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Gravitational waves: approximate methods and exact solutions

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I declare that my Ph.D. thesis has been elaborated personally and that I have used only the literature listed. I agree with loaning out my work.

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Contents

Introduction

The presented work concerns two wide branches of theoretical investigation of gravitational waves. Namely, the first three chapters concentrate on approximate techniques, although, in the second chapter the relation to exact solutions is mentioned. The final part is devoted to asymptotic behaviour in a certain class of exact radiative spacetimes.

In the first chapter the Efroimsky perturbation scheme for consistent treatment of gravitational waves and their influence on the background is summarized and compared with classical Isaacson's high-frequency approach. We demonstrate that the Efroimsky method in its present form is not compatible with the Isaacson limit of high-frequency gravitational waves, and we propose its natural generalization to resolve this drawback.

In the second chapter a formalism is introduced which may describe both standard linearized waves and gravitational waves in Isaacson's high-frequency limit. After emphasizing main differences between the two approximation techniques we generalize the Isaacson method to non-vacuum spacetimes. Then we present three large explicit classes of solutions for high-frequency gravitational waves in particular backgrounds. These involve non-expanding (plane, spherical or hyperbolical), cylindrical, and expanding (spherical) waves propagating in various universes which may contain a cosmological constant and electromagnetic field. Relations of high-frequency gravitational perturbations of these types to corresponding exact radiative spacetimes are described.

In the third part we concentrate on solving the wave equation describing the propagation of high-frequency waves which was derived by Isaacson [3]. Although the complete Isaacson formalism incorporates also the reaction of the background to the wave, we will not consider this effect here. Rather, we will explicitly present spectra of high-frequency waves which may propagate in some fundamental cosmological models, in particular the Friedmann–Robertson–Walker spacetimes and in the anisotropic Kasner universe.

The last chapter is devoted to the analysis of a class of exact type II solutions of the Robinson–Trautman family which contain pure radiation and (possibly) a cosmological constant. It is shown that these spacetimes exist for any sufficiently smooth initial data, and that they approach the spherically symmetric Vaidya–(anti–) de Sitter metric. We also investigate extensions of the metric, and we demonstrate that their order of smoothness is in general only finite. Some applications of the results are outlined.

Chapter 1

The Efroimsky formalism adapted to high-frequency perturbations

Some time ago Efroimsky introduced and developed new formalism for a consistent treatment of weak gravitational waves [1, 2]. This interesting mathematical framework is remarkable mainly due to the possibility to ascribe stress-energy tensor even to *low-frequency* gravitational waves influencing the background, which is in contrast to standard linearization approach where the background is kept fixed.

On the other hand, in a now classic paper [3] Isaacson (inspired by previous works [4,5]) presented a perturbation method which can be used for studies of *high-frequency* gravitational waves. Such waves also influence the cosmological background in which they propagate.

In our present work we first briefly summarize and compare the two above mentioned perturbation schemes. In particular, it is shown that the Efroimsky method is not consistent if high-frequency gravitational waves are considered. We propose a possible modification of the Efroimsky formalism which may resolve this drawback.

1.1 The formalism

Efroimsky's approach [1, 2] is based on introducing *three different* smooