac{2-k^2}{2r}\Psi_{\mathcal{H}}^{(\upsilon,k,T)},
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
\widehat{\overline{\Psi}}_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\widehat{\overline{\Psi}}_{\Pi}^{(\upsilon,k,V)}=&-\frac{3}{2}\phi_0\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\overline{\Psi}_{a}^{(\upsilon,k,V)}+\frac{1}{2}\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)}\\
&+\frac{1}{3}\overline{\Psi}_{\mathfrak{m}}^{(\upsilon,k,V)}+\frac{1}{4}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}+\frac{2-k^2}{2r}\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,T)}+\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,T)}\right),
\end{aligned}
\tag{112}
$$

$$
\begin{aligned}
\frac{1}{4}\widehat{\overline{\Psi}}_{\Pi}^{(\upsilon,k,V)}-\frac{1}{2}\widehat{\overline{\Psi}}_{\mathcal{E}}^{(\upsilon,k,V)}-\mathrm{i}\upsilon\Psi_{\mathcal{H}}^{(\upsilon,k,V)}=&-\frac{3}{4}\left(\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)}-\frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right)+\frac{2-k^2}{4r}\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,T)}\right.\\
&\left.-\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,T)}\right)-\frac{3}{2}\mathcal{E}_0\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)}+\frac{1}{4}\phi_0\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}\right.\\
&\left.-\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}\right)+\frac{3}{4}\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\overline{\Psi}_{a}^{(\upsilon,k,V)}+\mathcal{A}_0\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)},
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
\widehat{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}+\frac{1}{2}\widehat{\overline{\Psi}}_{Q}^{(\upsilon,k,V)}=&\frac{1}{2r}\Psi_{\mathcal{H}}^{(\upsilon,k,S)}-\frac{2-k^2}{2r}\Psi_{\mathcal{H}}^{(\upsilon,k,T)}+\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)}\\
&-\frac{3}{2}\phi_0\left(\Psi_{\mathcal{H}}^{(\upsilon,k,V)}+\frac{1}{6}\overline{\Psi}_{Q}^{(\upsilon,k,V)}\right)-\left(\mu_0+p_0+\frac{1}{4}\Pi_0-\frac{3}{2}\mathcal{E}_0\right)\Psi_{\Omega}^{(\upsilon,k,V)},
\end{aligned}
\tag{114}
$$

$$
\begin{aligned}
\mathrm{i}\upsilon\overline{\Psi}_{Q}^{(\upsilon,k,V)}+\widehat{\overline{\Psi}}_{\Pi}^{(\upsilon,k,V)}=&\frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}-\overline{\Psi}_{\mathfrak{p}}^{(\upsilon,k,V)}-\frac{3}{2}\Pi_0\overline{\Psi}_{a}^{(\upsilon,k,V)}+\frac{2-k^2}{2r}\overline{\Psi}_{\Pi}^{(\upsilon,k,T)}\\
&-\left(\frac{3}{2}\phi_0+\mathcal{A}_0\right)\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}-\left(\mu_0+p_0-\frac{1}{2}\Pi_0\right)\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)};
\end{aligned}
\tag{115}
$$

- Evolution and propagation equations for the harmonic coefficients of tensor quantities:

$$
\begin{aligned}
\mathrm{i}\upsilon\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,T)}+\frac{1}{2}\mathrm{i}\upsilon\overline{\Psi}_{\Pi}^{(\upsilon,k,T)}-\widehat{\Psi}_{\mathcal{H}}^{(\upsilon,k,T)}=&-\frac{1}{r}\left(\Psi_{\mathcal{H}}^{(\upsilon,k,V)}-\frac{1}{2}\overline{\Psi}_{Q}^{(\upsilon,k,V)}\right)+\left(\frac{1}{2}\phi_0+2\mathcal{A}_0\right)\Psi_{\mathcal{H}}^{(\upsilon,k,T)}\\
&-\frac{1}{2}\left(\mu_0+p_0+3\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\overline{\Psi}_{\Sigma}^{(\upsilon,k,T)},
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
\widehat{\overline{\Psi}}_{\mathcal{E}}^{(\upsilon,k,T)}-\frac{1}{2}\widehat{\overline{\Psi}}_{\Pi}^{(\upsilon,k,T)}-\mathrm{i}\upsilon\Psi_{\mathcal{H}}^{(\upsilon,k,T)}=&\frac{1}{r}\left(\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}-\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}\right)+\frac{1}{4}\phi_0\overline{\Psi}_{\Pi}^{(\upsilon,k,T)}\\
&+\frac{3}{2}\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\overline{\Psi}_{\zeta}^{(\upsilon,k,T)}-\left(\frac{1}{2}\phi_0+2\mathcal{A}_0\right)\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,T)};
\end{aligned}
\tag{117}
$$

- Constraint equations for the harmonic coefficients:

$$
\frac{k^2}{r}\left(\frac{1}{3}\overline{\Psi}_{\mathfrak{m}}^{(\upsilon,k,V)}-\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)}-\frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right)=-3\phi_0\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Psi_{\xi}^{(\upsilon,k,S)},
\tag{118}
$$

$$
\frac{k^2}{2r}\overline{\Psi}_{\mathfrak{p}}^{(\upsilon,k,V)}+\widehat{p}_0\Psi_{\xi}^{(\upsilon,k,S)}=0.
\tag{119}
$$

3.6.2. k-modes where $\left\{\mathcal{Q}_{\alpha}^{(k)},\bar{\mathcal{Q}}_{\alpha}^{(k)}\right\}\neq 0\wedge\left\{\mathcal{Q}_{\alpha\beta}^{(k)},\bar{\mathcal{Q}}_{\alpha\beta}^{(k)}\right\}=0$*:* $k^2=2$*.* As was discussed in the even sector, for the $k^2=2$ modes, the tensor harmonics are not defined, but the vector and scalar harmonics do not vanish necessarily. For those modes, we find the following equations for the harmonic coefficients.

3.6.2.1. Equations for the kinematical quantities associated with the timelike congruence

- Evolution and propagation equations for the harmonic coefficients of scalars and gradients of scalar quantities:

$$\widehat{\overline{\Psi}}_{\mathbb{A}}^{(\upsilon,k,V)} = -\widehat{\mathcal{A}}_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \mathcal{A}_0\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} - \left(\frac{3}{2}\phi_0 + 2\mathcal{A}_0\right)\overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} + \frac{1}{2}\left(\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)} + 3\overline{\Psi}_{\mathsf{p}}^{(\upsilon,k,V)}\right), \tag{120}$$

$$\mathrm{i}\upsilon\Psi_{\Omega}^{(\upsilon,k,S)} = \frac{1}{r}\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} + \mathcal{A}_0\Psi_{\xi}^{(\upsilon,k,S)}, \tag{121}$$

$$\widehat{\Psi}_{\Omega}^{(\upsilon,k,S)} = (\mathcal{A}_0 - \phi_0)\Psi_{\Omega}^{(\upsilon,k,S)} + \frac{2}{r}\Psi_{\Omega}^{(\upsilon,k,V)}; \tag{122}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$2\mathrm{i}\upsilon\Psi_{\Omega}^{(\upsilon,k,V)} - \widehat{\overline{\Psi}}_{\mathcal{A}}^{(\upsilon,k,V)} = \frac{1}{2}\phi_0\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} + \mathcal{A}_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}, \tag{123}$$

$$\mathrm{i}\upsilon\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right) = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)} + \frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}, \tag{124}$$

$$\widehat{\overline{\Psi}}_{\Sigma}^{(\upsilon,k,V)} - \widehat{\Psi}_{\Omega}^{(\upsilon,k,V)} = -\frac{1}{r}\Psi_{\Omega}^{(\upsilon,k,S)} - \frac{3}{2}\phi_0\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{Q}^{(\upsilon,k,V)} + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\Psi_{\Omega}^{(\upsilon,k,V)}; \tag{125}$$

- Constraint equations for the harmonic coefficients:

$$\frac{1}{r}\overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} + \widehat{\mathcal{A}}_0\Psi_{\xi}^{(\upsilon,k,S)} = 0, \tag{126}$$

$$-\Psi_{\mathcal{H}}^{(\upsilon,k,S)} + (\phi_0 - 2\mathcal{A}_0)\Psi_{\Omega}^{(\upsilon,k,S)} + \frac{2}{r}\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right) = 0, \tag{127}$$

$$\phi_0\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right) + \overline{\Psi}_{Q}^{(\upsilon,k,V)} + 2\Psi_{\mathcal{H}}^{(\upsilon,k,V)} + \frac{2}{r}\Psi_{\Omega}^{(\upsilon,k,S)} = 0, \tag{128}$$

$$\phi_0\overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)} + \mathcal{A}_0\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} - \frac{1}{3}\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathsf{p}}^{(\upsilon,k,V)} - \frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} = 0. \tag{129}$$

3.6.2.2. Equations for the kinematical quantities associated with the spacelike congruence

- Evolution and propagation equations for the harmonic coefficients of scalars and gradients of scalar quantities:

$$\mathrm{i}\upsilon\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} = -\widehat{\phi}_0\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} + \overline{\Psi}_{\alpha}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right), \tag{130}$$

$$\widehat{\overline{\Psi}}_{\mathbb{F}}^{(\upsilon,k,V)} = -\widehat{\phi}_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \frac{2}{3}\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)} - \frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} - \frac{3}{2}\phi_0\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)}, \tag{131}$$

$$\mathrm{i}\upsilon\Psi_{\xi}^{(\upsilon,k,S)} = \frac{1}{2}\Psi_{\mathcal{H}}^{(\upsilon,k,S)} + \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Psi_{\Omega}^{(\upsilon,k,S)} + \frac{1}{r}\overline{\Psi}_{\alpha}^{(\upsilon,k,V)}, \tag{132}$$

$$\widehat{\Psi}_{\xi}^{(\upsilon,k,S)} = -\phi_0\Psi_{\xi}^{(\upsilon,k,S)} + \frac{1}{r}\overline{\Psi}_{a}^{(\upsilon,k,V)}; \tag{133}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\begin{aligned}\widehat{\overline{\Psi}}_{\alpha}^{(\upsilon,k,V)} - \mathrm{i}\upsilon\overline{\Psi}_{a}^{(\upsilon,k,V)} &= -\Psi_{\mathcal{H}}^{(\upsilon,k,V)} - \left(\mathcal{A}_0 + \frac{1}{2}\phi_0\right)\overline{\Psi}_{\alpha}^{(\upsilon,k,V)} + \frac{1}{2}\overline{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)} \\ &\quad + \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} + \Psi_{\Omega}^{(\upsilon,k,V)}\right);\end{aligned} \tag{134}$$

- Constraint equations for the harmonic coefficients:

$$\frac{1}{r}\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} = -\widehat{\phi}_0\Psi_{\xi}^{(\upsilon,k,S)}, \tag{135}$$

$$\frac{2}{r}\Psi_{\xi}^{(\upsilon,k,S)} - \overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} - 2\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)} - \overline{\Psi}_{\Pi}^{(\upsilon,k,V)} = 0. \tag{136}$$

3.6.2.3. Equations for the Weyl tensor components and the matter variables

- Evolution and propagation equations for the harmonic coefficients of scalars and gradients of scalar quantities:

$$\mathrm{i}\upsilon\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)} = \widehat{\mu}_0\left(\Psi_{\Omega}^{(\upsilon,k,V)} - \overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\alpha}^{(\upsilon,k,V)}\right), \tag{137}$$

$$\mathrm{i}\upsilon\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\mathrm{i}\upsilon\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} = \left(\widehat{\mathcal{E}}_0 + \frac{1}{2}\widehat{\Pi}_0\right)\left(\Psi_{\Omega}^{(\upsilon,k,V)} - \overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \overline{\Psi}_{\alpha}^{(\upsilon,k,V)}\right), \tag{138}$$

$$\begin{aligned}\widehat{\overline{\Psi}}_{\mathbb{P}}^{(\upsilon,k,V)} + \widehat{\overline{\Psi}}_{\mathsf{p}}^{(\upsilon,k,V)} &= (\mu_0 + p_0)\left(\mathcal{A}_0\overline{\Psi}_{a}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}\right) - \mathcal{A}_0\left(\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathsf{p}}^{(\upsilon,k,V)}\right. \\ &\quad \left. + \overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right) - \frac{1}{2}\phi_0\left(\overline{\Psi}_{\mathsf{p}}^{(\upsilon,k,V)} + 4\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right) + \left(\frac{3}{2}\phi_0 + \mathcal{A}_0\right)\Pi_0 \\ &\quad \times \overline{\Psi}_{a}^{(\upsilon,k,V)} - \Pi_0\left(\frac{3}{2}\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} + \overline{\Psi}_{\mathbb{A}}^{(\upsilon,k,V)}\right),\end{aligned} \tag{139}$$

$$\begin{aligned}\widehat{\overline{\Psi}}_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\widehat{\overline{\Psi}}_{\mathbb{P}}^{(\upsilon,k,V)} - \frac{1}{3}\widehat{\overline{\Psi}}_{\mathsf{m}}^{(\upsilon,k,V)} &= -2\phi_0\left(\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)} + \frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)} - \frac{1}{12}\overline{\Psi}_{\mathsf{m}}^{(\upsilon,k,V)}\right) \\ &\quad - \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\left(\overline{\Psi}_{\mathbb{F}}^{(\upsilon,k,V)} - \phi_0\overline{\Psi}_{a}^{(\upsilon,k,V)}\right),\end{aligned} \tag{140}$$

$$\mathrm{i}\upsilon\Psi_{\mathcal{H}}^{(\upsilon,k,S)} = \frac{2}{r}\left(\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)} - \overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}\right) + 3\left(\frac{1}{2}\Pi_0 - \mathcal{E}_0\right)\Psi_{\xi}^{(\upsilon,k,S)}, \tag{141}$$

$$\widehat{\Psi}_{\mathcal{H}}^{(\upsilon,k,S)} = \frac{2}{r}\left(\Psi_{\mathcal{H}}^{(\upsilon,k,V)} - \frac{1}{2}\overline{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)}\right) - \frac{3}{2}\phi_0\Psi_{\mathcal{H}}^{(\upsilon,k,S)} - \left(3\mathcal{E}_0 + \mu_0 + p_0 - \frac{1}{2}\Pi_0\right)\Psi_{\Omega}^{(\upsilon,k,S)}; \tag{142}$$

- Evolution and propagation equations for the harmonic coefficients of vector quantities:

$$\begin{aligned}\mathrm{i}\upsilon\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)} + \frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}\right) &= \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\frac{1}{2}\overline{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)} + \Psi_{\mathcal{H}}^{(\upsilon,k,V)}\right) - \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\overline{\Psi}_{\alpha}^{(\upsilon,k,V)} \\ &\quad - \frac{1}{2}(\mu_0 + p_0 + \Pi_0)\left(\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)} - \Psi_{\Omega}^{(\upsilon,k,V)}\right) + \frac{1}{2r}\Psi_{\mathcal{H}}^{(\upsilon,k,S)},\end{aligned} \tag{143}$$

$$\begin{aligned}\widehat{\overline{\Psi}}_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\widehat{\overline{\Psi}}_{\Pi}^{(\upsilon,k,V)}=&-\frac{3}{2}\phi_0\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}+\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\overline{\Psi}_{a}^{(\upsilon,k,V)}\\&+\frac{1}{2}\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)}+\frac{1}{3}\overline{\Psi}_{\mathfrak{m}}^{(\upsilon,k,V)}+\frac{1}{4}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)},\end{aligned}\tag{144}$$

$$\begin{aligned}\frac{1}{4}\widehat{\overline{\Psi}}_{\Pi}^{(\upsilon,k,V)}-\frac{1}{2}\widehat{\overline{\Psi}}_{\mathcal{E}}^{(\upsilon,k,V)}-\mathrm{i}\upsilon\Psi_{\mathcal{H}}^{(\upsilon,k,V)}=&-\frac{3}{4}\left(\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)}-\frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right)+\frac{3}{4}\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\overline{\Psi}_{a}^{(\upsilon,k,V)}\\&-\frac{3}{2}\mathcal{E}_0\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)}+\frac{1}{4}\phi_0\left(\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)}-\frac{1}{2}\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}\right)\\&+\mathcal{A}_0\overline{\Psi}_{\mathcal{E}}^{(\upsilon,k,V)},\end{aligned}\tag{145}$$

$$\begin{aligned}\widehat{\Psi}_{\mathcal{H}}^{(\upsilon,k,V)}+\frac{1}{2}\widehat{\overline{\Psi}}_{\mathcal{Q}}^{(\upsilon,k,V)}=&\frac{1}{2r}\Psi_{\mathcal{H}}^{(\upsilon,k,S)}-\frac{3}{2}\phi_0\left(\Psi_{\mathcal{H}}^{(\upsilon,k,V)}+\frac{1}{6}\overline{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)}\right)+\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\overline{\Psi}_{\Sigma}^{(\upsilon,k,V)}\\&-\left(\mu_0+p_0+\frac{1}{4}\Pi_0-\frac{3}{2}\mathcal{E}_0\right)\Psi_{\Omega}^{(\upsilon,k,V)},\end{aligned}\tag{146}$$

$$\begin{aligned}\mathrm{i}\upsilon\overline{\Psi}_{\mathcal{Q}}^{(\upsilon,k,V)}+\widehat{\overline{\Psi}}_{\Pi}^{(\upsilon,k,V)}=&\frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}-\overline{\Psi}_{\mathfrak{p}}^{(\upsilon,k,V)}-\left(\frac{3}{2}\phi_0+\mathcal{A}_0\right)\overline{\Psi}_{\Pi}^{(\upsilon,k,V)}\\&-\frac{3}{2}\Pi_0\overline{\Psi}_{a}^{(\upsilon,k,V)}-\left(\mu_0+p_0-\frac{1}{2}\Pi_0\right)\overline{\Psi}_{\mathcal{A}}^{(\upsilon,k,V)};\end{aligned}\tag{147}$$

- Constraint equations for the harmonic coefficients:

$$\frac{2}{r}\left(\frac{1}{3}\overline{\Psi}_{\mathfrak{m}}^{(\upsilon,k,V)}-\overline{\Psi}_{\mathbb{E}}^{(\upsilon,k,V)}-\frac{1}{2}\overline{\Psi}_{\mathbb{P}}^{(\upsilon,k,V)}\right)=-3\phi_0\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Psi_{\xi}^{(\upsilon,k,S)},\tag{148}$$

$$\frac{1}{r}\overline{\Psi}_{\mathfrak{p}}^{(\upsilon,k,V)}+\widehat{p}_0\Psi_{\xi}^{(\upsilon,k,S)}=0.\tag{149}$$

3.6.3. k-modes where $\mathcal{Q}^{(k)}\neq 0\wedge\left\{\mathcal{Q}_{\alpha}^{(k)},\bar{\mathcal{Q}}_{\alpha}^{(k)}\right\}=0\wedge\left\{\mathcal{Q}_{\alpha\beta}^{(k)},\bar{\mathcal{Q}}_{\alpha\beta}^{(k)}\right\}=0$: $k=0$. Once again, as was discussed in the even sector, for *k*-modes where the vector and tensor harmonics are not defined, to describe the dynamics of the perturbations that are not along the directions on the sheets, we will use the relations in appendix D. The equations for the harmonic coefficients for the $k=0$ mode are:

$$\mathrm{i}\upsilon\Psi_{\Omega}^{(\upsilon,0,S)}=\mathcal{A}_0\Psi_{\xi}^{(\upsilon,0,S)},\tag{150}$$

$$\widehat{\Psi}_{\Omega}^{(\upsilon,0,S)}=(\mathcal{A}_0-\phi_0)\Psi_{\Omega}^{(\upsilon,0,S)},\tag{151}$$

$$\mathrm{i}\upsilon\Psi_{\xi}^{(\upsilon,0,S)}=0,\tag{152}$$

$$\widehat{\Psi}_{\xi}^{(\upsilon,0,S)}=-\phi_0\Psi_{\xi}^{(\upsilon,0,S)},\tag{153}$$

$$\mathrm{i}\upsilon\Psi_{\mathcal{H}}^{(\upsilon,0,S)}=-3\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\Psi_{\xi}^{(\upsilon,0,S)},\tag{154}$$

$$\widehat{\Psi}_{\mathcal{H}}^{(\upsilon,0,S)}=-\frac{3}{2}\phi_0\Psi_{\mathcal{H}}^{(\upsilon,0,S)}-\left(3\mathcal{E}_0+\mu_0+p_0-\frac{1}{2}\Pi_0\right)\Psi_{\Omega}^{(\upsilon,0,S)};\tag{155}$$

and the constraint

$$(2\mathcal{A}_0-\phi_0)\Psi_{\Omega}^{(\upsilon,0,S)}+\Psi_{\mathcal{H}}^{(\upsilon,0,S)}=0.\tag{156}$$

4. Adiabatic isotropic perturbations

As a first application of the system of covariant, gauge invariant equations presented in the previous section, in the remainder of the article we will consider linear isotropic perturbations of a static, spherically symmetric spacetime permeated by a perfect fluid, which we will call, for brevity and its immediate physical application, 'star'. Due to the scope of this article, here we will analyze the general properties of the system of differential equations and relegate finding and discussing the solutions for specific background spacetimes to another article.

4.1. The equilibrium spacetime

In the language of the 1+1+2 decomposition, static, spherically symmetric spacetimes with a perfect fluid source can be completely characterized by the six scalars $\{\mathcal{A}_0, \phi_0, \mathcal{E}_0, \mu_0, p_0, \Lambda\}$, such that all their 'dot derivatives' vanish. For simplicity, we will set the cosmological constant to zero: $\Lambda = 0$. Then, the covariant defined scalars verify (cf equations (23) and (24)):

$$\widehat{p}_0 = -(\mu_0 + p_0)\mathcal{A}_0\,, \tag{157}$$

$$\widehat{\phi}_0 = -\frac{1}{2}\phi_0^2 - \frac{2}{3}\mu_0 - \mathcal{E}_0\,, \tag{158}$$

$$\widehat{\mathcal{A}}_0 = \frac{3}{2}\mathcal{E}_0 + \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\mathcal{A}_0\,, \tag{159}$$

$$\widehat{\mathcal{E}}_0 + \frac{3}{2}\phi_0\mathcal{E}_0 = \frac{1}{3}\widehat{\mu}_0\,, \tag{160}$$

and the constraint

$$\frac{1}{3}(\mu_0 + 3p_0) - \mathcal{A}_0\phi_0 = \mathcal{E}_0\,. \tag{161}$$

We will consider the setup where two solutions of the Einstein field equations are smoothly matched at a common timelike hypersurface. The interior of the star is described by a static, spatially compact solution with a perfect fluid source, while the exterior spacetime is described by an asymptotically flat branch of the vacuum Schwarzschild solution with no event horizons.

4.2. Choice of frame in the perturbed spacetime

The sheets of the background spacetime have spherical symmetry. Hence, by choosing the e tangent vector field of the background spacetime to be aligned with the gradient of the circumferential radius, r, we can particularize the harmonics $\mathcal{Q}^{(k)}$ to be the spherical harmonics: Y_{lm}, where the eigenvalues k verify $k^2 = l(l+1)$, with $l \geqslant 0$, and $-l \leqslant m \leqslant l$. Moreover, this choice of frame implies that,

$$\mathcal{Q}^{(0)}_{\alpha} = 0, \quad \bar{\mathcal{Q}}^{(0)}_{\alpha} = 0, \quad \mathcal{Q}^{(0)}_{\alpha\beta} = \mathcal{Q}^{(1)}_{\alpha\beta} = 0, \quad \bar{\mathcal{Q}}^{(0)}_{\alpha\beta} = \bar{\mathcal{Q}}^{(1)}_{\alpha\beta} = 0. \tag{162}$$

The perturbed spacetime is assumed to maintain spherical symmetry, therefore, it is useful also to consider the spacelike vector field e to not have angular components and dependencies in the perturbed spacetime. Notice that, at this point, we are free to choose any smooth mapping between the background and the perturbed spacetimes given that covariantly defined tensors, vectors, and the quantities in equations (25) and (26) are identification gauge invariant. Considering the equations in sections 3.5 and 3.6, since the perturbations are spherically symmetric, all coefficients, in both even and odd sectors, with $l \geqslant 1$ vanish identically. Then, only the coefficients for the $l = 0$ mode, the monopole, may not be trivial.

In addition to the imposition that the perturbation maintains the spherical symmetry of the spacetime, for simplicity, we will further assume that the perturbation does not generate anisotropic pressure and is adiabatic, that is, in the comoving matter frame, the perturbation will not give rise to heat flows within the fluid. Moreover, we will also impose that the vorticity of the timelike congruence is identically zero in the perturbed spacetime. These extra conditions set the coefficients

$$\Psi_{\mathcal{P}}^{(\upsilon,0,S)},\ \Psi_{\Omega}^{(\upsilon,0,S)},\ \Psi_{\xi}^{(\upsilon,0,S)},\ \Psi_{\mathcal{H}}^{(\upsilon,0,S)} = 0\,, \tag{163}$$

and in the comoving frame

$$\Psi_{Q}^{(\upsilon,0,S)} = 0\,. \tag{164}$$

Now, the choice of frame is not completely determined by the choice of frame in the background spacetime and the choice that the e vector field does not have angular components and dependencies in the perturbed spacetime. Since the background spacetime is assumed to be static, comoving observers with the fluid in equilibrium are also static. Therefore, to describe the perturbed fluid, we can choose either to consider a congruence that describes the world-lines of observers locally comoving with the elements of the fluid or a congruence that describes the world-lines of static, $\dot{r} = 0$, observers. Both choices are valid, and both have advantages. For the former choice, the matter field is described by the source fluid in its rest frame, i.e. a perfect fluid model. Hence, it can be characterized simply by its energy density and pressure. Moreover, all quantities that characterize the observer directly characterize the elements of the fluid, allowing for a clear interpretation of the quantities and how their dependencies give rise to the various physical effects. In the latter case, we have the constraint $\frac{2}{3}\theta = \Sigma$ between the expansion and shear kinematical variables, equation (E2). However, for static observers, the fluid will no longer be perfect: such observers will measure momentum flows, and we have to consider a heat flow term in the perturbed stress-energy tensor: $Q \neq 0$ (cf appendix F). Nonetheless, the description of the problem in this frame is greatly simplified, allowing us to find some general properties of the solutions easily. Below, we show the system of equations for both frames.

In order to lighten the notation, here and in the following sections, we will indicate the perturbation-coefficients as

$$\Psi_{\chi}^{(\upsilon,0,S)} = \Psi_{\chi}\,. \tag{165}$$

where χ is a generic perturbation variable. This will not compromise clarity as only the monopole perturbations mode is nontrivial for the considered setup.

4.3. Comoving observers

In the comoving frame, the perturbed fluid can be modeled by a perfect fluid. Hence, it can be fully characterized by its energy density and pressure. To close the system, we have to provide a matter model for the perturbed fluid. As a simplifying assumption, we will consider that the perturbed matter fluid still verifies a barotropic equation of state such that

$$p = f(\mu)\,, \tag{166}$$

where f is assumed to be non-vanishing and of class $\mathcal{C}^1$ in some neighborhood containing μ_0. Then, at linear order

$$\mathsf{p} \approx f'(\mu_0)\,\mathsf{m}\,, \tag{167}$$

where prime represents derivative with respect to the function's parameter, so that $f'(\mu_0)$ represents the square of the adiabatic speed of sound, to be assumed non-vanishing in the interior of the perturbed star. Notice that the function f does not have to be equal to the equation of state of the background configuration, in particular, the equilibrium fluid is not even required to verify a barotropic equation of state.

Imposing the previous conditions in the equations of sections 3.5.3 and 3.6.3, we are left with the following system for the non-trivial coefficients of the perturbation variables

$$\widehat{\Psi}_{\mathsf{p}} + 2\mathcal{A}_0\left(1+\frac{1}{3f'(\mu_0)}\right)\Psi_{\mathsf{p}} = -(\mu_0+p_0)(\Psi_{\mathsf{A}}+\mathcal{A}_0\Psi_\Sigma), \tag{168}$$

$$\widehat{\Psi}_{\mathsf{A}} + \left(3\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Psi_{\mathsf{A}} = \frac{\mathcal{E}_0}{(\mu_0+p_0)f'(\mu_0)}\Psi_{\mathsf{p}} - \frac{3}{2}\left(\upsilon^2+\mathcal{A}_0^2 + \frac{1}{3}\mu_0 - 2\mathcal{E}_0\right)\Psi_\Sigma, \tag{169}$$

$$\widehat{\Psi}_\Sigma + \left(\frac{3}{2}\phi_0 - \frac{2\mathcal{A}_0}{3f'(\mu_0)}\right)\Psi_\Sigma = \frac{2}{3(\mu_0+p_0)f'(\mu_0)}\left[\frac{3f''(\mu_0)\widehat{\mu}_0+2\mathcal{A}_0}{3f'(\mu_0)} + \frac{\widehat{\mu}_0}{\mu_0+p_0}+\mathcal{A}_0\right]\Psi_{\mathsf{p}} + \frac{2}{3f'(\mu_0)}\Psi_{\mathsf{A}}, \tag{170}$$

and the constraints

$$\left(\upsilon^2+\mathcal{A}_0\phi_0+\mathcal{A}_0^2-p_0\right)\left(\frac{2}{3}\Psi_\theta-\Psi_\Sigma\right) = \Psi_{\mathsf{p}}-\phi_0\Psi_{\mathsf{A}}, \tag{171}$$

$$\Psi_{\mathsf{E}} = \mathcal{E}_0\left(\frac{3}{2}\Psi_\Sigma + \frac{\Psi_{\mathsf{p}}}{f'(\mu_0)(\mu_0+p_0)}\right) - \frac{1}{2}(\mu_0+p_0)\Psi_\Sigma, \tag{172}$$

$$\Psi_{\mathsf{F}} = \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\frac{2\Psi_{\mathsf{p}}}{3f'(\mu_0)(\mu_0+p_0)}+\Psi_\Sigma\right), \tag{173}$$

$$\Psi_{\mathsf{m}} = -(\mu_0+p_0)\Psi_\theta, \tag{174}$$

$$\Psi_{\mathsf{p}} = f'(\mu_0)\Psi_{\mathsf{m}}, \tag{175}$$

where equation (175) follows from equation (167). It can be readily shown that equations (168)–(171) are consistent with the relation (29).

To select the physically acceptable solutions and formalize the boundary value problem, we impose the following boundary conditions:

(i) The Energy density and the pressure perturbations at the center of the star must be finite in a neighborhood of the initial instant;
(ii) The spacetime interior to the perturbed star can be smoothly matched to an exterior vacuum Schwarzschild spacetime at a timelike hypersurface $\mathfrak{B}$, which will represent the 'surface of the star'.

From the point of view of the comoving observer, the boundary condition (ii) implies that the pressure of the perturbed fluid is identically zero at all times at the surface of the star and thus, p, hence Ψ_{p}, are also identically zero at the boundary $\mathfrak{B}$.

4.4. Static observers

An advantage of the adopted covariant formalism is the ability to readily change frames without deriving a new set of equations for the perturbations. Hence, an alternative way to describe isotropic adiabatic perturbations is to consider the frame of static observers.

As was discussed previously, static observers will not describe the perturbed fluid as a perfect fluid since, in general, these will measure a local net momentum flow. Therefore, the relation between the energy density, pressure and heat flow terms of the perturbed stress-energy tensor is more complex than that of the comoving frame above. It is shown in appendix F that a barotropic equation of state in the comoving frame leads to the following relation between the matter variables measured in a static frame

$$\mathsf{m} = \frac{1}{f'(\mu_0)}\mathsf{p} - \frac{1}{\mu_0 + p_0}\left(\widehat{\mu}_0 - \frac{\widehat{p}_0}{f'(\mu_0)}\right)Q\,, \tag{176}$$

where $f'(\mu_0)$ represents the square of the speed of sound measured in the comoving frame. We highlight that although we have kept the nomenclature of the previous subsection, m and p refer here, respectively, to the dot derivatives of the energy density and pressure, and Q to the scalar heat flow density, all three measured by a radially static observer. In this form, it is immediate to see that if the equilibrium and the perturbed fluids verify the same equation of state, the second term in the right-hand side of equation (176) vanishes. Therefore, the extra term accounts for the phase change between the two fluids. This is a very interesting result, showing that in the static frame the correction to the equation of state has to be taken into account already at linear level.

Imposing condition $\Psi_\Sigma = \frac{2}{3}\Psi_\theta$, equation (E2), and using equation (176) in the equations of sections 3.5.3 and 3.6.3, we find the following system for the non-trivial coefficients of the perturbation variables in the static frame

$$\begin{aligned}\widehat{\Psi}_{\mathsf{p}} + \left[\frac{\mu_0+p_0}{\phi_0} + \left(2 + \frac{1}{f'(\mu_0)}\right)\mathcal{A}_0\right]\Psi_{\mathsf{p}} = &\left[\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) + \frac{\mathcal{A}_0^2}{f'(\mu_0)}\right.\\ &\left. + \frac{\mathcal{A}_0\widehat{\mu}_0}{\mu_0+p_0} + \upsilon^2\right]\Psi_Q\,,\end{aligned} \tag{177}$$

$$\widehat{\Psi}_Q + \left(\phi_0 + 2\mathcal{A}_0 - \frac{\mathcal{A}_0}{f'(\mu_0)} - \frac{\widehat{\mu}_0}{\mu_0+p_0} - \frac{\mu_0+p_0}{\phi_0}\right)\Psi_Q = -\frac{1}{f'(\mu_0)}\Psi_{\mathsf{p}}\,; \tag{178}$$

and the constraints

$$\begin{aligned}\Psi_{\mathsf{m}} &= \frac{1}{f'(\mu_0)}\Psi_{\mathsf{p}} - \left(\frac{\mathcal{A}_0}{f'(\mu_0)} + \frac{\widehat{\mu}_0}{\mu_0+p_0}\right)\Psi_Q\,,\\ \Psi_{\mathsf{A}} &= \frac{1}{\phi_0}\left[\Psi_{\mathsf{p}} - \left(\frac{1}{2}\phi_0 + \mathcal{A}_0\right)\Psi_Q\right],\\ \Psi_{\mathsf{E}} &= \frac{1}{2}\phi_0\Psi_Q + \frac{1}{3}\Psi_{\mathsf{m}}\,,\\ \Psi_\theta &= -\frac{1}{\phi_0}\Psi_Q\,,\\ \Psi_\Sigma &= \frac{2}{3}\Psi_\theta\,,\\ \Psi_{\mathsf{F}} &= \Psi_Q\,.\end{aligned} \tag{179}$$

To select the physically relevant solutions, we impose the boundary conditions (i) and (ii). In particular, for the static observer, condition (ii) implies that at the surface of the star, at all times, we must have (cf, equation (F15))

$$\mathsf{p} - \mathcal{A}_0 Q|_{\mathfrak{B}} = 0\,, \tag{180}$$

hence

$$\Psi_{\mathsf{p}} - \mathcal{A}_0 \Psi_Q|_{\mathfrak{B}} = 0\,. \tag{181}$$

5. General solutions, Sturm–Liouville problem and properties of eigenfrequencies

The system (168)–(175) found for the comoving frame and the one found for the static frame, equations (177)–(179), both describe the same setup. Nonetheless, comparing the equations of each system, it is manifest that there is a trade-off when considering one picture over the other: in the comoving frame, the characterization of the matter fluid is simpler, but the dynamic description of the perturbed star is more complex, whereas from the point of view of static observers the description of the dynamics of the perturbed star is simpler, but the characterization of the matter fluid is more complex. The solutions of each system with appropriate boundary conditions are exactly the same, however when analyzing the properties and computing the solutions, from a technical point of view, it is easier to work in the static frame.

One key result that can be readily proven from equations (177) and (178) is that if the background spacetime is a solution of the Tolman–Oppenheimer–Volkoff equations for energy density and pressure functions, μ_0 and p_0, that are real analytic in the interior and boundary of the star, we can find real analytic solutions for the perturbations. Moreover, in those conditions the coefficients Ψ_Q verify a Sturm–Liouville eigenvalue problem with a limit-point-non-oscillating (LPNO) endpoint and a regular endpoint, with separated self-adjoint boundary conditions. This fact, in particular, allows us to relate the properties of the eigenfrequencies υ with the properties of the eigenvalues of the Sturm–Liouville problem. This section focuses on the derivation of these results.

5.1. General solutions for the perturbations

To find explicit solutions for the system (177) and (178), we must break covariance. In that regard, consider the parameter r defined in equation (E2). Following [22], if the sheets are isometric to 2-spheres, r^{-2} is, up to a multiplicative constant, equal to the Gauss curvature of the sheets. Therefore, the parameter r represents the circumferential radius function both in the background and the perturbed spacetime. Consequently, let $r = 0$ represents the center $\mathfrak{C}$ of the star, and we will consider that the background quantities $\{\mathcal{A}_0, \phi_0, \mathcal{E}_0, \mu_0, p_0\}$ and the coefficients Ψ_{p} and Ψ_Q are functions of r.

Now, we will impose that the following regularity constraints hold:

- The equilibrium fluid verifies the weak energy condition;
- The background spacetime is a solution of the Tolman–Oppenheimer–Volkoff equation for real analytic, non-trivial energy density and pressure functions for the whole range within the equilibrium star;
- The square of the speed of sound of the perturbed fluid, f', is positive and real analytic in the interior and at the boundary of the star.

Requiring real analytical background solutions is a rather strong constraint; nonetheless, to our knowledge, all classical exact solutions for compact astrophysical objects, verify this hypothesis in some open neighborhood of the center, at $r = 0$. Therefore, the following results are appropriate for treating perturbations of physically relevant setups. The radius of convergence of the power series, of course, may or may not be greater than the radius of the equilibrium star, in which case further treatment has to be carried out, nonetheless, for the following results, it suffices that the radius of convergence of the power series around the center is non-zero.

Imposing these conditions, equations (158), (161) and (E3) imply

$$\begin{aligned}\phi_0 &= \frac{2}{r}\sqrt{1-\frac{2M(r)}{r}}\,,\\ \mathcal{E}_0 &= \frac{1}{3}\mu_0 - \frac{2M(r)}{r^3}\,,\\ \mathcal{A}_0\phi_0 &= p_0 + \frac{2M(r)}{r^3}\,,\end{aligned} \tag{182}$$

where

$$M(r) := \frac{1}{2}\int_0^r \mu_0 x^2 \mathrm{d}x\,, \tag{183}$$

is usually dubbed the mass function, and equation (29) yields

$$\upsilon(r) = \lambda \mathrm{e}^{-\int \frac{2\mathcal{A}_0}{r\phi_0}\mathrm{d}r}\,, \tag{184}$$

relating the eigenfrequencies measured by an observer comoving with the fluid, υ, with the constant eigenfrequencies λ, measured by a free-falling observer at spatial infinity. If the functions μ_0 and p_0 verify the weak energy condition and are real analytic within the star, so are the functions $\mathcal{A}_0$ and $\mathcal{E}_0$, and ϕ_0 has a simple pole at the center $\mathfrak{C}$, but is otherwise real analytic in the interior and boundary of the star.

Hence, in the considered setup, equations (177) and (178) form a system of ODEs with real analytic coefficients in a neighborhood of $\mathfrak{C}$, with a simple pole at $r = 0$, and solutions can be found around the singular point.

Using equation (E2) to relate the hat derivatives with the derivatives with respect to r of first order quantities, the system of ODEs (177) and (178) is given by

$$\begin{aligned}\frac{r\phi_0}{2}\frac{\mathrm{d}\Psi_{\mathsf{p}}}{\mathrm{d}r} + \left[\frac{\mu_0+p_0}{\phi_0} + \left(2+\frac{1}{f'(\mu_0)}\right)\mathcal{A}_0\right]\Psi_{\mathsf{p}} = \Bigg[&\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\\ &+\frac{\mathcal{A}_0^2}{f'(\mu_0)} + \frac{r\mathcal{A}_0\phi_0}{2(\mu_0+p_0)}\frac{\mathrm{d}\mu_0}{\mathrm{d}r} + \upsilon^2\Bigg]\Psi_Q\,,\end{aligned} \tag{185}$$

$$\frac{r\phi_0}{2}\frac{\mathrm{d}\Psi_Q}{\mathrm{d}r} + \left(\phi_0 + 2\mathcal{A}_0 - \frac{\mathcal{A}_0}{f'(\mu_0)} - \frac{r\phi_0}{2(\mu_0+p_0)}\frac{\mathrm{d}\mu_0}{\mathrm{d}r} - \frac{\mu_0+p_0}{\phi_0}\right)\Psi_Q = -\frac{1}{f'(\mu_0)}\Psi_{\mathsf{p}}\,. \tag{186}$$

To find solutions for this system, it is useful to write it in the matrix form:

$$\frac{\mathrm{d}\mathbb{W}}{\mathrm{d}r} = \left(\frac{1}{r}\mathbb{R} + \Theta\right)\mathbb{W}\,, \tag{187}$$

where

$$\mathbb{W} = \begin{bmatrix} \Psi_{\mathsf{p}} \\ \Psi_Q \end{bmatrix}, \qquad \mathbb{R} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \tag{188}$$

and

$$\Theta = -\frac{2}{r\phi_0}\begin{bmatrix} \frac{\mu_0+p_0}{\phi_0} + 2\mathcal{A}_0 + \frac{\mathcal{A}_0}{f'(\mu_0)} & -\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) - \frac{\mathcal{A}_0^2}{f'(\mu_0)} - \frac{r\mathcal{A}_0\phi_0}{2(\mu_0+p_0)}\frac{\mathrm{d}\mu_0}{\mathrm{d}r} - \upsilon^2 \\ \frac{1}{f'(\mu_0)} & 2\mathcal{A}_0 - \frac{\mathcal{A}_0}{f'(\mu_0)} - \frac{r\phi_0}{2(\mu_0+p_0)}\frac{\mathrm{d}\mu_0}{\mathrm{d}r} - \frac{\mu_0+p_0}{\phi_0} \end{bmatrix}. \tag{189}$$

The regularity conditions that we have imposed on the thermodynamic variables of the equilibrium configuration, μ_0 and p_0, and on the equation of state of the perturbed fluid in the comoving frame, f, guarantee that the matrix Θ is analytic at $r=0$ and $r\phi_0$ does not vanish in the interior or boundary of the star. These conditions imply that $r=0$ is a regular singular point of the system.

To solve the system of ODEs, we will follow the formalism in [34]. Since the Θ is assumed to be a real analytic matrix at the center of the star, it can be expanded in a convergent power series of the form

$$\Theta(r) = \sum_{n=0}^{+\infty}\Theta_n r^n . \tag{190}$$

Then, the solution matrix $\mathbb{W}$ can be written in a power series guaranteed to converge to the solution in a neighborhood of $r=0$. The radius of convergence of the power series solution is equal, except for possibly at $r=0$, to the radius of convergence of the power series of Θ, which, of course, depends on the equilibrium background spacetime considered.

Before proceeding, we remark that the general form of the solutions of the boundary value problem (177)–(181) for a general static, spherically symmetric spacetime can be rather complicated. However, imposing the previous regularity conditions on μ_0 and p_0, and assuming $f'(\mu_0)$ is not zero in a neighborhood of the center of the star, it can be shown that various entries of the Θ_0 and Θ_1 coefficient-matrices of the series (190) are zero, which greatly simplifies the general family of solutions.

Taking into consideration the regularity of the background yields the power series solution

$$\begin{bmatrix} \Psi_{\mathsf{p}} \\ \Psi_Q \end{bmatrix} = \begin{bmatrix} -\frac{1}{r}(\Theta_0)_{12} & 1 \\ \frac{1}{r^2} & 0 \end{bmatrix}\mathbb{P}_{\mathbb{W}}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{191}$$

where c_1 and c_2 are integration constants, which might take different values for each υ and do not depend on the radial coordinate. The notation $(\Theta_n)_{ij}$ is to be interpreted as the ij-entry of the nth order coefficient of the power series of Θ, and $\mathbb{P}_{\mathbb{W}}$ is a real analytic matrix, such that

$$\begin{aligned} &\mathbb{P}_{\mathbb{W}}(r) = \sum_{n=0}^{+\infty}\mathbb{P}_n r^n \\ &\quad \mathbb{P}_0 = \mathbb{I}_2 \\ &\quad \mathbb{P}_k = \frac{1}{k}\sum_{j=0}^{k-1}\mathbb{A}_{k-1-j}\mathbb{P}_j, \quad \text{for } k \geqslant 1 \end{aligned} \tag{192}$$

where $\mathbb{I}_2$ represents the 2×2 identity matrix, the matrix $\mathbb{A}$ is given by

$$\mathbb{A} = \begin{bmatrix} \Theta_{22} - r(\Theta_0)_{12}\Theta_{21} & r^2\Theta_{21} \\ \frac{\Theta_{12}-(\Theta_0)_{12}}{r^2} + \frac{(\Theta_0)_{12}(\Theta_{22}-\Theta_{11})}{r} - (\Theta_0)_{12}^2\Theta_{21} & \Theta_{11} + r(\Theta_0)_{12}\Theta_{21} \end{bmatrix}, \tag{193}$$

and $\mathbb{A}_n$ represents the nth order coefficient of its power series, that is, $\mathbb{A}(r) = \sum_{n=0}^{+\infty} \mathbb{A}_n r^n$.

The general family of solutions of the system, for $0 < r < a$, where $a \in \mathbb{R}_{>0}$, is given by equations (191) and (192). To select the physically acceptable solutions and extend the domain to $r = 0$, we will impose the boundary conditions (i) and (ii) of section 4.3. Considering $\mathbb{P}_0 = \mathbb{I}_2$, we can directly compute the lower order coefficients of the power series expansion of $\mathbb{W}$. Imposing the boundary condition at the center sets the coefficient c_1 to be zero so that the perturbations do not diverge at $r = 0$ at all times. Then, we find

$$\begin{bmatrix} \Psi_{\mathsf{p}} \\ \Psi_{\mathcal{Q}} \end{bmatrix} = \begin{bmatrix} c_2 + \mathcal{O}\left(r^2\right) \\ \mathcal{O}\left(r\right) \end{bmatrix}. \tag{194}$$

Provided the background spacetime, the equation of state of the perturbed fluid, and the values of the eigenfrequencies, these results allow us to find analytic solutions for the perturbations that verify the boundary conditions. The eigenfrequencies themselves cannot be restricted directly from these results. Nonetheless, in the following subsection, we will show that general useful properties for the eigenfrequencies can be derived by relating υ with the eigenvalues of a Sturm–Liouville problem.

5.2. Sturm–Liouville eigenvalue problem

We do not need to break covariance for the following result. Therefore, to keep the discussion independent of a local coordinate system, we will consider the hat derivatives without associating them with derivatives in a specific coordinate system. Let ℓ be an affine parameter of the e congruence, and without loss of generality, we set that $\ell = 0$ represents the center of the star and $\ell = \ell_{\mathfrak{B}}$ the boundary of the equilibrium star.

Taking the hat derivative of equation (178) we find

$$D_e\left[\exp\left(\int_{\ell_0}^{\ell} F(x)\,\mathrm{d}x\right)\widehat{\Psi}_{\mathcal{Q}}\right] + \exp\left(\int_{\ell_0}^{\ell} F(x)\,\mathrm{d}x\right) G(\ell)\,\Psi_{\mathcal{Q}} = -\frac{1}{f'(\mu_0)}\exp\left(\int_{\ell_0}^{\ell} F(x)\,\mathrm{d}x\right)\lambda^2\Psi_{\mathcal{Q}}, \tag{195}$$

where, for notational convenience, we have indicated the hat derivatives as $D_e \equiv e^{\alpha} D_{\alpha}$, $\ell_0 \in \mathbb{R}$, λ is an integration constant following from equation (29) and

$$\begin{aligned} F(\ell) &= \phi_0 + 4\mathcal{A}_0 - \frac{\widehat{\mu}_0}{\mu_0 + p_0} + \frac{f''(\mu_0)\,\widehat{\mu}_0}{f'(\mu_0)}, \\ G(\ell) &= \frac{f''(\mu_0)\,\widehat{\mu}_0}{f'(\mu_0)}\left(\phi_0 + 2\mathcal{A}_0 - \frac{\widehat{\mu}_0}{\mu_0 + p_0} - \frac{\mu_0 + p_0}{\phi_0}\right) + \frac{1}{f'(\mu_0)}\left(2\mathcal{A}_0\phi_0 - p_0 + \mathcal{A}_0^2\right) \\ &\quad + \mu_0 + 3p_0 + \mathcal{A}_0\phi_0 + 2\mathcal{A}_0^2 - \frac{1}{2}\phi_0^2 - \frac{2\mathcal{A}_0\widehat{\mu}_0}{\mu_0 + p_0} - \frac{\mu_0 + p_0}{\phi_0}\left(\frac{\widehat{\mu}_0}{\mu_0 + p_0} + \frac{\mu_0 + p_0}{\phi_0}\right) \\ &\quad - \left[D_e\left(\frac{\widehat{\mu}_0}{\mu_0 + p_0}\right) + D_e\left(\frac{\mu_0 + p_0}{\phi_0}\right)\right]. \end{aligned} \tag{196}$$

Moreover, conditions (i) and (ii) of section 4.3 imply the following set of separated self-adjoint boundary conditions:

$$\begin{cases} \Psi_{\mathcal{Q}}\big|_{\mathfrak{C}} = 0 \\ \widehat{\Psi}_{\mathcal{Q}} + \left(\phi_0 + 2\mathcal{A}_0 - \dfrac{\widehat{\mu}_0}{\mu_0 + p_0} - \dfrac{\mu_0 + p_0}{\phi_0}\right)\Psi_{\mathcal{Q}}\Big|_{\mathfrak{B}} = 0. \end{cases} \tag{197}$$

Therefore, provided the background spacetime and f' are sufficiently regular, Ψ_Q verifies a formal Sturm–Liouville eigenvalue problem with weight function

$$w(\ell) = \frac{1}{f'(\mu_0)} \exp\left(\int_{\ell_0}^{\ell} F(x)\,\mathrm{d}x\right), \tag{198}$$

and eigenvalues λ^2. This last assumption, however, is not trivial, and we have to specify the conditions under which the associated Sturm–Liouville operator is self-adjoint.

Considering the discussion in the previous subsection, in particular the regularity conditions in section 5.1, the function ϕ_0 has a simple pole at $\mathfrak{C}$. This endpoint, at $\ell = 0$, is limit-point since we have found solutions that are not in $L^2(]0,\delta[, w)$, for any $\delta \in \mathbb{R}_{>0}$. However, in the conditions considered in the previous subsection, we have explicitly shown that all solutions for Ψ_Q are real analytic in $]0,\varepsilon[$, for some $\varepsilon \in \mathbb{R}_{>0}$, hence there are no non-trivial solutions that have an infinite number of zeros in a right-neighborhood of $\ell = 0$. We can then conclude that the endpoint $\ell = 0$ is LPNO. Moreover, the regularity conditions guarantee that the coefficients of the differential equation verify

$$w(\ell) > 0, \quad \exp\left(\int_{\ell_0}^{\ell} F(x)\,\mathrm{d}x\right) > 0, \tag{199}$$

almost everywhere and

$$\exp\left(-\int_{\ell_0}^{\ell} F(x)\,\mathrm{d}x\right), \ \exp\left(\int_{\ell_0}^{\ell} F(x)\,\mathrm{d}x\right) G(\ell), \ w(\ell) \in L_{\mathrm{loc}}(]0,\ell_{\mathfrak{B}}[,\mathbb{R}) \tag{200}$$

where $L_{\mathrm{loc}}(]0,\ell_{\mathfrak{B}}[,\mathbb{R})$ is the set of locally Lebesgue integrable real functions on $]0,\ell_{\mathfrak{B}}[$. Given the previous considerations, we can apply the results in [35, 36] to equation (195) together with the boundary conditions (197), concluding that the eigenvalues verify several very useful properties. In particular, the eigenvalues λ^2 are (i) real, (ii) simple, (iii) countable, and (iv) have a minimum and are unbounded from above, that is $\lambda^2 = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$, with $\lambda_0 < \lambda_1 < \lambda_2 < \ldots \to +\infty$, and $\lambda_0 \in \mathbb{R}$. Indeed, in the following subsection, we show explicitly that, in the considered setup, the eigenfrequencies are such that υ^2 are bounded from below and derive necessary constraints for the minimum value.

5.3. Lower bound for the square of the eigenfrequencies

The result that Ψ_Q verifies a Sturm–Liouville eigenvalue problem is of great importance for the description of the behavior of the perturbed star. The fact that the eigenvalues λ^2, hence υ^2, are bounded from below sets that either in the oscillating case or in the continuous collapse or expanding scenarios, the eigenfrequencies of the excitable eigenmodes are not arbitrarily small. In addition to those results, in the considered setup, we will prove the following:

Proposition 1. *Let the boundary conditions (i) and (ii) of section 4.3 and the regularity conditions in section 5.1 hold. If, in addition, for the interior, background spacetime $\widehat{\mathcal{A}}_0(\ell_{\mathfrak{B}}) \geqslant 0$, then non-trivial $\mathcal{C}^1$ solutions of the system (177) and (178) exist only if*

$$\upsilon^2 \geqslant -\mathcal{A}_0\phi_0 - \frac{1}{2}\mu_0\bigg|_{\ell_{\mathfrak{B}}}. \tag{201}$$

Using equation (182) we can rewrite inequality (201) as $\upsilon^2 \geqslant -p_0 - \frac{2M}{r^3} - \frac{\mu_0}{2}\big|_{\mathfrak{B}}$, showing explicitly that the right-hand side is always negative.

The premises of proposition 1 can be significantly relaxed, leading to the following result:

Proposition 2. *Let the boundary conditions (i) and (ii) of section 4.3 hold and consider the following regularity conditions*

- *The equilibrium fluid verifies the weak energy condition;*
- *The background spacetime is a solution of the Tolman–Oppenheimer–Volkoff equation characterized by non-trivial, $\mathcal{C}^1$ functions $\mathcal{A}_0$, μ_0 and p_0, and ϕ_0 has a simple pole at the center, but is otherwise of class $\mathcal{C}^1$ within the domain of the solution;*
- *$f'(\mu_0)$ is positive in the interior of the perturbed star.*

Then, non-trivial $\mathcal{C}^1$ solutions of the system (177) and (178) exist only if

$$\max_{\ell\in]0,\ell_{\mathfrak{B}}[} \upsilon^2 > - \max_{\ell\in]0,\ell_{\mathfrak{B}}[} \left[\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0+2\mathcal{A}_0\right)+\frac{\mathcal{A}_0^2}{f'(\mu_0)}+\frac{\mathcal{A}_0\widehat{\mu}_0}{\mu_0+p_0}\right]. \tag{202}$$

Lemma 1. *If the premises of proposition 2 hold, a non-trivial $\mathcal{C}^1$ solution Ψ_{p} has at least one root.*

Notice that (201) is much simpler than (202). However, even under the conditions where both inequalities can be used, it should be stressed that without specifying the background spacetime and the equation of state of the perturbed fluid, there is no way to infer which provides a tighter bound. Indeed, the determination of the tighter bound depends critically of the background spacetime. Moreover, the inequalities in propositions 1 and 2 are not tight enough to provide a criteria for stability or instability based on the background spacetime and the equation of state of the perturbed fluid. Nonetheless, finding a constraint on the minimum allowed values of υ^2 is of great importance when searching for solutions to the system. Since the results in [36] also associate the number of nodes of the eigenfunctions with the ordinal number of the eigenvalues in the sequence $(\lambda_n)_{n\in\mathbb{N}}$, these constraints provide a baseline to search for the eigenfrequencies algorithmically. Lemma 1 provides a further criterion useful to find the fundamental eigenfrequency.

As in the previous subsection, in the following proofs, we will consider ℓ to be an affine parameter of the congruence associated with the integral curves of the e vector field, and without loss of generality, we set that $\ell=0$ represents the center of the star and $\ell=\ell_{\mathfrak{B}}$ the boundary of the equilibrium star.

Proof for proposition 1. The function ϕ_0 has a simple pole at $\ell=0$. Therefore, imposing regularity of Ψ_Q and Ψ_{p} and their derivatives in the interior and boundary of the star, equation (178) implies

$$\Psi_Q(0)=0\,. \tag{203}$$

In that case, we can disregard the solution with $\Psi_{\mathsf{p}}(0)=0$ since this corresponds to the trivial solution. Then, depending on the initial conditions, either $\Psi_{\mathsf{p}}(0)>0$ or $\Psi_{\mathsf{p}}(0)<0$. As expected, the reasoning for one case applies similarly to the other. Therefore, we will explicitly treat only the case $\Psi_{\mathsf{p}}(0)>0$. Then, assume $\Psi_{\mathsf{p}}(0)>0$. Imposing $f'(\mu_0)>0$ and given that ϕ_0 is positive, equation (178) further implies

$$\widehat{\Psi}_Q(0)\,\Psi_{\mathsf{p}}(0)<0\,. \tag{204}$$

Therefore, for all $\ell\in]0,\epsilon[$, for some $0<\epsilon<\ell_{\mathfrak{B}}$, we have $\Psi_{\mathsf{p}}(\ell)\,\Psi_Q(\ell)<0$, that is, for our choice of initial data

$$\Psi_Q(\ell)<0\ ,\text{for all } \ell\in]0,\epsilon[\,. \tag{205}$$

Now, the values of the eigenfrequencies are those that verify the boundary conditions (i) and (ii), in particular equation (181). Imposing that the regularity conditions in section 5.1 hold, then $\mathcal{A}_0 \geqslant 0$ within the equilibrium star, being zero only at the center: $\ell = 0$. Therefore, if $\mathcal{C}^1$ solutions exist, for the fundamental mode, that is, for the lowest value of υ^2, choosing $\Psi_{\mathsf{p}}(0) > 0$, we must have

$$\Psi_{\mathsf{p}} - \mathcal{A}_0 \Psi_Q > 0 \,, \text{for } \ell \in [0, \ell_{\mathfrak{B}}[\,, \tag{206}$$

and

$$\Psi_{\mathsf{p}} - \mathcal{A}_0 \Psi_Q = 0 \,, \text{at } \ell = \ell_{\mathfrak{B}} \,. \tag{207}$$

These results then imply that at the boundary

$$D_e \left(\Psi_{\mathsf{p}} - \mathcal{A}_0 \Psi_Q\right)\Big|_{\ell_{\mathfrak{B}}} = \widehat{\Psi}_{\mathsf{p}} - \mathcal{A}_0 \widehat{\Psi}_Q - \widehat{\mathcal{A}}_0 \Psi_Q\Big|_{\ell_{\mathfrak{B}}} \leqslant 0 \,, \tag{208}$$

where, for notational convenience, we have indicated the hat derivative as $D_e \equiv e^{\alpha} D_{\alpha}$. We remark that the derivative in equation (208) is taken only considering quantities of the interior spacetime and evaluated at $\ell_{\mathfrak{B}}$ considering an observer in the interior spacetime. The analogous derivative for the full, matched spacetime might or not exist, since the Israel-Darmois junction conditions do not impose any constraint on the derivatives of the involved quantities. In a sense, equation (208) can be considered a one-sided derivative for the full spacetime.

Continuing, since Ψ_Q verifies a Sturm–Liouville eigenvalue problem, for the fundamental eigenmode Ψ_Q has no zeros. Therefore, $\Psi_Q(\ell) < 0$, for all $\ell \in \,]0, \ell_{\mathfrak{B}}]$. Then, further imposing that the interior, background spacetime is such that $\widehat{\mathcal{A}}_0(\ell_{\mathfrak{B}}) \geqslant 0$, inequality (208) is verified if

$$\widehat{\Psi}_{\mathsf{p}} - \mathcal{A}_0 \widehat{\Psi}_Q\Big|_{\ell_{\mathfrak{B}}} \leqslant 0 \,. \tag{209}$$

At the boundary, $\ell = \ell_{\mathfrak{B}}$, equations (177) and (178) reduce to

$$\begin{aligned} \widehat{\Psi}_{\mathsf{p}} &= \left[\frac{\mu_0}{\phi_0}\left(\frac{1}{2}\phi_0 + \mathcal{A}_0\right) + \frac{\mathcal{A}_0 \widehat{\mu}_0}{\mu_0} - 2\mathcal{A}_0^2 + \upsilon^2\right] \Psi_Q \,, \\ \widehat{\Psi}_Q &= -\left(\phi_0 + 2\mathcal{A}_0 - \frac{\widehat{\mu}_0}{\mu_0} - \frac{\mu_0}{\phi_0}\right) \Psi_Q \,. \end{aligned} \tag{210}$$

Substituting equation (210) in the inequality (209) we find

$$\left(\frac{1}{2}\mu_0 + \phi_0 \mathcal{A}_0 + \upsilon^2\right) \Psi_Q\Big|_{\ell_{\mathfrak{B}}} \leqslant 0 \,. \tag{211}$$

Remarking once again that for the fundamental eigenmode $\Psi_Q(\ell) < 0$, for all $\ell \in \,]0, \ell_{\mathfrak{B}}]$, inequality (211) implies inequality (201).

□

Proof for proposition 2 and lemma 1. For clarity, we will repeat some of the intermediate results found in the proof of proposition 1.

Consider the premises of proposition 2. In those conditions, $\mathcal{A}_0 \geqslant 0$ within the equilibrium star, being zero only at the center: $\ell = 0$. Now, if $\mathcal{C}^1$ solutions of the boundary value problem (177), (178) and (181) exist for all $\ell \in [0, \ell_{\mathfrak{B}}]$, then

$$\Psi_{\mathsf{p}}(\ell_{\mathfrak{B}})\, \Psi_Q(\ell_{\mathfrak{B}}) \geqslant 0 \,. \tag{212}$$

On the other hand, since ϕ_0 has a simple pole at $\ell = 0$, from equation (178), regularity of $\widehat{\Psi}_Q(0)$ implies

$$\Psi_Q(0) = 0\,. \tag{213}$$

In particular, this allows us to disregard the solution with $\Psi_{\mathsf{p}}(0) = 0$, since this corresponds to the trivial solution. Moreover, imposing $f'(\mu_0) > 0$, given that ϕ_0 is positive, equation (178) leads to the conclusion that[6]

$$\widehat{\Psi}_Q(0)\,\Psi_{\mathsf{p}}(0) < 0\,. \tag{214}$$

Therefore, for all $\ell \in]0,\epsilon[$, for some $0 < \epsilon < \ell_{\mathfrak{B}}$, we have

$$\Psi_{\mathsf{p}}(\ell)\,\Psi_Q(\ell) < 0\,. \tag{215}$$

Considering Bolzano's theorem, from equations (212) and (215) we conclude that there is a point $0 < a \leqslant \ell_{\mathfrak{B}}$ where either $\Psi_{\mathsf{p}}(\ell)$ or $\Psi_Q(\ell)$ vanishes. Now, either $\Psi_{\mathsf{p}}(0) > 0$ or $\Psi_{\mathsf{p}}(0) < 0$. As expected, the reasoning for one case applies similarly to the other. Hence, we will explicitly treat only the case $\Psi_{\mathsf{p}}(0) > 0$. In that case, equation (215) implies that for all $\ell \in]0,\epsilon[$, $\Psi_Q(\ell) < 0$. Continuing, assume $a \in]0,\ell_{\mathfrak{B}}[$ is such that

$$\begin{aligned} \Psi_Q(a) &= 0\,,\\ \widehat{\Psi}_Q(a) &\geqslant 0\,,\\ \Psi_{\mathsf{p}}(\ell \leqslant a) &> 0\,. \end{aligned} \tag{216}$$

If several such values exist, we take a to be the smallest one that verifies equation (216). Notice that if $\Psi_{\mathsf{p}}(a) = 0$, we would recover the trivial solution, hence in light of equation (181), we can exclude the case $a = \ell_{\mathfrak{B}}$. At $\ell = a$, equation (178) reads

$$\widehat{\Psi}_Q = -\frac{1}{f'(\mu_0)}\Psi_{\mathsf{p}}\,, \tag{217}$$

however, further assuming $f'(\mu_0) > 0$, the above equation contradicts the hypothesis. This result, in particular, proves lemma 1, that is, either Ψ_Q or Ψ_{p} must have a root in the interior of the star, but independently of υ^2, we have shown that Ψ_Q can only have a root at $\ell = a$ if Ψ_{p} has a root at some $\ell < a$.

From the previous result, we conclude that for a valid solution of the boundary value problem to exist, there must be either a point $\ell = b < a$ or, if there is no value a that verifies equation (216), a value $b \in]0,\ell_{\mathfrak{B}}[$, such that

$$\begin{aligned} \Psi_{\mathsf{p}}(b) &= 0\,,\\ \widehat{\Psi}_{\mathsf{p}}(b) &\leqslant 0\,,\\ \Psi_Q(\ell \leqslant b) &< 0\,. \end{aligned} \tag{218}$$

Once again, notice that if $\Psi_Q(b) = 0$, we would recover the trivial solution, hence in light of equation (181), we can exclude the case $b = \ell_{\mathfrak{B}}$. At $\ell = b$, equation (177) then reads

$$\widehat{\Psi}_{\mathsf{p}} = \left[\frac{\mu_0 + p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) + \frac{\mathcal{A}_0^2}{f'(\mu_0)} + \frac{\mathcal{A}_0\widehat{\mu}_0}{\mu_0 + p_0} + \upsilon^2\right]\Psi_Q\,. \tag{219}$$

[6] Notice that equation (214) is consistent with equation (194), found in the case of real analytic background spacetimes around the center of the star. Nonetheless, equation (214) is more general, following from simply assuming the solutions are continuously differentiable around the center.

However, for υ^2 negative and sufficiently small we would have

$$\frac{\mu_0 + p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) + \frac{\mathcal{A}_0^2}{f'(\mu_0)} + \frac{\mathcal{A}_0\widehat{\mu}_0}{\mu_0 + p_0} + \upsilon^2\bigg|_{\ell=b} < 0\,, \tag{220}$$

hence $\widehat{\Psi}_{\mathsf{p}}(b) \geqslant 0$, which would contradict the hypothesis.

The case where $\Psi_{\mathsf{p}}(0) < 0$ follows exactly the same reasoning, confirming that in the considered setup, regular solutions of the system with the considered boundary conditions exist only if the value of υ^2 is bounded from below.

Based on these results, we can find a lower bound for the value of υ^2. Since the quantity

$$\frac{\mu_0 + p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) + \frac{\mathcal{A}_0^2}{f'(\mu_0)} + \frac{\mathcal{A}_0\widehat{\mu}_0}{\mu_0 + p_0} + \upsilon^2\,, \tag{221}$$

cannot be negative at all interior points of the perturbed star, non-trivial $\mathcal{C}^1$ solutions of the boundary value problem necessarily verify inequality (202). We remark, however, that inequality (202) is rather conservative, so much so that even if υ^2 verifies the inequality, the quantity in (221) might still be negative at all points within the star. □

6. Conclusions

We developed a framework to describe general first-order perturbations of static, locally rotationally symmetric of class II spacetimes within the theory of GR. The new system of equations is completely general and applicable to any equilibrium configuration. Moreover, by construction, the framework is covariant and identification gauge invariant. This last point is of pivotal importance when compared with perturbation theory available in the literature. The classical metric-based approaches are intrinsically dependent on a coordinate system and the gauge, which makes it challenging to understand the dynamics of the perturbed spacetime unambiguously. Indeed, the adopted covariant spacetime decomposition formalism provides a manifest advantage over the standard theory: all quantities are geometrically and physically motivated, which clearly shows the dependencies and the source of the various physical properties of the perturbations. In addition, the framework leads to a natural separation between even and odd parity components of the perturbations, such that the systems of differential equations for each parity are completely decoupled at the linear level.

As a first application, we considered the study of linear, isotropic, and adiabatic perturbations. To do so, we have explicitly shown the importance of choosing frames to describe the unperturbed and the perturbed spacetimes. To our knowledge, this cannot be done in general in the metric-based linearized theory available in the literature and makes the covariant gauge-invariant approach all the more powerful. Given the symmetries of the problem, we considered two meaningful frames: a frame associated with observers locally comoving with the elements of volume of the fluid and a frame associated with static observers with respect to an observer at spatial infinity. These frames represent the classical Lagrangian and Eulerian pictures, respectively. The adopted formalism makes it quite simple to change between frames and evaluate the advantages of one frame over the other. In particular, we were able to derive the relation between the equations of state of the comoving and static frames in the perturbed spacetime, showing that, even at linear level, we have to account for phase changes between the equilibrium and the perturbed fluids. Moreover, we have shown that, since a perfect fluid in the comoving frame will not be perceived as perfect in the static frame, and the net momentum flow must be included in the equation of state to account for the correction to the rate of change of the energy density and the pressure when compared to those measured in the comoving frame.

Focusing on analyzing the general properties of linear, isotropic, and adiabatic perturbations, we have rigorously shown that the problem can be cast in the form of a singular Sturm–Liouville eigenvalue problem, inheriting the standard properties for the eigenfunctions and eigenvalues. Moreover, under rather general regularity conditions, we have found lower bounds for the values of the eigenfrequencies. The application of the methods developed in this article, the determination of the analytical solutions to the system of equations, and the discussion regarding the stability of selected solutions of the theory will be done elsewhere.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Definition of the 1+1+2 potentials

In this appendix we present the definition of the 1+1+2 potentials, namely the kinematical quantities of the congruences formed by the integral curves of the vector fields u and e, the electric and magnetic parts of the Weyl tensor, and the components of the stress-energy tensor.

A.1. Kinematical quantities

A.1.1. Decomposition of the covariant derivatives on the sheet W. Using the definitions of the projector operators onto the surfaces V and W, introduced in section 2, the covariant derivatives of the vector fields u and e can be uniquely decomposed in their components along u, e and in W.

The covariant derivatives of the tensor fields u and e on the sheet can be decomposed as

$$\delta_\alpha u_\beta \equiv N_\alpha{}^\mu N_\beta{}^\nu \nabla_\mu u_\nu = \frac{1}{2} N_{\alpha\beta}\tilde{\theta} + \Sigma_{\alpha\beta} + \varepsilon_{\alpha\beta}\Omega\,, \tag{A1}$$

where

$$\begin{aligned} \tilde{\theta} &= \delta_\alpha u^\alpha\,, \\ \Sigma_{\alpha\beta} &= \delta_{\{\alpha} u_{\beta\}}\,, \\ \Omega &= \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu u_\nu\,, \end{aligned} \tag{A2}$$

and

$$\delta_\alpha e_\beta = \frac{1}{2} N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi\,, \tag{A3}$$

with

$$\begin{aligned} \phi &= \delta_\mu e^\mu \,, \\ \zeta_{\alpha\beta} &= \delta_{\{\alpha} e_{\beta\}} \,, \\ \xi &= \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu e_\nu \,, \end{aligned} \tag{A4}$$

where the curly brackets represent the projected symmetric part without trace of a tensor in W, that is, for a tensor $\chi_{\alpha\beta}$,

$$\chi_{\{\alpha\beta\}} = \left[N^\mu{}_{(\alpha}N_{\beta)}{}^\nu - \frac{N_{\alpha\beta}}{2}N^{\mu\nu}\right]\chi_{\mu\nu} \,. \tag{A5}$$

Using the 2-form $\varepsilon_{\alpha\beta}$, a completely anti-symmetric tensor defined on the sheet, $\chi_{\alpha\beta} = \chi_{[\alpha\beta]} = N^\mu{}_{[\alpha}N_{\beta]}{}^\nu\chi_{\mu\nu}$, can be written as

$$\chi_{[\alpha\beta]} = \varepsilon_{\alpha\beta}\left(\frac{1}{2}\varepsilon^{\mu\nu}\chi_{\mu\nu}\right) \,. \tag{A6}$$

A.1.2. Decomposition of the covariant derivatives on V. The decomposition of the projected covariant derivatives of u onto V is given by

$$D_\alpha u_\beta = h_\alpha{}^\mu h_\beta{}^\nu \nabla_\mu u_\nu = \frac{1}{3}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \,, \tag{A7}$$

with

$$\theta = h^{\mu\nu}D_\mu u_\nu \,, \tag{A8}$$

$$\sigma_{\alpha\beta} = D_{\langle\alpha}u_{\beta\rangle} \,, \tag{A9}$$

$$\omega_{\alpha\beta} = h^\mu{}_{[\alpha}h_{\beta]}{}^\nu D_\mu u_\nu \,, \tag{A10}$$

where we used the angular brackets to represent the projected symmetric part without trace of a tensor on V, that is, for a tensor, $\chi_{\alpha\beta}$,

$$\chi_{\langle\alpha\beta\rangle} = \left[h^\mu{}_{(\alpha}h_{\beta)}{}^\nu - \frac{h_{\alpha\beta}}{3}h^{\mu\nu}\right]\chi_{\mu\nu} \,. \tag{A11}$$

The scalar and tensor quantities in equations (A8)–(A10) can themselves be further decomposed in their contributions exclusively on W and along e, such that

$$\theta = \tilde{\theta} + \vartheta \,, \tag{A12}$$

where $\tilde{\theta}$ is defined in equation (A2) and

$$\vartheta = -u^\mu\left(e^\nu D_\nu e_\mu\right) = -u^\mu\widehat{e}_\mu \,; \tag{A13}$$

$$\sigma_{\alpha\beta} = \Sigma_{\alpha\beta} + 2\Sigma_{(\alpha}e_{\beta)} + \Sigma\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) \,, \tag{A14}$$

with

$$\begin{aligned} \Sigma_{\alpha\beta} &= \sigma_{\{\alpha\beta\}}\,, \\ \Sigma_{\alpha} &= N_{\alpha}{}^{\mu} e^{\nu} \sigma_{\mu\nu}\,, \\ \Sigma &= e^{\mu} e^{\nu} \sigma_{\mu\nu} = -N^{\mu\nu} \sigma_{\mu\nu}\,, \end{aligned} \tag{A15}$$

and

$$\omega_{\alpha\beta} = \varepsilon_{\alpha\beta\mu} \left(\Omega e^{\mu} + \Omega^{\mu}\right)\,, \tag{A16}$$

where Ω is given in equation (A2) and

$$\Omega^{\alpha} = \frac{1}{2} N_{\gamma}{}^{\alpha} \varepsilon^{\mu\nu\gamma} D_{\mu} u_{\nu}\,. \tag{A17}$$

The quantities θ, Σ, $\tilde{\theta}$ and ϑ are not independent, in fact:

$$\tilde{\theta} = \frac{2}{3}\theta - \Sigma\,, \tag{A18}$$

$$\vartheta = \frac{1}{3}\theta + \Sigma\,; \tag{A19}$$

as such, when setting up the 1+1+2 formalism, only two of those quantities are chosen. The convention followed here uses the variables θ and Σ.

For the projected covariant derivative of the vector field e on V we have

$$D_{\alpha} e_{\beta} = h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \nabla_{\mu} e_{\nu} = \delta_{\alpha} e_{\beta} + e_{\alpha} a_{\beta}\,, \tag{A20}$$

where $\delta_{\alpha} e_{\beta}$ is given by equation (A3) and

$$a_{\alpha} = e^{\mu} D_{\mu} e_{\alpha} = \widehat{e}_{\alpha}\,. \tag{A21}$$

A.1.3. Decomposition of the covariant derivatives on the full manifold. Finally, we can decompose the total covariant derivatives of u^{α} and e^{α}, such that

$$\nabla_{\alpha} u_{\beta} = -u_{\alpha}\left(\mathcal{A} e_{\beta} + \mathcal{A}_{\beta}\right) + D_{\alpha} u_{\beta}\,, \tag{A22}$$

with

$$\begin{aligned} \mathcal{A} &= -u_{\mu} u^{\nu} \nabla_{\nu} e^{\mu} = -u_{\mu} \dot{e}^{\mu}\,, \\ \mathcal{A}_{\alpha} &= N_{\alpha\mu} \dot{u}^{\mu}\,, \end{aligned} \tag{A23}$$

and

$$\nabla_{\alpha} e_{\beta} = D_{\alpha} e_{\beta} - u_{\alpha} \alpha_{\beta} - \mathcal{A} u_{\alpha} u_{\beta} + \left(\frac{1}{3}\theta + \Sigma\right) e_{\alpha} u_{\beta} + \left(\Sigma_{\alpha} - \varepsilon_{\alpha\mu} \Omega^{\mu}\right) u_{\beta}\,, \tag{A24}$$

where

$$\alpha_{\alpha} = h_{\alpha}{}^{\mu} \dot{e}_{\mu}\,. \tag{A25}$$

A.2. Covariant derivatives of the projectors and the Levi–Civita volume form

From the previous relations, we find the following expressions for the covariant derivatives of the projector tensors:

$$\begin{aligned} \delta_\alpha N_{\beta\gamma} &= 0\,, & \nabla_\rho \varepsilon_{\alpha\beta\gamma\delta} &= 0\,, \\ e^\mu D_\mu N_{\alpha\beta} &= -2e_{(\alpha}a_{\beta)}\,, & u^\mu \nabla_\mu \varepsilon_{\alpha\beta\gamma} &= \varepsilon_{\alpha\beta\gamma\mu}\left(\mathcal{A}e^\mu + \mathcal{A}^\mu\right), \\ u^\mu \nabla_\mu N_{\alpha\beta} &= 2u_{(\alpha}\mathcal{A}_{\beta)} - 2e_{(\alpha}\alpha_{\beta)}\,, & D_\alpha \varepsilon_{\beta\gamma\delta} &= 0\,, \\ D_\alpha h_{\beta\gamma} &= 0\,, & u^\mu \nabla_\mu h_{\alpha\beta} &= 2u_{(\alpha|}\left(\mathcal{A}e_{|\beta)} + \mathcal{A}_{|\beta)}\right). \end{aligned} \tag{A26}$$

A.3. Weyl and stress-energy tensors

In four spacetime dimensions, the Riemann tensor is characterized by the Ricci and the Weyl tensors, according with equation (12). Remarkably, the Weyl 4-tensor can be completely characterized by two 2-tensors, defined as

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}u^\mu u^\nu\,, \tag{A27}$$

$$H_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\mu\nu\beta\delta}u^\delta\,, \tag{A28}$$

respectively referred as the 'electric' and 'magnetic' part of the Weyl tensor, both symmetric and traceless tensors, such that

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = &-\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_\alpha E_{\beta[\gamma}u_{\delta]} + 2u_\beta E_{\alpha[\gamma}u_{\delta]} - 2\varepsilon_{\alpha\beta\mu}H^\mu{}_{[\gamma}u_{\delta]} \\ &- 2\varepsilon_{\mu\gamma\delta}H^\mu{}_{[\alpha}u_{\beta]}\,. \end{aligned} \tag{A29}$$

Them the Weyl tensor is completely characterized by the 1+1+2 decomposition of the tensors $E_{\alpha\beta}$ and $H_{\alpha\beta}$, given by

$$\begin{aligned} E_{\alpha\beta} &= \mathcal{E}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{E}_\alpha e_\beta + e_\alpha \mathcal{E}_\beta + \mathcal{E}_{\alpha\beta}\,, \\ H_{\alpha\beta} &= \mathcal{H}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{H}_\alpha e_\beta + e_\alpha \mathcal{H}_\beta + \mathcal{H}_{\alpha\beta}\,, \end{aligned} \tag{A30}$$

where

$$\begin{aligned} \mathcal{E} &= E_{\mu\nu}e^\mu e^\nu = -N^{\mu\nu}E_{\mu\nu}\,, & \mathcal{H} &= e^\mu e^\nu H_{\mu\nu} = -N^{\mu\nu}H_{\mu\nu}\,, \\ \mathcal{E}_\alpha &= N_\alpha{}^\mu e^\nu E_{\mu\nu} = e^\mu N_\alpha{}^\nu E_{\mu\nu}\,, & \mathcal{H}_\alpha &= N_\alpha{}^\mu e^\nu H_{\mu\nu} = e^\mu N_\alpha{}^\nu H_{\mu\nu}\,, \\ \mathcal{E}_{\alpha\beta} &= E_{\{\alpha\beta\}}\,, & \mathcal{H}_{\alpha\beta} &= H_{\{\alpha\beta\}}\,. \end{aligned} \tag{A31}$$

We shall also decompose the metric stress-energy tensor, with components $T_{\alpha\beta}$ in a local coordinate system, in terms of its pointwise projections onto u, e, and W:

$$\begin{aligned} T_{\alpha\beta} = &\,\mu u_\alpha u_\beta + (p + \Pi)e_\alpha e_\beta + \left(p - \frac{1}{2}\Pi\right)N_{\alpha\beta} + 2Qe_{(\alpha}u_{\beta)} + 2Q_{(\alpha}u_{\beta)} \\ &+ 2\Pi_{(\alpha}e_{\beta)} + \Pi_{\alpha\beta}\,, \end{aligned} \tag{A32}$$

with

$$\begin{aligned} \mu &= u^\mu u^\nu T_{\mu\nu}\,, & Q_\alpha &= -N_\alpha{}^\mu u^\nu T_{\mu\nu} = -u^\mu N_\alpha{}^\nu T_{\mu\nu}\,,\\ p &= \frac{1}{3}\left(e^\mu e^\nu + N^{\mu\nu}\right) T_{\mu\nu}\,, & \Pi_\alpha &= N_\alpha{}^\mu e^\nu T_{\mu\nu} = e^\mu N_\alpha{}^\nu T_{\mu\nu}\,,\\ \Pi &= \frac{1}{3} T_{\alpha\beta}\left(2e^\alpha e^\beta - N^{\alpha\beta}\right)\,, & \Pi_{\alpha\beta} &= T_{\{\alpha\beta\}}\,.\\ Q &= -e^\mu u^\nu T_{\mu\nu} = -u^\mu e^\nu T_{\mu\nu}\,, \end{aligned} \tag{A33}$$

Appendix B. Field equations in the 1+1+2 formalism

Here, we present the Einstein field equations for the theory of General Relativity, written in the 1+1+2 covariant spacetime decomposition formalism introduced in section 2. Although the complete set of equations has already been presented in literature, these expressions are known to contain several typographical errors. To our knowledge, these errors do not affect published works. However, the corrected equations are essential in studying general linear perturbations of static LRS II spacetimes, hence the importance of presenting them here.

B.1. Equations for the kinematical quantities associated with the timelike congruence

We find the following evolution and propagation equations for the kinematical quantities associated with the u congruence.

Scalar equations:

$$\begin{aligned} \widehat{\mathcal{A}} - \dot{\theta} = {}& \frac{1}{2}(\mu + 3p) - \Lambda + \frac{1}{3}\theta^2 + \frac{3}{2}\Sigma^2 - 2\Omega^2 + 2\Sigma_\mu\Sigma^\mu - \mathcal{A}(\mathcal{A} + \phi)\\ & - 2\Omega_\mu\Omega^\mu + \mathcal{A}_\mu(a^\mu - \mathcal{A}^\mu) + \Sigma_{\mu\nu}\Sigma^{\mu\nu} - \delta_\mu\mathcal{A}^\mu\,, \end{aligned} \tag{B1}$$

$$\begin{aligned} \dot{\Sigma} - \frac{2}{3}\widehat{\mathcal{A}} = {}& \frac{1}{2}\Pi - \mathcal{E} + \frac{1}{3}(2\mathcal{A} - \phi)\mathcal{A} - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Sigma - \frac{2}{3}\Omega^2 + \Sigma_\mu\left(2\alpha^\mu - \frac{1}{3}\Sigma^\mu\right)\\ & - \frac{1}{3}\mathcal{A}_\mu(\mathcal{A}^\mu + 2a^\mu) + \frac{1}{3}\Omega_\mu\Omega^\mu + \frac{1}{3}\Sigma_{\mu\nu}\Sigma^{\mu\nu} - \frac{1}{3}\delta_\mu\mathcal{A}^\mu\,, \end{aligned} \tag{B2}$$

$$\dot{\Omega} = \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu\mathcal{A}_\nu + \mathcal{A}\xi + \left(\Sigma - \frac{2}{3}\theta\right)\Omega + \Omega_\mu(\Sigma^\mu + \alpha^\mu)\,, \tag{B3}$$

$$\widehat{\Omega} = \Omega(\mathcal{A} - \phi) + \Omega_\mu(\mathcal{A}^\mu + a^\mu) - \delta_\mu\Omega^\mu\,, \tag{B4}$$

$$\frac{2}{3}\widehat{\theta} - \widehat{\Sigma} = Q + \frac{3}{2}\Sigma\phi + 2\Omega\xi + \delta_\mu\Sigma^\mu + \varepsilon_{\mu\nu}\delta^\mu\Omega^\nu - 2\Sigma_\mu a^\mu + 2\varepsilon_{\mu\nu}\mathcal{A}^\mu\Omega^\nu - \zeta_{\mu\nu}\Sigma^{\mu\nu}\,; \tag{B5}$$

Vector equations:

$$\begin{aligned} N_\alpha{}^\mu\dot{\Omega}_\mu + \frac{1}{2}\varepsilon_\alpha{}^\mu\widehat{\mathcal{A}}_\mu = {}& \frac{1}{2}\xi\mathcal{A}_\alpha - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Omega_\alpha + \Omega(\Sigma_\alpha - \alpha_\alpha)\\ & - \frac{1}{2}\varepsilon_\alpha{}^\mu\left(\frac{1}{2}\phi\mathcal{A}_\mu + \mathcal{A}a_\mu - \delta_\mu\mathcal{A}\right) - \frac{1}{2}\varepsilon_\alpha{}^\mu\zeta_{\mu\nu}\mathcal{A}^\nu + \Sigma_{\alpha\mu}\Omega^\mu\,, \end{aligned} \tag{B6}$$

$$\begin{aligned} N_\alpha{}^\mu\dot{\Sigma}_\mu - \frac{1}{2}N_\alpha{}^\mu\widehat{\mathcal{A}}_\mu = {}& \frac{1}{2}\delta_\alpha\mathcal{A} + \left(\mathcal{A} - \frac{1}{4}\phi\right)\mathcal{A}_\alpha - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Sigma_\alpha + \frac{1}{2}\mathcal{A}a_\alpha - \frac{3}{2}\Sigma\alpha_\alpha\\ & - \Omega\Omega_\alpha - \frac{1}{2}(\zeta_{\alpha\mu} + \varepsilon_{\alpha\mu}\xi)\mathcal{A}^\mu + \Sigma_{\alpha\mu}(\alpha^\mu - \Sigma^\mu) - \mathcal{E}_\alpha + \frac{1}{2}\Pi_\alpha\,, \end{aligned} \tag{B7}$$

$$
\begin{aligned}
N_\alpha{}^\mu \widehat{\Sigma}_\mu - \varepsilon_\alpha{}^\mu \widehat{\Omega}_\mu = &\frac{1}{2}\delta_\alpha \Sigma + \frac{2}{3}\delta_\alpha \theta - \varepsilon_\alpha{}^\mu \delta_\mu \Omega - \frac{3}{2}\phi \Sigma_\alpha + \varepsilon_\alpha{}^\mu \Sigma_\mu \xi - \Omega_\alpha \xi \\
&+ \left(\frac{1}{2}\phi + 2\mathcal{A}\right) \varepsilon_\alpha{}^\mu \Omega_\mu - \frac{3}{2}\Sigma a_\alpha + \varepsilon_\alpha{}^\mu \Omega \left(a_\mu - 2\mathcal{A}_\mu\right) \\
&- \delta_\mu \Sigma_\alpha{}^\mu - \zeta_\alpha{}^\mu \Sigma_\mu + \Sigma_\alpha{}^\mu a_\mu + \varepsilon_{\alpha\mu} \zeta^{\mu\nu} \Omega_\nu - Q_\alpha ;
\end{aligned}
\tag{B8}
$$

Tensor equations:

$$
\begin{aligned}
N^\mu{}_{\{\alpha} N_{\beta\}}{}^\nu \dot{\Sigma}_{\mu\nu} = &\delta_{\{\alpha} \mathcal{A}_{\beta\}} + \mathcal{A}_{\{\alpha} \mathcal{A}_{\beta\}} - \Sigma_{\{\alpha} \Sigma_{\beta\}} - 2\Sigma_{\{\alpha} \alpha_{\beta\}} - \Omega_{\{\alpha} \Omega_{\beta\}} \\
&+ \mathcal{A} \zeta_{\alpha\beta} - \left(\frac{2}{3}\theta - \Sigma\right) \Sigma_{\alpha\beta} - \Sigma_{\{\alpha|}{}^\mu \Sigma_{\mu|\beta\}} - \mathcal{E}_{\alpha\beta} + \frac{1}{2}\Pi_{\alpha\beta} ,
\end{aligned}
\tag{B9}
$$

$$
\begin{aligned}
N^\mu{}_{\{\alpha} N_{\beta\}}{}^\nu \widehat{\Sigma}_{\mu\nu} = &\delta_{\{\alpha} \Sigma_{\beta\}} + \frac{1}{2}\varepsilon_{\{\alpha|}{}^\mu \delta_\mu \Omega_{|\beta\}} + \frac{1}{2}\varepsilon_{\{\alpha|}{}^\mu \delta_{|\beta\}} \Omega_\mu - \frac{1}{2}\Sigma_{\alpha\beta}\phi + \varepsilon^\mu{}_{\{\alpha} \Sigma_{\beta\}\mu} \xi \\
&+ \frac{3}{2}\Sigma \zeta_{\alpha\beta} - \varepsilon_{\mu\{\alpha} \zeta_{\beta\}}{}^\mu \Omega - 2a_{\{\alpha} \Sigma_{\beta\}} + \varepsilon_{\{\alpha|}{}^\mu \Omega_\mu \mathcal{A}_{|\beta\}} + \varepsilon_{\{\alpha|}{}^\mu \Omega_{|\beta\}} \mathcal{A}_\mu \\
&- \Sigma_{\mu\{\alpha} \zeta_{\beta\}}{}^\mu - \varepsilon_{\mu\{\alpha} \mathcal{H}_{\beta\}}{}^\mu ;
\end{aligned}
\tag{B10}
$$

Constraint equations:

$$
\delta_\mu \Omega^\mu + \varepsilon^{\mu\nu} \delta_\mu \Sigma_\nu = (2\mathcal{A} - \phi)\Omega - 3\Sigma\xi + \mathcal{H} + \varepsilon^{\mu\nu} \zeta_\mu{}^\gamma \Sigma_{\gamma\nu} ,
\tag{B11}
$$

$$
\begin{aligned}
\delta_\alpha \Sigma - \frac{2}{3}\delta_\alpha \theta + 2\varepsilon_\alpha{}^\mu \delta_\mu \Omega + 2\delta_\mu \Sigma_\alpha{}^\mu = &\phi \left(\varepsilon_\alpha{}^\mu \Omega_\mu - \Sigma_\alpha\right) - 2\xi \left(\Omega_\alpha - 3\varepsilon_\alpha{}^\mu \Sigma_\mu\right) - 4\varepsilon_\alpha{}^\mu \Omega \mathcal{A}_\mu \\
&+ 2\zeta_{\alpha\mu} \Sigma^\mu + 2\varepsilon_\alpha{}^\mu \Omega^\nu \zeta_{\mu\nu} - 2\varepsilon_\alpha{}^\mu \mathcal{H}_\mu - Q_\alpha .
\end{aligned}
\tag{B12}
$$

B.2. Equations for the kinematical quantities associated with the spacelike congruence

We also find the following evolution and propagation equations for the kinematical quantities associated with the *e* congruence.

Scalar equations:

$$
\begin{aligned}
\dot{\phi} = &Q + \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)(2\mathcal{A} - \phi) + 2\Omega\xi + \delta_\mu \alpha^\mu - \alpha_\mu \left(a^\mu - \mathcal{A}^\mu\right) \\
&- \left(\mathcal{A}^\nu + a^\nu\right)\left(\Sigma_\nu + \varepsilon_{\mu\nu}\Omega^\mu\right) - \Sigma^{\mu\nu}\zeta_{\mu\nu} ,
\end{aligned}
\tag{B13}
$$

$$
\begin{aligned}
\widehat{\phi} = &\left(\frac{1}{3}\theta + \Sigma\right)\left(\frac{2}{3}\theta - \Sigma\right) - \frac{1}{2}\phi^2 + 2\xi^2 - \frac{2}{3}(\mu + \Lambda) - \frac{1}{2}\Pi - \mathcal{E} \\
&+ \delta_\mu a^\mu - a_\mu a^\mu - \Sigma_\mu \Sigma^\mu + \Omega_\mu \Omega^\mu + 2\varepsilon_{\mu\nu}\alpha^\mu \Omega^\nu - \zeta_{\mu\nu}\zeta^{\mu\nu} ,
\end{aligned}
\tag{B14}
$$

$$
\begin{aligned}
\dot{\xi} = &\left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\xi + \frac{1}{2}\mathcal{H} + \Omega\left(\mathcal{A} - \frac{1}{2}\phi\right) + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu \alpha_\nu \\
&+ \frac{1}{2}\left(\mathcal{A}_\mu + a_\mu\right)\left[\Omega^\mu + \varepsilon^{\mu\nu}\left(\Sigma_\nu + \alpha_\nu\right)\right] - \frac{1}{2}\varepsilon^{\mu\nu}\Sigma_{\mu\gamma}\zeta^\gamma{}_\nu ,
\end{aligned}
\tag{B15}
$$

$$
\widehat{\xi} = \left(\frac{1}{3}\theta + \Sigma\right)\Omega - \phi\xi + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu a_\nu + \Omega^\mu\left(\alpha_\mu + \Sigma_\mu\right) - \frac{1}{2}\varepsilon^{\mu\nu}\zeta_{\mu\gamma}\zeta^\gamma{}_\nu ;
\tag{B16}
$$

Vector equations:

$$
\begin{aligned}
N^{\mu}{}_{\alpha}\widehat{\alpha}_{\mu} - N^{\mu}{}_{\alpha}\dot{a}_{\mu} = {} & \varepsilon_{\mu\alpha}\mathcal{H}^{\mu} - \left(\mathcal{A} + \frac{1}{2}\phi\right)\alpha_{\alpha} + \varepsilon_{\alpha\mu}\alpha^{\mu}\xi + \left(\frac{1}{3}\theta + \Sigma\right)(\mathcal{A}_{\alpha} + a_{\alpha}) \\
& + \left(\frac{1}{2}\phi - \mathcal{A}\right)(\Sigma_{\alpha} + \varepsilon_{\alpha\mu}\Omega^{\mu}) + (\Omega_{\alpha} - \varepsilon_{\alpha\mu}\Sigma^{\mu})\xi \\
& + (\Sigma^{\mu} + \varepsilon^{\mu\nu}\Omega_{\nu} - \alpha^{\mu})\zeta_{\mu\alpha} + \frac{1}{2}\mathcal{Q}_{\alpha};
\end{aligned}
\tag{B17}
$$

Tensor equations:

$$
\begin{aligned}
N^{\mu}{}_{\{\alpha}N_{\beta\}}{}^{\nu}\dot{\zeta}_{\mu\nu} = {} & \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\zeta_{\alpha\beta} + \varepsilon_{\mu\{\alpha|}\zeta^{\mu}{}_{|\beta\}}\Omega + \left(\mathcal{A} - \frac{1}{2}\phi\right)\Sigma_{\alpha\beta} - \Sigma_{\{\alpha|\mu}\left(\zeta^{\mu}{}_{|\beta\}} + \varepsilon^{\mu}{}_{|\beta\}}\xi\right) \\
& + \delta_{\{\alpha}\alpha_{\beta\}} + \left(\mathcal{A}_{\{\alpha} - a_{\{\alpha}\right)\alpha_{\beta\}} - \left(\Sigma_{\{\alpha} - \varepsilon_{\{\alpha|\mu}\Omega^{\mu}\right)\left(\mathcal{A}_{\beta\}} + a_{\beta\}}\right) - \varepsilon^{\mu}{}_{\{\alpha}\mathcal{H}_{\beta\}\mu},
\end{aligned}
\tag{B18}
$$

$$
\begin{aligned}
N^{\mu}{}_{\{\alpha}N_{\beta\}}{}^{\nu}\widehat{\zeta}_{\mu\nu} = {} & \delta_{\{\alpha}a_{\beta\}} - \phi\zeta_{\alpha\beta} - \zeta_{\{\alpha|\mu}\zeta^{\mu}{}_{|\beta\}} - a_{\{\alpha}a_{\beta\}} - \Sigma_{\{\alpha}\Sigma_{\beta\}} - \Omega_{\{\alpha}\Omega_{\beta\}} \\
& + 2\alpha_{\{\alpha}\varepsilon_{\beta\}\mu}\Omega^{\mu} + \left(\frac{1}{3}\theta + \Sigma\right)\Sigma_{\alpha\beta} - \mathcal{E}_{\alpha\beta} - \frac{1}{2}\Pi_{\alpha\beta};
\end{aligned}
\tag{B19}
$$

Constraint equation:

$$
\begin{aligned}
\frac{1}{2}\delta_{\alpha}\phi - \delta_{\mu}\zeta_{\alpha}{}^{\mu} - \varepsilon_{\alpha}{}^{\mu}\delta_{\mu}\xi = {} & -2\varepsilon_{\alpha\mu}a^{\mu}\xi - \Omega\left(\Omega_{\alpha} + \varepsilon_{\alpha\mu}\Sigma^{\mu} - 2\varepsilon_{\alpha\mu}\alpha^{\mu}\right) \\
& + \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)(\Sigma_{\alpha} - \varepsilon_{\alpha\mu}\Omega^{\mu}) - \mathcal{E}_{\alpha} \\
& - \frac{1}{2}\Pi_{\alpha} - \Sigma_{\alpha}{}^{\mu}\left(\Sigma_{\mu} - \varepsilon_{\mu\nu}\Omega^{\nu}\right).
\end{aligned}
\tag{B20}
$$

B.3. Electric and magnetic parts of the Weyl tensor

Moreover, we have the following evolution and propagation equations the electric and magnetic part of the Weyl tensor.

Scalar equations:

$$
\begin{aligned}
\dot{\mathcal{E}} + \frac{1}{2}\dot{\Pi} + \frac{1}{3}\widehat{\mathcal{Q}} = {} & 3\mathcal{H}\xi + \mathcal{E}\left(\frac{3}{2}\Sigma - \theta\right) - \frac{1}{2}\Pi\left(\frac{1}{3}\theta + \frac{1}{2}\Sigma\right) + \frac{1}{3}\mathcal{Q}\left(\frac{1}{2}\phi - 2\mathcal{A}\right) \\
& - \frac{1}{2}(\mu + p)\Sigma + \frac{1}{6}\delta_{\mu}\mathcal{Q}^{\mu} + \varepsilon^{\mu\nu}\delta_{\mu}\mathcal{H}_{\nu} + \frac{1}{3}\mathcal{Q}_{\mu}(a^{\mu} + \mathcal{A}^{\mu}) + 2\varepsilon_{\mu\nu}\mathcal{A}^{\mu}\mathcal{H}^{\nu} \\
& + \mathcal{E}^{\mu}(\Sigma_{\mu} + 2\alpha_{\mu} - \varepsilon_{\mu\nu}\Omega^{\nu}) - \frac{1}{2}\left(\frac{1}{3}\Sigma_{\mu} - 2\alpha_{\mu} + \varepsilon_{\mu\nu}\Omega^{\nu}\right)\Pi^{\mu} \\
& - \left(\mathcal{E}_{\mu\nu} - \frac{1}{6}\Pi_{\mu\nu}\right)\Sigma^{\mu\nu} + \varepsilon_{\mu\nu}\mathcal{H}_{\alpha}{}^{\mu}\zeta^{\nu\alpha},
\end{aligned}
\tag{B21}
$$

$$
\begin{aligned}
\widehat{\mathcal{E}} - \frac{1}{3}\widehat{\mu} + \frac{1}{2}\widehat{\Pi} = {} & -\delta_{\mu}\mathcal{E}^{\mu} - \frac{1}{2}\delta_{\mu}\Pi^{\mu} - \frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\mathcal{Q} + 3\mathcal{H}\Omega \\
& + 2\left(\mathcal{E}_{\mu} + \frac{1}{2}\Pi_{\mu}\right)a^{\mu} + \frac{1}{2}\Sigma_{\mu}\mathcal{Q}^{\mu} + 3\mathcal{H}_{\mu}\Omega^{\mu} + \frac{3}{2}\varepsilon_{\mu\nu}\mathcal{Q}^{\mu}\Omega^{\nu} \\
& + \varepsilon^{\mu\nu}\Sigma_{\mu}\mathcal{H}_{\nu} + \varepsilon^{\mu\nu}\Sigma_{\mu\gamma}\mathcal{H}_{\nu}{}^{\gamma} + \left(\mathcal{E}^{\mu\nu} + \frac{1}{2}\Pi^{\mu\nu}\right)\zeta_{\mu\nu},
\end{aligned}
\tag{B22}
$$

$$\dot{\mathcal{H}} = \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu\Pi_\nu - \varepsilon^{\mu\nu}\delta_\mu\mathcal{E}_\nu + \Omega Q - 3\left(\mathcal{E} - \frac{1}{2}\Pi\right)\xi - \left(\theta - \frac{3}{2}\Sigma\right)\mathcal{H}$$
$$- 2\varepsilon^{\mu\nu}\mathcal{A}_\mu\mathcal{E}_\nu + \mathcal{H}^\mu\left(2\alpha_\mu + \Sigma_\mu - \varepsilon_{\mu\nu}\Omega^\nu\right) - \mathcal{H}^{\mu\nu}\Sigma_{\mu\nu} \tag{B23}$$
$$- \frac{1}{2}\left(\Omega_\mu + \varepsilon_{\mu\nu}\Sigma^\nu\right)Q^\mu + \varepsilon^{\mu\nu}\mathcal{E}_{\nu\gamma}\zeta_\mu{}^\gamma + \frac{1}{2}\varepsilon_\mu{}^\nu\Pi^{\gamma\mu}\zeta_{\nu\gamma}\,,$$

$$\widehat{\mathcal{H}} = -\delta_\mu\mathcal{H}^\mu - \frac{1}{2}\varepsilon_{\mu\nu}\delta^\mu Q^\nu - \frac{3}{2}\mathcal{H}\phi - \left(3\mathcal{E} + \mu + p - \frac{1}{2}\Pi\right)\Omega - Q\xi$$
$$+ 2\mathcal{H}^\mu a_\mu - \left(3\mathcal{E}^\mu - \frac{1}{2}\Pi^\mu\right)\Omega_\mu + \varepsilon_{\mu\nu}\Sigma^\nu\left(\mathcal{E}^\mu + \frac{1}{2}\Pi^\mu\right) \tag{B24}$$
$$- \varepsilon_{\nu\gamma}\Sigma^{\nu\mu}\left(\mathcal{E}_\mu{}^\gamma + \frac{1}{2}\Pi_\mu{}^\gamma\right) + \mathcal{H}^{\mu\nu}\zeta_{\mu\nu}\,;$$

Vector equations:

$$N_\alpha{}^\mu\dot{\mathcal{E}}_\mu + \frac{1}{2}N_\alpha{}^\mu\dot{\Pi}_\mu = \left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\frac{1}{2}Q_\alpha + \varepsilon_{\alpha\mu}\mathcal{H}^\mu\right) - \frac{1}{2}Q\mathcal{A}_\alpha + \frac{3}{2}\mathcal{E}_\alpha\Sigma + 3\mathcal{H}_\alpha\xi - \frac{3}{2}\left(\mathcal{E}\right.$$
$$\left.+\frac{1}{2}\Pi\right)\alpha_\alpha + \varepsilon_{\alpha\mu}\Omega\left(\mathcal{E}^\mu - \frac{1}{2}\Pi^\mu\right) - \left(\mathcal{E}_\alpha + \frac{1}{6}\Pi_\alpha\right)\theta + \frac{3}{2}\varepsilon_{\alpha\mu}\mathcal{A}^\mu\mathcal{H}$$
$$+ \frac{1}{4}\Sigma\Pi_\alpha - \frac{1}{2}\left(\mu + p + \Pi\right)\left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu\right) + \frac{1}{2}\left(\zeta_{\alpha\mu} + \varepsilon_{\alpha\mu}\xi\right)Q^\mu$$
$$+ \mathcal{E}_{\alpha\mu}\left(2\Sigma^\mu + \alpha^\mu - 2\varepsilon^{\mu\nu}\Omega_\nu\right) + \Sigma_{\alpha\mu}\left(\mathcal{E}^\mu - \frac{1}{2}\Pi^\mu\right) \tag{B25}$$
$$- \varepsilon^{\mu\nu}\mathcal{H}_{\alpha\mu}\mathcal{A}_\nu - \varepsilon^{\mu\nu}\mathcal{H}_\mu\zeta_{\nu\alpha} + \frac{1}{2}\Pi_{\alpha\mu}\alpha^\mu$$
$$- \frac{1}{2}\delta_\alpha Q + \frac{1}{2}\varepsilon_\alpha{}^\mu\delta_\mu\mathcal{H} + \varepsilon^{\mu\nu}\delta_\mu\mathcal{H}_{\nu\alpha}\,,$$

$$N^\mu{}_\alpha\widehat{\mathcal{E}}_\mu + \frac{1}{2}N^\mu{}_\alpha\widehat{\Pi}_\mu = \frac{1}{2}Q\Sigma_\alpha + 3\mathcal{H}_\alpha\Omega - \frac{3}{2}\mathcal{H}\Omega_\alpha - \left(\frac{1}{3}\theta + \frac{1}{4}\Sigma\right)Q_\alpha - \frac{3}{2}\left(\mathcal{E}_\alpha + \frac{1}{2}\Pi_\alpha\right)\phi$$
$$- \frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right)a_\alpha + \frac{3}{2}\varepsilon_{\alpha\mu}\left(\Sigma^\mu\mathcal{H} - \Sigma\mathcal{H}^\mu\right) + \varepsilon_{\alpha\mu}\left(\mathcal{E}^\mu + \frac{1}{2}\Pi^\mu\right)\xi$$
$$+ \frac{3}{2}\varepsilon_{\alpha\mu}\left(Q^\mu\Omega - Q\Omega^\mu\right) + \frac{1}{2}\Sigma_{\alpha\mu}Q^\mu + \varepsilon_{\alpha\nu}\left(\mathcal{H}_\mu\Sigma^{\mu\nu} - \Sigma_\mu\mathcal{H}^{\mu\nu}\right) \tag{B26}$$
$$+ \left(\mathcal{E}_{\alpha\mu} + \frac{1}{2}\Pi_{\alpha\mu}\right)a^\mu - \left(\mathcal{E}^\mu + \frac{1}{2}\Pi^\mu\right)\zeta_{\mu\alpha} + 3\mathcal{H}_{\alpha\mu}\Omega^\mu$$
$$+ \frac{1}{2}\delta_\alpha\mathcal{E} + \frac{1}{3}\delta_\alpha\mu + \frac{1}{4}\delta_\alpha\Pi - \delta_\mu\mathcal{E}_\alpha{}^\mu - \frac{1}{2}\delta_\mu\Pi_\alpha{}^\mu\,,$$

$$\frac{1}{2}\varepsilon_\alpha{}^\mu\widehat{\mathcal{E}}_\mu - N_\alpha{}^\mu\dot{\mathcal{H}}_\mu - \frac{1}{4}\varepsilon_\alpha{}^\mu\widehat{\Pi}_\mu$$
$$= \frac{3}{4}\varepsilon_\alpha{}^\mu\delta_\mu\mathcal{E} - \frac{3}{8}\varepsilon_\alpha{}^\mu\delta_\mu\Pi + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu\mathcal{E}_{\nu\alpha} - \frac{1}{4}\varepsilon^{\mu\nu}\delta_\mu\Pi_{\nu\alpha} - \frac{3}{4}\mathcal{H}\left(\Sigma_\alpha - 2\alpha_\alpha\right) \tag{B27}$$
$$+ \frac{3}{2}\varepsilon_{\alpha\mu}\mathcal{A}^\mu\mathcal{E} + \frac{3}{8}\varepsilon_{\alpha\mu}Q^\mu\Sigma - \frac{1}{4}\varepsilon_{\alpha\mu}\Sigma^\mu Q - \frac{3}{4}\varepsilon_{\alpha\mu}\Omega^\mu\mathcal{H} + \frac{5}{2}\xi\left(\mathcal{E}_\alpha - \frac{1}{2}\Pi_\alpha\right)$$

$$
\begin{aligned}
&-\frac{3}{4}\left(Q_\alpha\Omega+Q\Omega_\alpha\right)-\frac{3}{4}\varepsilon_{\alpha\mu}a^\mu\left(\mathcal{E}-\frac{1}{2}\Pi\right)-\varepsilon_{\alpha\mu}\mathcal{E}^\mu\mathcal{A}+\mathcal{H}_\alpha\left(\theta-\frac{3}{4}\Sigma\right)\\
&-\frac{1}{4}\varepsilon_{\alpha\mu}\left(\mathcal{E}^\mu-\frac{1}{2}\Pi^\mu\right)\phi+\frac{1}{2}\varepsilon_{\alpha\mu}\mathcal{H}^\mu\Omega+\frac{1}{2}\varepsilon_{\alpha\mu}a_\nu\left(\mathcal{E}^{\mu\nu}-\frac{1}{2}\Pi^{\mu\nu}\right)\\
&-\frac{1}{2}\varepsilon^{\mu\nu}\zeta_{\nu\alpha}\left(\mathcal{E}_\mu-\frac{1}{2}\Pi_\mu\right)-\varepsilon_{\alpha\mu}\zeta^{\mu\nu}\left(\mathcal{E}_\nu-\frac{1}{2}\Pi_\nu\right)+\varepsilon^{\mu\nu}\mathcal{A}_\mu\mathcal{E}_{\nu\alpha}\\
&-\frac{1}{2}\Sigma_{\alpha\mu}\left(3\mathcal{H}^\mu+\frac{1}{2}\varepsilon^{\mu\nu}Q_\nu\right)-\mathcal{H}_{\alpha\mu}\left(\frac{3}{2}\Sigma^\mu+\alpha^\mu-\frac{1}{2}\varepsilon^{\mu\nu}\Omega_\nu\right),
\end{aligned}
$$

$$
\begin{aligned}
N^\mu{}_\alpha\widehat{\mathcal{H}}_\mu-\frac{1}{2}\varepsilon_\alpha{}^\mu\widehat{Q}_\mu=&\frac{1}{2}\delta_\alpha\mathcal{H}-\delta_\mu\mathcal{H}_\alpha{}^\mu-\frac{1}{2}\varepsilon_\alpha{}^\mu\delta_\mu Q-\frac{3}{2}\varepsilon_{\alpha\mu}\Sigma^\mu\left(\mathcal{E}+\frac{1}{2}\Pi\right)-\frac{3}{2}\mathcal{H}a_\alpha\\
&+\frac{1}{2}\varepsilon_{\alpha\mu}Qa^\mu-\Omega\left(3\mathcal{E}_\alpha-\frac{1}{2}\Pi_\alpha\right)+\frac{3}{2}\varepsilon_{\alpha\mu}\Sigma\left(\mathcal{E}^\mu+\frac{1}{2}\Pi^\mu\right)\\
&+\left(\varepsilon_{\alpha\mu}\mathcal{H}^\mu-\frac{1}{2}Q_\alpha\right)\xi-\Omega_\alpha\left(\mu+p+\frac{1}{4}\Pi-\frac{3}{2}\mathcal{E}\right)\\
&-\frac{3}{2}\left(\mathcal{H}_\alpha-\frac{1}{6}\varepsilon_{\alpha\mu}Q^\mu\right)\phi-\Omega^\mu\left(3\mathcal{E}_{\alpha\mu}-\frac{1}{2}\Pi_{\alpha\mu}\right)\\
&+\mathcal{H}_{\alpha\mu}a^\mu-\mathcal{H}^\mu\zeta_{\alpha\mu}+\varepsilon_{\alpha\nu}\Sigma_\mu\left(\mathcal{E}^{\mu\nu}+\frac{1}{2}\Pi^{\mu\nu}\right)\\
&-\varepsilon_{\alpha\nu}\Sigma_\mu{}^\nu\left(\mathcal{E}^\mu+\frac{1}{2}\Pi^\mu\right)+\frac{1}{2}\varepsilon_{\alpha\mu}Q_\nu\zeta^{\mu\nu};
\end{aligned}
\tag{B28}
$$

Tensor equations:

$$
\begin{aligned}
&N^\mu{}_{\{\alpha}N_{\beta\}}{}^\nu\dot{\mathcal{E}}_{\mu\nu}+\frac{1}{2}N^\mu{}_{\{\alpha}N_{\beta\}}{}^\nu\dot{\Pi}_{\mu\nu}+N^\mu{}_{\{\alpha}\varepsilon_{\beta\}}{}^\nu\widehat{\mathcal{H}}_{\mu\nu}\\
&=\varepsilon_{\{\alpha|}{}^\nu\delta_\nu\mathcal{H}_{|\beta\}}-\frac{1}{2}\delta_{\{\alpha}Q_{\beta\}}-\frac{1}{2}\left(\mu+p+3\mathcal{E}-\frac{1}{2}\Pi\right)\Sigma_{\alpha\beta}\\
&-\frac{1}{2}Q\zeta_{\alpha\beta}-\frac{3}{2}\mathcal{H}\varepsilon^\nu{}_{\{\alpha}\zeta_{\beta\}\nu}-\left(\theta+\frac{3}{2}\Sigma\right)\mathcal{E}_{\alpha\beta}+\mathcal{H}_{\alpha\beta}\xi\\
&+\varepsilon_{\mu\{\alpha}\mathcal{E}_{\beta\}}{}^\mu\Omega-\left(\frac{1}{6}\theta-\frac{1}{4}\Sigma\right)\Pi_{\alpha\beta}+\frac{1}{2}\varepsilon^\mu{}_{\{\alpha}\Pi_{\beta\}\mu}\Omega\\
&+\left(\frac{1}{2}\phi+2\mathcal{A}\right)\varepsilon_{\mu\{\alpha}\mathcal{H}_{\beta\}}{}^\mu+\left(3\mathcal{E}_{\{\alpha}-\frac{1}{2}\Pi_{\{\alpha}\right)\Sigma_{\beta\}}\\
&-\mathcal{A}_{\{\alpha}Q_{\beta\}}-\left(2\mathcal{E}_{\{\alpha}+\Pi_{\{\alpha}\right)\left(\alpha_{\beta\}}-\frac{1}{2}\varepsilon_{\beta\}}{}^\mu\Omega_\mu\right)\\
&+\mathcal{H}_{\{\alpha}\varepsilon_{\beta\}}{}^\mu\left(2\mathcal{A}_\mu-a_\mu\right)+\mathcal{H}_\mu\varepsilon^\mu{}_{\{\alpha}a_{\beta\}}\\
&-\mathcal{H}^\mu{}_{\{\alpha}\varepsilon_{\beta\}}{}^\nu\zeta_{\mu\nu},
\end{aligned}
\tag{B29}
$$

$$
\begin{aligned}
& N^{\mu}{}_{\{\alpha\varepsilon\beta\}}{}^{\nu}\widehat{\mathcal{E}}_{\mu\nu}-\frac{1}{2}N^{\mu}{}_{\{\alpha\varepsilon\beta\}}{}^{\nu}\widehat{\Pi}_{\mu\nu}-\dot{\mathcal{H}}_{\{\alpha\beta\}} \\
&\quad =\varepsilon_{\{\alpha|}{}^{\mu}\delta_{\mu}\mathcal{E}_{|\beta\}}-\frac{1}{2}\varepsilon_{\{\alpha|}{}^{\mu}\delta_{\mu}\Pi_{|\beta\}}+\xi\left(\mathcal{E}_{\alpha\beta}-\frac{1}{2}\Pi_{\alpha\beta}\right) \\
&\qquad +\frac{3}{2}\mathcal{H}\Sigma_{\alpha\beta}+\frac{1}{2}Q\varepsilon_{\mu\{\alpha}\Sigma_{\beta\}}{}^{\mu}+\left(\theta+\frac{3}{2}\Sigma\right)\mathcal{H}_{\alpha\beta} \\
&\qquad -\varepsilon_{\mu\{\alpha}\mathcal{H}_{\beta\}}{}^{\mu}\Omega-\frac{3}{2}Q_{\{\alpha}\Omega_{\beta\}}-\frac{1}{4}\phi\varepsilon_{\mu\{\alpha}\Pi_{\beta\}}{}^{\mu} \\
&\qquad -\frac{1}{2}a^{\mu}\varepsilon_{\mu\{\alpha}\Pi_{\beta\}}-\left(3\mathcal{H}_{\{\alpha}+\frac{1}{2}\varepsilon_{\mu\{\alpha}Q^{\mu}\right)\Sigma_{\beta\}} \\
&\qquad +\Omega^{\mu}\varepsilon_{\mu\{\alpha}\mathcal{H}_{\beta\}}-\frac{3}{2}\left(\mathcal{E}-\frac{1}{2}\Pi\right)\varepsilon_{\mu\{\alpha}\zeta_{\beta\}}{}^{\mu} \\
&\qquad +\varepsilon_{\mu\{\alpha}a_{\beta\}}\left(\mathcal{E}^{\mu}-\frac{1}{2}\Pi^{\mu}\right)-3\mathcal{H}_{\mu\{\alpha}\Sigma_{\beta\}}{}^{\mu} \\
&\qquad +2\mathcal{H}_{\{\alpha}\alpha_{\beta\}}+\left(\frac{1}{2}\phi+2\mathcal{A}\right)\varepsilon_{\mu\{\alpha}\mathcal{E}_{\beta\}}{}^{\mu} \\
&\qquad +\zeta_{\mu\nu}\varepsilon^{\mu}{}_{\{\alpha}\left(\mathcal{E}^{\nu}{}_{\beta\}}-\frac{1}{2}\Pi^{\nu}{}_{\beta\}}\right) \\
&\qquad -\varepsilon_{\mu\{\alpha}\mathcal{E}_{\beta\}}\left(2\mathcal{A}^{\mu}-a^{\mu}\right),
\end{aligned} \tag{B30}
$$

where we have used the fact that for any two symmetric and traceless 2-tensors on the sheet, A and B, that is $A_{\mu\nu}=N_{\{\mu}{}^{\alpha}N_{\nu\}}{}^{\beta}A_{\alpha\beta}$ and $B_{\mu\nu}=N_{\{\mu}{}^{\alpha}N_{\nu\}}{}^{\beta}B_{\alpha\beta}$, the contraction $A^{\delta}{}_{\{a}B_{b\}\delta}=0$.

B.4. Matter conservation laws

Lastly, the energy and momentum conservation equations are given by

$$
\dot{\mu}+\widehat{Q}=Q^{\mu}\left(a_{\mu}-2\mathcal{A}_{\mu}\right)-2\Pi^{\mu}\Sigma_{\mu}-\left(\phi+2\mathcal{A}\right)Q-\frac{3}{2}\Pi\Sigma-\theta\left(\mu+p\right)-\delta_{\mu}Q^{\mu}-\Pi^{\mu\nu}\Sigma_{\mu\nu}, \tag{B31}
$$

$$
\begin{aligned}
\dot{Q}+\widehat{p}+\hat{\Pi}=&-\delta_{\mu}\Pi^{\mu}-\left(\frac{3}{2}\phi+\mathcal{A}\right)\Pi-\left(\frac{4}{3}\theta+\Sigma\right)Q-\left(\mu+p\right)\mathcal{A} \\
&+\left(\alpha_{\mu}-\Sigma_{\mu}+\varepsilon_{\mu\nu}\Omega^{\nu}\right)Q^{\mu}+\Pi^{\mu}\left(2a_{\mu}-\mathcal{A}_{\mu}\right)+\Pi_{\mu\nu}\zeta^{\mu\nu},
\end{aligned} \tag{B32}
$$

$$
\begin{aligned}
N_{\alpha}{}^{\mu}\dot{Q}_{\mu}+N_{\alpha}{}^{\mu}\widehat{\Pi}_{\mu}=&-\delta_{\alpha}p+\frac{1}{2}\delta_{\alpha}\Pi-\delta_{\mu}\Pi_{\alpha}{}^{\mu}-Q\left(\alpha_{\alpha}+\Sigma_{\alpha}+\varepsilon_{\alpha\beta}\Omega^{\beta}\right)+\varepsilon_{\alpha\mu}Q^{\mu}\Omega \\
&-\frac{3}{2}\Pi a_{\alpha}-\left(\frac{4}{3}\theta-\frac{1}{2}\Sigma\right)Q_{\alpha}-\left(\frac{3}{2}\phi+\mathcal{A}\right)\Pi_{\alpha}-\Pi^{\mu}\left(\zeta_{\alpha\mu}-\varepsilon_{\alpha\mu}\xi\right) \\
&-\left(\mu+p-\frac{1}{2}\Pi\right)\mathcal{A}_{\alpha}-Q^{\mu}\Sigma_{\alpha\mu}+\Pi_{\alpha\mu}\left(a^{\mu}-\mathcal{A}^{\mu}\right).
\end{aligned} \tag{B33}
$$

Appendix C. Linearized 1+1+2 equations for the angular gradients variables

In this appendix, we present a set of covariant, gauge invariant equations for the linear order perturbations of a background spacetime assumed to be static LRS II and permeated by a general matter fluid, such that the equilibrium configuration is characterized by the scalars $\{\phi_0,\mathcal{A}_0,\mathcal{E}_0,\mu_0,p_0,\Pi_0,\Lambda\}$. Considering the variables in equation (25) and using the commutation relations (17), we find the following set of linearized equations for the various quantities that characterize the perturbed spacetime.

C.1. Linearized equations for the kinematical quantities associated with the timelike congruence

- Evolution and propagation equations for scalars and gradients of scalar quantities:

$$(\delta_\alpha\theta)^{\cdot} - \widehat{\mathbb{A}}_\alpha = \left[\frac{1}{2}(\mu_0 + 3p_0) - \Lambda - \mathcal{A}_0(\mathcal{A}_0 + \phi_0)\right] a_\alpha - \frac{1}{2}(\mathfrak{m}_\alpha + 3\mathfrak{p}_\alpha) + \mathcal{A}_0\mathbb{F}_\alpha + \left(\frac{3}{2}\phi_0 + 2\mathcal{A}_0\right)\mathbb{A}_\alpha + \delta_\alpha(\delta_\mu\mathcal{A}^\mu), \tag{C1}$$

$$\frac{2}{3}(\delta_\alpha\theta)^{\cdot} - (\delta_\alpha\Sigma)^{\cdot} = \phi_0\mathbb{A}_\alpha + \mathcal{A}_0(\mathbb{F}_\alpha + \phi_0\mathcal{A}_\alpha) - \left(\frac{1}{3}\mu_0 + p_0 - \frac{2}{3}\Lambda + \frac{1}{2}\Pi_0 - \mathcal{E}_0\right)\mathcal{A}_\alpha - \frac{1}{3}\mathfrak{m}_\alpha - \mathfrak{p}_\alpha - \frac{1}{2}\mathbb{P}_\alpha + \mathbb{E}_\alpha + \delta_\alpha(\delta_\mu\mathcal{A}^\mu), \tag{C2}$$

$$\frac{3}{2}\widehat{\delta_\alpha\Sigma} - \widehat{\delta_\alpha\theta} = \left(\mu_0 + 3p_0 - 2\Lambda + \frac{3}{2}\Pi_0 - 3\mathcal{E}_0 - 3\mathcal{A}_0\phi_0\right)\varepsilon_{\alpha\mu}\Omega^\mu + \frac{1}{2}\phi_0(\delta_\alpha\theta - 6\delta_\alpha\Sigma) - \frac{3}{2}\delta_\alpha\mathcal{Q} - \frac{3}{2}\delta_\alpha(\delta_\mu\Sigma^\mu) - \frac{3}{2}\varepsilon_{\mu\nu}\delta_\alpha(\delta^\mu\Omega^\nu), \tag{C3}$$

$$\dot{\Omega} = \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu\mathcal{A}_\nu + \mathcal{A}_0\xi, \tag{C4}$$

$$\hat{\Omega} = (\mathcal{A}_0 - \phi_0)\Omega - \delta_\mu\Omega^\mu; \tag{C5}$$

- Evolution and propagation equations for vector quantities:

$$\dot{\Omega}_\alpha + \frac{1}{2}\varepsilon_\alpha{}^\mu\widehat{\mathcal{A}}_\mu = -\frac{1}{2}\varepsilon_\alpha{}^\mu\left(\frac{1}{2}\phi_0\mathcal{A}_\mu + \mathcal{A}_0 a_\mu - \mathbb{A}_\mu\right), \tag{C6}$$

$$\dot{\Sigma}_\alpha - \frac{1}{2}\widehat{\mathcal{A}}_\alpha = \left(\mathcal{A}_0 - \frac{1}{4}\phi_0\right)\mathcal{A}_\alpha + \frac{1}{2}\mathcal{A}_0 a_\alpha + \frac{1}{2}\mathbb{A}_\alpha - \mathcal{E}_\alpha + \frac{1}{2}\Pi_\alpha, \tag{C7}$$

$$\widehat{\Sigma}_\alpha - \varepsilon_\alpha{}^\mu\widehat{\Omega}_\mu = \frac{1}{2}\delta_\alpha\Sigma + \frac{2}{3}\delta_\alpha\theta - \frac{3}{2}\phi_0\Sigma_\alpha + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\varepsilon_\alpha{}^\mu\Omega_\mu - \varepsilon_\alpha{}^\mu\delta_\mu\Omega - \delta_\mu\Sigma_\alpha{}^\mu - \mathcal{Q}_\alpha; \tag{C8}$$

- Evolution and propagation equations for tensor quantities:

$$\dot{\Sigma}_{\alpha\beta} = \delta_{\{\alpha}\mathcal{A}_{\beta\}} + \mathcal{A}_0\zeta_{\alpha\beta} - \mathcal{E}_{\alpha\beta} + \frac{1}{2}\Pi_{\alpha\beta}, \tag{C9}$$

$$\widehat{\Sigma}_{\alpha\beta} = \delta_{\{\alpha}\Sigma_{\beta\}} + \frac{1}{2}\varepsilon_{\{\alpha|}{}^\mu\delta_\mu\Omega_{|\beta\}} + \frac{1}{2}\varepsilon_{\{\alpha|}{}^\mu\delta_{|\beta\}}\Omega_\mu - \frac{1}{2}\phi_0\Sigma_{\alpha\beta} - \varepsilon_{\mu\{\alpha}\mathcal{H}_{\beta\}}{}^\mu; \tag{C10}$$

- Constraint equations:

$$\varepsilon^{\mu\nu}\delta_\mu\mathbb{A}_\nu = -2\widehat{\mathcal{A}}_0\xi, \tag{C11}$$

$$\delta_\mu\Omega^\mu + \varepsilon^{\mu\nu}\delta_\mu\Sigma_\nu = (2\mathcal{A}_0 - \phi_0)\Omega + \mathcal{H}, \tag{C12}$$

$$\varepsilon_\alpha{}^\mu\delta_\mu\Omega + \delta_\mu\Sigma_\alpha{}^\mu = \frac{1}{2}\phi_0(\varepsilon_\alpha{}^\mu\Omega_\mu - \Sigma_\alpha) - \varepsilon_\alpha{}^\mu\mathcal{H}_\mu + \frac{1}{3}\delta_\alpha\theta - \frac{1}{2}\delta_\alpha\Sigma - \frac{1}{2}\mathcal{Q}_\alpha. \tag{C13}$$

C.2. Linearized equations for the kinematical quantities associated with the spacelike congruence

- Evolution and propagation equations for scalars and gradients of scalar quantities:

$$\begin{aligned}\dot{\mathbb{F}}_\alpha &= \left[\frac{1}{2}\phi_0^2 + \frac{2}{3}(\mu_0+\Lambda) + \frac{1}{2}\Pi_0 + \mathcal{E}_0\right](\Sigma_\alpha + \alpha_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu) \\ &\quad + \delta_\alpha \mathcal{Q} + \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\left(\frac{2}{3}\delta_\alpha\theta - \delta_\alpha\Sigma\right) + \delta_\alpha(\delta_\mu\alpha^\mu)\,,\end{aligned} \tag{C14}$$

$$\begin{aligned}\widehat{\mathbb{F}}_\alpha &= \left[\frac{1}{2}\phi_0^2 + \frac{2}{3}(\mu_0+\Lambda) + \frac{1}{2}\Pi_0 + \mathcal{E}_0\right]a_\alpha + \delta_\alpha(\delta_\mu a^\mu) \\ &\quad - \frac{2}{3}\mathfrak{m}_\alpha - \frac{1}{2}\mathbb{P}_\alpha - \mathbb{E}_\alpha - \frac{3}{2}\phi_0\mathbb{F}_\alpha\,,\end{aligned} \tag{C15}$$

$$\dot{\xi} = \frac{1}{2}\mathcal{H} + \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Omega + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu\alpha_\nu\,, \tag{C16}$$

$$\widehat{\xi} = -\phi_0\xi + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu a_\nu\,; \tag{C17}$$

- Evolution and propagation equations for vector quantities:

$$\widehat{\alpha}_\alpha - \dot{a}_\alpha = \varepsilon_{\mu\alpha}\mathcal{H}^\mu - \left(\mathcal{A}_0 + \frac{1}{2}\phi_0\right)\alpha_\alpha + \frac{1}{2}\mathcal{Q}_\alpha + \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)(\Sigma_\alpha + \varepsilon_{\alpha\mu}\Omega^\mu)\,; \tag{C18}$$

- Evolution and propagation equations for tensor quantities:

$$\dot{\zeta}_{\alpha\beta} = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Sigma_{\alpha\beta} + \delta_{\{\alpha}\alpha_{\beta\}} - \varepsilon^\mu{}_{\{\alpha}\mathcal{H}_{\beta\}\mu}\,, \tag{C19}$$

$$\widehat{\zeta}_{\alpha\beta} = \delta_{\{\alpha}a_{\beta\}} - \phi_0\zeta_{\alpha\beta} - \mathcal{E}_{\alpha\beta} - \frac{1}{2}\Pi_{\alpha\beta}\,; \tag{C20}$$

- Constraint equations:

$$\varepsilon^{\mu\nu}\delta_\mu\mathbb{F}_\nu = \left[\phi_0^2 + \frac{4}{3}(\mu_0+\Lambda) + \Pi_0 + 2\mathcal{E}_0\right]\xi\,, \tag{C21}$$

$$\delta_\mu\zeta_\alpha{}^\mu + \varepsilon_\alpha{}^\mu\delta_\mu\xi = \frac{1}{2}\mathbb{F}_\alpha + \mathcal{E}_\alpha + \frac{1}{2}\Pi_\alpha\,. \tag{C22}$$

C.3. Linearized equations for the Weyl tensor components and the matter variables

- Evolution and propagation equations for scalars and gradients of scalar quantities:

$$\begin{aligned}\widehat{\delta_\alpha\mathcal{Q}} + \dot{\mathfrak{m}}_\alpha &= \widehat{\mu}_0(\varepsilon_{\alpha\mu}\Omega^\mu - \Sigma_\alpha - \alpha_\alpha) - \frac{3}{2}\Pi_0\delta_\alpha\Sigma - \delta_\alpha(\delta_\mu\mathcal{Q}^\mu) \\ &\quad - (\mu_0 + p_0)\delta_\alpha\theta - \left(\frac{3}{2}\phi_0 + 2\mathcal{A}_0\right)\delta_\alpha\mathcal{Q}\,,\end{aligned} \tag{C23}$$

$$\begin{aligned}\frac{1}{3}\dot{\mathfrak{m}}_\alpha - \frac{1}{2}\dot{\mathbb{P}}_\alpha - \dot{\mathbb{E}}_\alpha &= \frac{3}{2}\phi_0\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)(\varepsilon_{\alpha\mu}\Omega^\mu - \Sigma_\alpha - \alpha_\alpha) - \frac{1}{2}\phi_0\delta_\alpha\mathcal{Q} \\ &\quad + \left(\mu_0 + p_0 - \frac{1}{2}\Pi_0 - 3\mathcal{E}_0\right)\left(\frac{1}{2}\delta_\alpha\Sigma - \frac{1}{3}\delta_\alpha\theta\right) \\ &\quad - \frac{1}{2}\delta_\alpha(\delta_\mu\mathcal{Q}^\mu) - \varepsilon^{\mu\nu}\delta_\alpha(\delta_\mu\mathcal{H}_\nu)\,,\end{aligned} \tag{C24}$$

$$
\begin{aligned}
(\delta_\alpha Q)^{\cdot} + \widehat{\mathbb{P}}_\alpha + \widehat{\mathfrak{p}}_\alpha = {} & (\mu_0 + p_0)(\mathcal{A}_0 a_\alpha - \mathbb{A}_\alpha) - \mathcal{A}_0 (\mathfrak{m}_\alpha + \mathfrak{p}_\alpha + \mathbb{P}_\alpha) \\
& - \frac{1}{2}\phi_0 (\mathfrak{p}_\alpha + 4\mathbb{P}_\alpha) + \left(\frac{3}{2}\phi_0 + \mathcal{A}_0\right)\Pi_0 a_\alpha \\
& - \Pi_0 \left(\frac{3}{2}\mathbb{F}_\alpha + \mathbb{A}_\alpha\right) - \delta_\alpha (\delta_\mu \Pi^\mu) ,
\end{aligned} \tag{C25}
$$

$$
\begin{aligned}
\frac{3}{2}\widehat{\mathbb{P}}_\alpha + 3\widehat{\mathbb{E}}_\alpha - \widehat{\mathfrak{m}}_\alpha = {} & -6\phi_0 \left(\mathbb{E}_\alpha + \frac{1}{2}\mathbb{P}_\alpha - \frac{1}{12}\mathfrak{m}_\alpha\right) - \frac{9}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right) \\
& \times (\mathbb{F}_\alpha - \phi_0 a_\alpha) - 3\delta_\alpha (\delta_\mu \mathcal{E}^\mu) - \frac{3}{2}\delta_\alpha (\delta_\mu \Pi^\mu) ,
\end{aligned} \tag{C26}
$$

$$
\dot{\mathcal{H}} = \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu \Pi_\nu - \varepsilon^{\mu\nu}\delta_\mu \mathcal{E}_\nu - 3\left(\mathcal{E}_0 - \frac{1}{2}\Pi_0\right)\xi , \tag{C27}
$$

$$
\widehat{\mathcal{H}}* = -\delta_\mu \mathcal{H}^\mu - \frac{1}{2}\varepsilon_{\mu\nu}\delta^\mu Q^\nu - \frac{3}{2}\phi_0 \mathcal{H} - \left(3\mathcal{E}_0 + \mu_0 + p_0 - \frac{1}{2}\Pi_0\right)\Omega ; \tag{C28}
$$

- Evolution and propagation equations for vector quantities:

$$
\begin{aligned}
\dot{\mathcal{E}}_\alpha + \frac{1}{2}\dot{\Pi}_\alpha = {} & \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\frac{1}{2}Q_\alpha + \varepsilon_{\alpha\mu}\mathcal{H}^\mu\right) - \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\alpha_\alpha \\
& - \frac{1}{2}(\mu_0 + p_0 + \Pi_0)(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu) - \frac{1}{2}\delta_\alpha Q \\
& + \frac{1}{2}\varepsilon_\alpha{}^\mu \delta_\mu \mathcal{H} + \varepsilon^{\mu\nu}\delta_\mu \mathcal{H}_{\nu\alpha} ,
\end{aligned} \tag{C29}
$$

$$
\begin{aligned}
\widehat{\mathcal{E}}_\alpha + \frac{1}{2}\widehat{\Pi}_\alpha = {} & -\frac{3}{2}\phi_0 \left(\mathcal{E}_\alpha + \frac{1}{2}\Pi_\alpha\right) - \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)a_\alpha - \frac{1}{2}\delta_\mu \Pi_\alpha{}^\mu \\
& + \frac{1}{2}\mathbb{E}_\alpha + \frac{1}{3}\mathfrak{m}_\alpha + \frac{1}{4}\mathbb{P}_\alpha - \delta_\mu \mathcal{E}_\alpha{}^\mu ,
\end{aligned} \tag{C30}
$$

$$
\begin{aligned}
\frac{1}{2}\varepsilon_\alpha{}^\mu \widehat{\mathcal{E}}_\mu - \dot{\mathcal{H}}_\alpha - \frac{1}{4}\varepsilon_\alpha{}^\mu \widehat{\Pi}_\mu = {} & \frac{3}{4}\varepsilon_\alpha{}^\mu \mathbb{E}_\mu - \frac{3}{8}\varepsilon_\alpha{}^\mu \mathbb{P}_\mu + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu \mathcal{E}_{\nu\alpha} - \frac{1}{4}\varepsilon^{\mu\nu}\delta_\mu \Pi_{\nu\alpha} \\
& + \frac{3}{2}\mathcal{E}_0 \varepsilon_{\alpha\mu}\mathcal{A}^\mu - \frac{3}{4}\left(\mathcal{E}_0 - \frac{1}{2}\Pi_0\right)\varepsilon_{\alpha\mu}a^\mu \\
& - \mathcal{A}_0 \varepsilon_{\alpha\mu}\mathcal{E}^\mu - \frac{1}{4}\phi_0 \varepsilon_{\alpha\mu}\left(\mathcal{E}^\mu - \frac{1}{2}\Pi^\mu\right) ,
\end{aligned} \tag{C31}
$$

$$
\begin{aligned}
\widehat{\mathcal{H}}_\alpha - \frac{1}{2}\varepsilon_\alpha{}^\mu \widehat{Q}_\mu = {} & \frac{1}{2}\delta_\alpha \mathcal{H} - \delta_\mu \mathcal{H}_\alpha{}^\mu - \frac{1}{2}\varepsilon_\alpha{}^\mu \delta_\mu Q - \frac{3}{2}\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\varepsilon_{\alpha\mu}\Sigma^\mu \\
& - \left(\mu_0 + p_0 + \frac{1}{4}\Pi_0 - \frac{3}{2}\mathcal{E}_0\right)\Omega_\alpha - \frac{3}{2}\phi_0 \left(\mathcal{H}_\alpha - \frac{1}{6}\varepsilon_{\alpha\mu}Q^\mu\right) ,
\end{aligned} \tag{C32}
$$

$$
\begin{aligned}
\dot{Q}_\alpha + \widehat{\Pi}_\alpha = {} & \frac{1}{2}\mathbb{P}_\alpha - \mathfrak{p}_\alpha - \frac{3}{2}\Pi_0 a_\alpha - \left(\frac{3}{2}\phi_0 + \mathcal{A}_0\right)\Pi_\alpha \\
& - \delta_\mu \Pi_\alpha{}^\mu - \left(\mu_0 + p_0 - \frac{1}{2}\Pi_0\right)\mathcal{A}_\alpha ;
\end{aligned} \tag{C33}
$$

- Evolution and propagation equations for tensor quantities:

$$
\begin{aligned}
\dot{\mathcal{E}}_{\alpha\beta} + \frac{1}{2}\dot{\Pi}_{\alpha\beta} + N^\mu{}_{\{\alpha}\varepsilon_{\beta\}}{}^\nu \widehat{\mathcal{H}}_{\mu\nu} = {} & \varepsilon_{\{\alpha|}{}^\nu \delta_\nu \mathcal{H}_{|\beta\}} - \frac{1}{2}\delta_{\{\alpha}Q_{\beta\}} + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\varepsilon_{\mu\{\alpha}\mathcal{H}_{\beta\}}{}^\mu \\
& - \frac{1}{2}\left(\mu_0 + p_0 + 3\mathcal{E}_0 - \frac{1}{2}\Pi_0\right)\Sigma_{\alpha\beta} ,
\end{aligned} \tag{C34}
$$

$$N^{\mu}{}_{\{\alpha\varepsilon\beta\}}{}^{\nu}\widehat{\mathcal{E}}_{\mu\nu} - \frac{1}{2}N^{\mu}{}_{\{\alpha\varepsilon\beta\}}{}^{\nu}\widehat{\Pi}_{\mu\nu} - \dot{\mathcal{H}}_{\alpha\beta} = \varepsilon_{\{\alpha|}{}^{\mu}\delta_{\mu}\mathcal{E}_{|\beta\}} - \frac{1}{2}\varepsilon_{\{\alpha|}{}^{\mu}\delta_{\mu}\Pi_{|\beta\}} - \frac{3}{2}\left(\mathcal{E}_0 - \frac{1}{2}\Pi_0\right) \times \varepsilon_{\mu\{\alpha}\zeta_{\beta\}}{}^{\mu} + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\varepsilon_{\mu\{\alpha}\mathcal{E}_{\beta\}}{}^{\mu} - \frac{1}{4}\phi_0\varepsilon_{\mu\{\alpha}\Pi_{\beta\}}{}^{\mu}\,; \quad \text{(C35)}$$

- Constraint equations:

$$\frac{1}{3}\varepsilon^{\mu\nu}\delta_{\mu}\mathfrak{m}_{\nu} - \varepsilon^{\mu\nu}\delta_{\mu}\mathbb{E}_{\nu} - \frac{1}{2}\varepsilon^{\mu\nu}\delta_{\mu}\mathbb{P}_{\nu} = -3\phi_0\left(\mathcal{E}_0 + \frac{1}{2}\Pi_0\right)\xi\,, \quad \text{(C36)}$$

$$\varepsilon^{\mu\nu}\delta_{\mu}\mathfrak{p}_{\nu} = -2\widehat{p}_0\xi\,, \quad \text{(C37)}$$

Appendix D. Linearized 1+1+2 equations for the dot-derivatives variables

In this appendix, we present a set of covariant, gauge invariant equations for the linear order perturbations of a background spacetime assumed to be static LRS II and permeated by a general matter fluid, such that the equilibrium configuration is characterized by the scalars $\{\phi_0, \mathcal{A}_0, \mathcal{E}_0, \mu_0, p_0, \Pi_0, \Lambda\}$. Considering the variables in (26) and using the commutation relations (17), we find the following set of linearized equations for the various quantities that characterize the perturbed spacetime. Here, we also show that the variables in equations (25) and (26) are not fully independent.

D.1. Linearized equations for the kinematical quantities associated with the timelike congruence

- Evolution and propagation equations for scalars and dot-derivatives of scalar quantities:

$$\widehat{\mathsf{A}} - \ddot{\theta} = \frac{1}{2}(\mathsf{m} + 3\mathsf{p}) + \widehat{\mathcal{A}}_0\left(\frac{1}{3}\theta + \Sigma\right) - (3\mathcal{A}_0 + \phi_0)\,\mathsf{A} - \mathcal{A}_0\mathsf{F} - (\delta_{\mu}\mathcal{A}^{\mu})^{\cdot}\,, \quad \text{(D1)}$$

$$\ddot{\Sigma} - \frac{2}{3}\ddot{\theta} = \frac{1}{3}(\mathsf{m} + 3\mathsf{p}) + \frac{1}{2}\mathcal{P} - \mathsf{E} - \mathcal{A}_0\mathsf{F} - \phi_0\mathsf{A} - (\delta_{\mu}\mathcal{A}^{\mu})^{\cdot}\,, \quad \text{(D2)}$$

$$\dot{\Omega} = \frac{1}{2}\varepsilon^{\mu\nu}\delta_{\mu}\mathcal{A}_{\nu} + \mathcal{A}_0\xi\,, \quad \text{(D3)}$$

$$\widehat{\Omega} = (\mathcal{A}_0 - \phi_0)\,\Omega - \delta_{\mu}\Omega^{\mu}\,, \quad \text{(D4)}$$

$$\frac{2}{3}\widehat{\theta} - \widehat{\Sigma} = \mathcal{Q} + \frac{3}{2}\phi_0\Sigma + \delta_{\mu}\Sigma^{\mu} + \varepsilon_{\mu\nu}\delta^{\mu}\Omega^{\nu}\,; \quad \text{(D5)}$$

- Evolution and propagation equations for vector quantities:

$$\dot{\Omega}_{\alpha} + \frac{1}{2}\varepsilon_{\alpha}{}^{\mu}\widehat{\mathcal{A}}_{\mu} = -\frac{1}{2}\varepsilon_{\alpha}{}^{\mu}\left(\frac{1}{2}\phi_0\mathcal{A}_{\mu} + \mathcal{A}_0 a_{\mu} - \mathbb{A}_{\mu}\right), \quad \text{(D6)}$$

$$\dot{\Sigma}_{\alpha} - \frac{1}{2}N_{\alpha}{}^{\mu}\widehat{\mathcal{A}}_{\alpha} = \frac{1}{2}\mathbb{A}_{\alpha} + \left(\mathcal{A}_0 - \frac{1}{4}\phi_0\right)\mathcal{A}_{\alpha} + \frac{1}{2}\mathcal{A}_0 a_{\alpha} - \mathcal{E}_{\alpha} + \frac{1}{2}\Pi_{\alpha}\,, \quad \text{(D7)}$$

$$\hat{\Sigma}_\alpha - \varepsilon_\alpha{}^\mu \widehat{\Omega}_\mu = \frac{1}{2}\delta_\alpha \Sigma + \frac{2}{3}\delta_\alpha \theta - \varepsilon_\alpha{}^\mu \delta_\mu \Omega + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\varepsilon_\alpha{}^\mu \Omega_\mu - \frac{3}{2}\phi_0 \Sigma_\alpha - \delta_\mu \Sigma_\alpha{}^\mu - Q_\alpha ; \tag{D8}$$

- Evolution and propagation equations for tensor quantities:

$$\dot{\Sigma}_{\alpha\beta} = \delta_{\{\alpha}\mathcal{A}_{\beta\}} + \mathcal{A}_0 \zeta_{\alpha\beta} - \mathcal{E}_{\alpha\beta} + \frac{1}{2}\Pi_{\alpha\beta} , \tag{D9}$$

$$\widehat{\Sigma}_{\alpha\beta} = \delta_{\{\alpha}\Sigma_{\beta\}} + \frac{1}{2}\varepsilon_{\{\alpha|}{}^\mu \delta_\mu \Omega_{|\beta\}} + \frac{1}{2}\varepsilon_{\{\alpha|}{}^\mu \delta_{|\beta\}}\Omega_\mu - \frac{1}{2}\phi_0 \Sigma_{\alpha\beta} - \varepsilon_{\mu\{\alpha}\mathcal{H}_{\beta\}}{}^\mu ; \tag{D10}$$

- Constraint equations:

$$\delta_\mu \Omega^\mu + \varepsilon^{\mu\nu}\delta_\mu \Sigma_\nu = (2\mathcal{A}_0 - \phi_0)\,\Omega + \mathcal{H} , \tag{D11}$$

$$\delta_\alpha \Sigma - \frac{2}{3}\delta_\alpha \theta + 2\varepsilon_\alpha{}^\mu \delta_\mu \Omega + 2\delta_\mu \Sigma_\alpha{}^\mu = \phi_0\left(\varepsilon_\alpha{}^\mu \Omega_\mu - \Sigma_\alpha\right) - 2\varepsilon_\alpha{}^\mu \mathcal{H}_\mu - Q_\alpha . \tag{D12}$$

D.2. Linearized equations for the kinematical quantities associated with the spacelike congruence

- Evolution and propagation equations for scalars and dot-derivatives of scalar quantities:

$$\mathsf{F} = Q + (2\mathcal{A}_0 - \phi_0)\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) + \delta_\mu \alpha^\mu , \tag{D13}$$

$$\widehat{\mathsf{F}} = \hat{\phi}_0\left(\frac{1}{3}\theta + \Sigma\right) - (\mathcal{A}_0 + \phi_0)\,\mathsf{F} - \frac{2}{3}\mathsf{m} - \frac{1}{2}\mathcal{P} - \mathsf{E} + (\delta_\mu a^\mu)^{\cdot} , \tag{D14}$$

$$\dot{\xi} = \frac{1}{2}\mathcal{H} + \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Omega + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu \alpha_\nu , \tag{D15}$$

$$\widehat{\xi} = -\phi_0 \xi + \frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu a_\nu ; \tag{D16}$$

- Evolution and propagation equations for vector quantities:

$$\widehat{\alpha}_\alpha - \dot{a}_\alpha = \varepsilon_{\mu\alpha}\mathcal{H}^\mu - \left(\mathcal{A}_0 + \frac{1}{2}\phi_0\right)\alpha_\alpha + \frac{1}{2}Q_\alpha + \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\Sigma_\alpha + \varepsilon_{\alpha\mu}\Omega^\mu\right) ; \tag{D17}$$

- Evolution and propagation equations for tensor quantities:

$$\dot{\zeta}_{\alpha\beta} = \left(\mathcal{A}_0 - \frac{1}{2}\phi_0\right)\Sigma_{\alpha\beta} + \delta_{\{\alpha}\alpha_{\beta\}} - \varepsilon^\mu{}_{\{\alpha}\mathcal{H}_{\beta\}\mu} , \tag{D18}$$

$$\widehat{\zeta}_{\alpha\beta} = \delta_{\{\alpha}a_{\beta\}} - \phi_0 \zeta_{\alpha\beta} - \mathcal{E}_{\alpha\beta} - \frac{1}{2}\Pi_{\alpha\beta} ; \tag{D19}$$

- Constraint equation:

$$\delta_\mu \zeta_\alpha{}^\mu + \varepsilon_\alpha{}^\mu \delta_\mu \xi - \frac{1}{2}\mathbb{F}_\alpha = \mathcal{E}_\alpha + \frac{1}{2}\Pi_\alpha . \tag{D20}$$

D.3. Linearized equations for the Weyl tensor components and the matter variables

- Evolution and propagation equations for scalars and dot-derivatives of scalar quantities:

$$\begin{aligned}\mathsf{E}+\frac{1}{2}\mathcal{P}+\frac{1}{3}\widehat{\mathcal{Q}}=&\ \mathcal{E}_0\left(\frac{3}{2}\Sigma-\theta\right)-\frac{1}{2}\Pi_0\left(\frac{1}{3}\theta+\frac{1}{2}\Sigma\right)+\frac{1}{3}\left(\frac{1}{2}\phi_0-2\mathcal{A}_0\right)\mathcal{Q}\\&-\frac{1}{2}\left(\mu_0+p_0\right)\Sigma+\frac{1}{6}\delta_\mu\mathcal{Q}^\mu+\varepsilon^{\mu\nu}\delta_\mu\mathcal{H}_\nu\,,\end{aligned} \tag{D21}$$

$$\begin{aligned}\widehat{\mathsf{E}}-\frac{1}{3}\widehat{\mathsf{m}}+\frac{1}{2}\widehat{\mathcal{P}}=&\left(\widehat{\mathcal{E}}_0-\frac{1}{3}\hat{\mu}_0+\frac{1}{2}\widehat{\Pi}_0\right)\left(\frac{1}{3}\theta+\Sigma\right)-\left(\delta_\mu\mathcal{E}^\mu\right)^\cdot-\frac{1}{2}\left(\delta_\mu\Pi^\mu\right)^\cdot\\&+\frac{1}{3}\mathcal{A}_0\mathsf{m}-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\mathsf{F}-\left(\mathcal{A}_0+\frac{3}{2}\phi_0\right)\left(\mathsf{E}+\frac{1}{2}\mathcal{P}\right),\end{aligned} \tag{D22}$$

$$\dot{\mathcal{H}}=\frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu\Pi_\nu-\varepsilon^{\mu\nu}\delta_\mu\mathcal{E}_\nu-3\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)\xi\,, \tag{D23}$$

$$\widehat{\mathcal{H}}=-\delta_\mu\mathcal{H}^\mu-\frac{1}{2}\varepsilon_{\mu\nu}\delta^\mu\mathcal{Q}^\nu-\frac{3}{2}\phi_0\mathcal{H}-\left(3\mathcal{E}_0+\mu_0+p_0-\frac{1}{2}\Pi_0\right)\Omega\,, \tag{D24}$$

$$\mathsf{m}+\widehat{\mathcal{Q}}=-\left(\phi_0+2\mathcal{A}_0\right)\mathcal{Q}-\frac{3}{2}\Pi_0\Sigma-\left(\mu_0+p_0\right)\theta-\delta_\mu\mathcal{Q}^\mu\,, \tag{D25}$$

$$\begin{aligned}\ddot{\mathcal{Q}}+\widehat{\mathsf{p}}+\widehat{\mathcal{P}}=&\left(\widehat{p}_0+\widehat{\Pi}_0\right)\left(\frac{1}{3}\theta+\Sigma\right)-\Pi_0\left(\frac{3}{2}\mathsf{F}+\mathsf{A}\right)-\left(\frac{3}{2}\phi_0+2\mathcal{A}_0\right)\mathcal{P}\\&-\mathcal{A}_0\left(\mathsf{m}+2\mathsf{p}\right)-\left(\mu_0+p_0\right)\mathsf{A}-\left(\delta_\mu\Pi^\mu\right)^\cdot\,;\end{aligned} \tag{D26}$$

- Evolution and propagation equations for vector quantities:

$$\begin{aligned}\dot{\mathcal{E}}_\alpha+\frac{1}{2}\dot{\Pi}_\alpha=&\left(\frac{1}{2}\phi_0-\mathcal{A}_0\right)\left(\frac{1}{2}\mathcal{Q}_\alpha+\varepsilon_{\alpha\mu}\mathcal{H}^\mu\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\alpha_\alpha\\&-\frac{1}{2}\left(\mu_0+p_0+\Pi_0\right)\left(\Sigma_\alpha-\varepsilon_{\alpha\mu}\Omega^\mu\right)\\&-\frac{1}{2}\delta_\alpha\mathcal{Q}+\frac{1}{2}\varepsilon_\alpha{}^\mu\delta_\mu\mathcal{H}+\varepsilon^{\mu\nu}\delta_\mu\mathcal{H}_{\nu\alpha}\,,\end{aligned} \tag{D27}$$

$$\begin{aligned}\widehat{\mathcal{E}}_\alpha+\frac{1}{2}\widehat{\Pi}_\alpha=&-\frac{3}{2}\phi_0\left(\mathcal{E}_\alpha+\frac{1}{2}\Pi_\alpha\right)-\frac{3}{2}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)a_\alpha\\&+\frac{1}{2}\mathbb{E}_\alpha+\frac{1}{3}\mathfrak{m}_\alpha+\frac{1}{4}\mathbb{P}_\alpha-\delta_\mu\mathcal{E}_\alpha{}^\mu-\frac{1}{2}\delta_\mu\Pi_\alpha{}^\mu\,,\end{aligned} \tag{D28}$$

$$\begin{aligned}\frac{1}{2}\varepsilon_\alpha{}^\mu\widehat{\mathcal{E}}_\mu-\dot{\mathcal{H}}_\alpha-\frac{1}{4}\varepsilon_\alpha{}^\mu\widehat{\Pi}_\mu=&\ \frac{3}{4}\varepsilon_\alpha{}^\mu\mathbb{E}_\mu-\frac{3}{8}\varepsilon_\alpha{}^\mu\mathbb{P}_\mu+\frac{1}{2}\varepsilon^{\mu\nu}\delta_\mu\mathcal{E}_{\nu\alpha}-\frac{1}{4}\varepsilon^{\mu\nu}\delta_\mu\Pi_{\nu\alpha}\\&+\frac{3}{2}\varepsilon_{\alpha\mu}\mathcal{E}_0\mathcal{A}^\mu-\frac{3}{4}\varepsilon_{\alpha\mu}\left(\mathcal{E}_0-\frac{1}{2}\Pi_0\right)a^\mu-\varepsilon_{\alpha\mu}\mathcal{E}^\mu\mathcal{A}\\&-\frac{1}{4}\varepsilon_{\alpha\mu}\phi_0\left(\mathcal{E}^\mu-\frac{1}{2}\Pi^\mu\right),\end{aligned} \tag{D29}$$

$$\begin{aligned}\widehat{\mathcal{H}}_\alpha-\frac{1}{2}\varepsilon_\alpha{}^\mu\widehat{\mathcal{Q}}_\mu=&\ \frac{1}{2}\delta_\alpha\mathcal{H}-\delta_\mu\mathcal{H}_\alpha{}^\mu-\frac{1}{2}\varepsilon_\alpha{}^\mu\delta_\mu\mathcal{Q}-\frac{3}{2}\varepsilon_{\alpha\mu}\left(\mathcal{E}_0+\frac{1}{2}\Pi_0\right)\Sigma^\mu\\&-\left(\mu_0+p_0+\frac{1}{4}\Pi_0-\frac{3}{2}\mathcal{E}_0\right)\Omega_\alpha-\frac{3}{2}\phi_0\left(\mathcal{H}_\alpha-\frac{1}{6}\varepsilon_{\alpha\mu}\mathcal{Q}^\mu\right),\end{aligned} \tag{D30}$$

$$\dot{\mathcal{Q}}_\alpha+\widehat{\Pi}_\alpha=-\mathfrak{p}_\alpha+\frac{1}{2}\mathbb{P}_\alpha-\delta_\mu\Pi_\alpha{}^\mu-\frac{3}{2}\Pi_0a_\alpha-\left(\frac{3}{2}\phi_0+\mathcal{A}_0\right)\Pi_\alpha-\left(\mu_0+p_0-\frac{1}{2}\Pi_0\right)\mathcal{A}_\alpha\,; \tag{D31}$$

- Evolution and propagation equations for tensor quantities:

$$\dot{\mathcal{E}}_{\alpha\beta} + \frac{1}{2}\dot{\Pi}_{\alpha\beta} + \varepsilon_{\{\alpha|}{}^{\nu}\widehat{\mathcal{H}}_{|\beta\}\nu} = \varepsilon_{\{\alpha|}{}^{\nu}\delta_{\nu}\mathcal{H}_{|\beta\}} - \frac{1}{2}\left(\mu_0 + p_0 + 3\mathcal{E}_0 - \frac{1}{2}\Pi_0\right)\Sigma_{\alpha\beta} + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\varepsilon_{\mu\{\alpha}\mathcal{H}_{\beta\}}{}^{\mu} - \frac{1}{2}\delta_{\{\alpha}\mathcal{Q}_{\beta\}}\,, \tag{D32}$$

$$\varepsilon_{\{\alpha|}{}^{\nu}\widehat{\mathcal{E}}_{|\beta\}\nu} - \frac{1}{2}\varepsilon_{\{\alpha|}{}^{\nu}\widehat{\Pi}_{|\beta\}\nu} - \dot{\mathcal{H}}_{\{\alpha\beta\}} = \varepsilon_{\{\alpha|}{}^{\mu}\delta_{\mu}\mathcal{E}_{|\beta\}} - \frac{1}{2}\varepsilon_{\{\alpha|}{}^{\mu}\delta_{\mu}\Pi_{|\beta\}} - \frac{1}{4}\phi_0\varepsilon_{\mu\{\alpha}\Pi_{\beta\}}{}^{\mu} - \frac{3}{2}\left(\mathcal{E}_0 - \frac{1}{2}\Pi_0\right)\varepsilon_{\mu\{\alpha}\zeta_{\beta\}}{}^{\mu} + \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right)\varepsilon_{\mu\{\alpha}\mathcal{E}_{\beta\}}{}^{\mu}\,. \tag{D33}$$

D.4. Extra equations

The variables $\{\mathfrak{m},\mathfrak{p},\mathbb{P},\mathbb{F},\mathbb{A},\mathbb{E}\}$ and $\{\mathsf{m},\mathsf{p},\mathcal{P},\mathsf{F},\mathsf{A},\mathsf{E}\}$ are not fully independent. Using the commutation relations (17) we find

$$\begin{aligned}
\delta_\alpha\mathsf{m} - \dot{\mathfrak{m}}_\alpha &= \widehat{\mu}_0\left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu + \alpha_\alpha\right),\\
\delta_\alpha\mathsf{p} - \dot{\mathfrak{p}}_\alpha &= \widehat{p}_0\left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu + \alpha_\alpha\right),\\
\delta_\alpha\mathcal{P} - \dot{\mathbb{P}}_\alpha &= \widehat{\Pi}_0\left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu + \alpha_\alpha\right),\\
\delta_\alpha\mathsf{F} - \dot{\mathbb{F}}_\alpha &= \widehat{\phi}_0\left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu + \alpha_\alpha\right),\\
\delta_\alpha\mathsf{A} - \dot{\mathbb{A}}_\alpha &= \widehat{\mathcal{A}}_0\left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu + \alpha_\alpha\right),\\
\delta_\alpha\mathsf{E} - \dot{\mathbb{E}}_\alpha &= \widehat{\mathcal{E}}_0\left(\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu + \alpha_\alpha\right).
\end{aligned} \tag{D34}$$

Moreover, we also have

$$\begin{aligned}
\varepsilon^{\alpha\beta}\delta_\alpha\mathfrak{m}_\beta &= -2\widehat{\mu}_0\xi\,,\\
\varepsilon^{\alpha\beta}\delta_\alpha\mathfrak{p}_\beta &= -2\widehat{p}_0\xi\,,\\
\varepsilon^{\alpha\beta}\delta_\alpha\mathbb{P}_\beta &= -2\widehat{\Pi}_0\xi\,,\\
\varepsilon^{\alpha\beta}\delta_\alpha\mathbb{F}_\beta &= -2\widehat{\phi}_0\xi\,,\\
\varepsilon^{\alpha\beta}\delta_\alpha\mathbb{A}_\beta &= -2\widehat{\mathcal{A}}_0\xi\,,\\
\varepsilon^{\alpha\beta}\delta_\alpha\mathbb{E}_\beta &= -2\widehat{\mathcal{E}}_0\xi\,.
\end{aligned} \tag{D35}$$

Appendix E. Scalar, vector and tensor harmonics

We list some of the properties of the scalar, vector, and tensor eigenfunctions of the covariantly defined Laplace–Beltrami operator on 2-hypersurfaces. For concreteness, we consider a locally rotationally symmetric of class II spacetime, such that it admits the existence of spatial sections that can be described by 2-hypersurfaces.

E.1. Scalar harmonics

Let $\mathcal{Q}^{(k)}$ represent the scalar eigenfunctions of the covariantly defined Laplace–Beltrami operator $\delta^2 \equiv \delta^\mu \delta_\mu$, where the δ operator was defined in equation (A1), such that

$$\begin{aligned} \delta^2 \mathcal{Q}^{(k)} &= -\frac{k^2}{r^2}\mathcal{Q}^{(k)}\,, \\ \widehat{\mathcal{Q}}^{(k)} &= \dot{\mathcal{Q}}^{(k)} = 0\,, \end{aligned} \tag{E1}$$

where we have assumed that the harmonic index verifies $k^2 \geqslant 0$ and the function r is covariantly defined as

$$\begin{aligned} \frac{\hat{r}}{r} &= \frac{1}{2}\phi\,, \\ \frac{\dot{r}}{r} &= \frac{1}{3}\theta - \frac{1}{2}\Sigma\,, \\ \delta_\alpha r &= 0\,, \end{aligned} \tag{E2}$$

or, using the 1+1+2 equations in appendix B in the particular case of an LRS II spacetime, we have:

$$\frac{1}{r^2} = \frac{\phi^2}{4} - \mathcal{E} + \frac{1}{3}(\mu+\Lambda) - \frac{\Pi}{2} - \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)^2. \tag{E3}$$

E.2. Vector harmonics

From the scalar harmonics $\mathcal{Q}^{(k)}$, we may define a complete set of 1-tensors that can be used as a basis for sufficiently smooth 1-tensor fields defined on the 2-hypersurfaces. The vector harmonics are defined as

$$\begin{aligned} \mathcal{Q}^{(k)}_\alpha &= r\delta_\alpha \mathcal{Q}^{(k)}\,, \\ \bar{\mathcal{Q}}^{(k)}_\alpha &= r\varepsilon_{\alpha\mu}\delta^\mu \mathcal{Q}^{(k)}\,, \end{aligned} \tag{E4}$$

where $\mathcal{Q}^{(k)}_\alpha$ are referred as the 'even' vector harmonics and $\bar{\mathcal{Q}}^{(k)}_\alpha$ as the 'odd' vector harmonics. These have the following properties

$$\begin{aligned} u^\mu \nabla_\mu \mathcal{Q}^{(k)}_\alpha &= 0\,, \\ e^\mu D_\mu \mathcal{Q}^{(k)}_\alpha &= 0\,, \\ \delta^\mu \mathcal{Q}^{(k)}_\mu &= -\frac{k^2}{r}\mathcal{Q}^{(k)}\,, \\ \varepsilon^{\mu\nu}\delta_\mu \mathcal{Q}^{(k)}_\nu &= 0\,, \\ \delta^2 \mathcal{Q}^{(k)}_\alpha &= \frac{1-k^2}{r^2}\mathcal{Q}^{(k)}_\alpha\,; \end{aligned} \tag{E5}$$

and

$$\begin{aligned}u^{\mu}\nabla_{\mu}\bar{\mathcal{Q}}^{(k)}_{\alpha} &= 0\,,\\ e^{\mu}D_{\mu}\bar{\mathcal{Q}}^{(k)}_{\alpha} &= 0\,,\\ \delta^{\mu}\bar{\mathcal{Q}}^{(k)}_{\mu} &= 0\,,\\ \varepsilon^{\mu\nu}\delta_{\mu}\bar{\mathcal{Q}}^{(k)}_{\nu} &= \frac{k^2}{r}\mathcal{Q}^{(k)}\,,\\ \delta^{2}\bar{\mathcal{Q}}^{(k)}_{\alpha} &= \frac{1-k^2}{r^2}\bar{\mathcal{Q}}^{(k)}_{\alpha}\,.\end{aligned} \tag{E6}$$

Moreover, we have

$$\begin{aligned}\bar{\mathcal{Q}}^{(k)}_{\alpha} &= \varepsilon_{\alpha}{}^{\mu}\mathcal{Q}^{(k)}_{\mu}\,,\\ \mathcal{Q}^{(k)}_{\alpha} &= -\varepsilon_{\alpha}{}^{\mu}\bar{\mathcal{Q}}^{(k)}_{\mu}\,,\end{aligned} \tag{E7}$$

which can be used to readily verify $N^{\mu\nu}\mathcal{Q}^{(k)}_{\mu}\bar{\mathcal{Q}}^{(k)}_{\nu}=0$, valid for each value of k.

E.3. Tensor harmonics

The scalar and vector harmonics introduced in the previous subsection can themselves be used to define a complete set of symmetric, traceless 2-tensors that form a basis for sufficiently smooth, symmetric, traceless 2-tensor fields defined on the 2-hypersurfaces. The tensor harmonics are defined as

$$\begin{aligned}\mathcal{Q}^{(k)}_{\alpha\beta} &= r^2\delta_{\{\alpha}\delta_{\beta\}}\mathcal{Q}^{(k)}\,,\\ \bar{\mathcal{Q}}^{(k)}_{\alpha\beta} &= r^2\varepsilon_{\mu\{\alpha|}\delta^{\mu}\delta_{|\beta\}}\mathcal{Q}^{(k)}\,,\end{aligned} \tag{E8}$$

where, similarly to the vector harmonics above, $\mathcal{Q}^{(k)}_{\alpha\beta}$ are referred as the 'even' tensor harmonics and $\bar{\mathcal{Q}}^{(k)}_{\alpha\beta}$ are referred to as the 'odd' tensor harmonics. These have the following properties

$$\begin{aligned}u^{\mu}\nabla_{\mu}\mathcal{Q}^{(k)}_{\alpha\beta} &= 0\,,\\ e^{\mu}D_{\mu}\mathcal{Q}^{(k)}_{\alpha\beta} &= 0\,,\\ \delta^{\beta}\mathcal{Q}^{(k)}_{\alpha\beta} &= \frac{2-k^2}{2r}\mathcal{Q}^{(k)}_{\alpha}\,,\\ \varepsilon^{\mu\nu}\delta_{\mu}\mathcal{Q}^{(k)}_{\nu\alpha} &= \frac{2-k^2}{2r}\bar{\mathcal{Q}}^{(k)}_{\alpha}\,;\end{aligned} \tag{E9}$$

and

$$\begin{aligned}u^{\mu}\nabla_{\mu}\bar{\mathcal{Q}}^{(k)}_{\alpha\beta} &= 0\,,\\ e^{\mu}D_{\mu}\bar{\mathcal{Q}}^{(k)}_{\alpha\beta} &= 0\,,\\ \delta^{\mu}\bar{\mathcal{Q}}^{(k)}_{\alpha\mu} &= -\frac{2-k^2}{2r}\bar{\mathcal{Q}}^{(k)}_{\alpha}\,,\\ \varepsilon^{\mu\nu}\delta_{\mu}\bar{\mathcal{Q}}^{(k)}_{\nu\alpha} &= \frac{2-k^2}{2r}\mathcal{Q}^{(k)}_{\alpha}\,.\end{aligned} \tag{E10}$$

Moreover, we have

$$\begin{aligned}\mathcal{Q}^{(k)}_{\alpha\beta} &= -\varepsilon^{\mu}{}_{\{\alpha}\bar{\mathcal{Q}}^{(k)}_{\beta\}\mu}\,,\\ \bar{\mathcal{Q}}^{(k)}_{\alpha\beta} &= \varepsilon^{\mu}{}_{\{\alpha}\mathcal{Q}^{(k)}_{\beta\}\mu}\,,\end{aligned}\tag{E11}$$

which can be used to readily prove that $N^{\alpha\mu}N^{\beta\nu}\mathcal{Q}^{(k)}_{\alpha\beta}\bar{\mathcal{Q}}^{(k)}_{\mu\nu}=0$, valid for each value of k.

Appendix F. Change in the equation of state under isotropic frame transformations

In this appendix, we make use of generalized Lorentz boosts in the 1+1+2 formalism to derive the transformation law for the equation of state when changing between the comoving frame and the static frame.

F.1. Isotropic frame transformations and projectors

Let the dyad (u,e) be formed, respectively, by a timelike and a spacelike vector field in a spacetime, such that $u^\alpha u_\alpha=-1$ and $e^\alpha e_\alpha=1$, and let

$$\begin{aligned}\bar{u}^\alpha &= (1+a)\,u^\alpha + be^\alpha + m^\alpha\,,\\ \bar{e}^\alpha &= cu^\alpha + (1+d)\,e^\alpha + k^\alpha\,,\end{aligned}\tag{F1}$$

represent the components in some local coordinate system of a dyad $(\bar{u},\bar{e})$, with $\{a,b,c,d\}$ being scalar coefficients and

$$\begin{aligned}u^\alpha m_\alpha &= 0\,,\quad u^\alpha k_\alpha = 0\,,\\ e^\alpha m_\alpha &= 0\,,\quad e^\alpha k_\alpha = 0\,.\end{aligned}\tag{F2}$$

In the expressions above, we can assume, without loss of generality, that the vector fields m and k are associated with rotational displacements. Then, for our purposes, it suffices to consider that the frame transformation is isotropic, such that $m^\alpha=0$ and $k^\alpha=0$.

Imposing that the new dyad verifies

$$\begin{aligned}\bar{u}^\alpha\bar{u}_\alpha &= -1\,,\\ \bar{e}^\alpha\bar{e}_\alpha &= +1\,,\\ \bar{u}^\alpha\bar{e}_\alpha &= 0\,,\end{aligned}\tag{F3}$$

we find that the transformation coefficients verify

$$\begin{aligned}(1+d)^2 &= (1+a)^2\\ (1+a)^2 &= 1+c^2\\ b^2 &= c^2\,.\end{aligned}\tag{F4}$$

Assuming, $b,c>0$, hence $a,d>0$, that is, the transformation does not swap the direction between the corresponding vector fields, i.e. imposing that $u^\alpha\bar{u}_\alpha\leqslant 0$ and $e^\alpha\bar{e}_\alpha\geqslant 0$, we see that the relation between a and c can be made obvious by introducing a single hyperbolic angle β such that

$$\begin{aligned}1+a &= \cosh\beta\,,\\ c &= \sinh\beta\,.\end{aligned}\tag{F5}$$

Then, we have the following relation between the two dyads

$$\begin{aligned} \bar{u}^\alpha &= u^\alpha \cosh\beta + e^\alpha \sinh\beta\,, \\ \bar{e}^\alpha &= u^\alpha \sinh\beta + e^\alpha \cosh\beta\,. \end{aligned} \tag{F6}$$

Given relations (F6), the projectors h and N transform accordingly as

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} + (u_\alpha u_\beta + e_\alpha e_\beta)\sinh^2\beta + \frac{1}{2}(u_\alpha e_\beta + e_\alpha u_\beta)\sinh(2\beta)\,, \tag{F7}$$

$$\bar{N}_{\alpha\beta} = N_{\alpha\beta}\,, \tag{F8}$$

which verify $\bar{h}_{\alpha\beta}\bar{u}^\beta = 0$, $\bar{N}_{\alpha\beta}\bar{u}^\beta = 0$ and $\bar{N}_{\alpha\beta}\bar{e}^\beta = 0$, as expected.

F.2. Fluid variables transformations

Consider now a general symmetric stress-energy tensor decomposed accordingly with equation (15). Under the transformation (F6) we have

$$\begin{aligned} \bar{\mu} &= \mu - Q\sinh(2\beta) + (\mu + p + \Pi)\sinh^2\beta\,, \\ \bar{p} &= p - \frac{1}{3}Q\sinh(2\beta) + \frac{1}{3}(\mu + p + \Pi)\sinh^2\beta\,, \\ \bar{Q} &= Q\cosh(2\beta) - \frac{1}{2}(\mu + p + \Pi)\sinh(2\beta)\,, \\ \bar{\Pi} &= \Pi\left(1 + \frac{2}{3}\sinh^2\beta\right) - \frac{2}{3}Q\sinh(2\beta) + \frac{2}{3}(\mu + p)\sinh^2\beta\,, \end{aligned} \tag{F9}$$

where we have used the intuitive notation of keeping an overline to indicate variables measured in the frame associated with the dyad $(\bar{u}, \bar{e})$. Moreover, we have the following relations between the scalar kinematical quantities

$$\begin{aligned} \bar{\theta} &= \theta\cosh\beta + (\phi + \mathcal{A})\sinh\beta + \nabla_u\cosh\beta + \nabla_e\sinh\beta\,, \\ \bar{\Sigma} &= \Sigma\cosh\beta - \frac{1}{3}(\phi - 2\mathcal{A})\sinh\beta + \frac{2}{3}\nabla_u\cosh\beta + \frac{2}{3}\nabla_e\sinh\beta\,, \end{aligned} \tag{F10}$$

where $\nabla_u \equiv u^\alpha\nabla_\alpha$ and $\nabla_e \equiv e^\alpha\nabla_\alpha$.

F.3. Relation between equations of state of the comoving and static observers

The comoving and static observer frames considered in section 4 are related by an isotropic change of frame. Then, we can use the results in the previous subsections of this appendix to find the equation of state experienced by a static observer, knowing the equation of state measured by the comoving observer.

Assuming the dyad (u, e) to be associated to the comoving frame and the dyad $(\bar{u}, \bar{e})$ to be associated with the static frame, we can particularize equation (F9) to the case when $Q = 0$ and $\Pi = 0$, such that

$$\begin{aligned} \bar{\mu} &= \mu + (\mu + p)\sinh^2\beta\,, \\ \bar{p} &= p + \frac{1}{3}(\mu + p)\sinh^2\beta\,, \\ \bar{Q} &= -\frac{1}{2}(\mu + p)\sinh(2\beta)\,, \\ \bar{\Pi} &= \frac{2}{3}(\mu + p)\sinh^2\beta\,, \end{aligned} \tag{F11}$$

relating the nontrivial matter variables of the two frames. Moreover, since in the static frame—the one associated with the barred variables—we have $\frac{3}{2}\bar{\Sigma} = \bar{\theta}$, then

$$\tanh\beta = \frac{1}{\phi}\left(\Sigma - \frac{2}{3}\theta\right), \tag{F12}$$

which upon substitution in equation (F11) yields

$$\bar{Q} = -\frac{\mu + p}{\phi}\left(\Sigma - \frac{2}{3}\theta\right)\cosh^2\beta. \tag{F13}$$

Assuming the background spacetime to be static and LRS II, equation (F12) explicitly shows that β is a first-order quantity with respect to the background, i.e. β vanishes in the background. This is expected since comoving observers are also static observers for a static LRS II spacetime. Moreover, these results confirm the expectation that the anisotropic pressure measured by the static observer, $\bar{\Pi}$, is zero at linear level for adiabatic, radial perturbations.

We can now find the linear order correction with respect to the background of the equation of state in the static frame. Taking the derivative along $\bar{u}$ of $\bar{p}$ and $\bar{\mu}$, and using equations (F6) and (F11) we find, in general,

$$\begin{aligned}
\nabla_{\bar{u}}\bar{\mu} &= \dot{\mu}\cosh\beta + \widehat{\mu}\sinh\beta + \left[\left(\nabla_u \sinh^2\beta\right)(\mu+p) + \sinh^2\beta\,(\dot{\mu}+\dot{p})\right]\cosh\beta \\
&\quad + \left[\left(\nabla_e \sinh^2\beta\right)(\mu+p) + \sinh^2\beta\,(\widehat{\mu}+\widehat{p})\right]\sinh\beta, \\
\nabla_{\bar{u}}\bar{p} &= \dot{p}\cosh\beta + \widehat{p}\sinh\beta + \frac{1}{3}\left[\left(\nabla_u \sinh^2\beta\right)(\mu+p) + \sinh^2\beta\,(\dot{\mu}+\dot{p})\right]\cosh\beta \\
&\quad + \frac{1}{3}\left[\left(\nabla_e \sinh^2\beta\right)(\mu+p) + \sinh^2\beta\,(\widehat{\mu}+\widehat{p})\right]\sinh\beta,
\end{aligned} \tag{F14}$$

where $\dot{\mu} = u^\alpha\nabla_\alpha\mu$ and $\widehat{\mu} = e^\alpha\nabla_\alpha\mu$, and similarly for p. Using the fact that β is a first-order quantity with respect to the background, in first-order perturbation theory equation (F14) simplifies to

$$\begin{aligned}
\nabla_{\bar{u}}\bar{\mu} &= \dot{\mu} + \widehat{\mu}_0\beta, \\
\nabla_{\bar{u}}\bar{p} &= \dot{p} + \widehat{p}_0\beta.
\end{aligned} \tag{F15}$$

Also, from equation (F11), at first order, we have

$$\beta = -\frac{\bar{Q}}{\mu_0 + p_0}. \tag{F16}$$

Gathering the intermediate results and using the equations that characterize the background solution (cf section 4.1) we find the transformed equation of state for the static frame

$$\nabla_{\bar{u}}\bar{\mu} = \frac{1}{f'(\mu_0)}\nabla_{\bar{u}}\bar{p} - \frac{1}{\mu_0 + p_0}\left(\widehat{\mu}_0 - \frac{\widehat{p}_0}{f'(\mu_0)}\right)Q, \tag{F17}$$

where $\dot{p} = f'(\mu_0)\,\dot{\mu}$, that is, $f'(\mu_0)$ represents the square of the adiabatic speed of sound measured in the comoving frame.

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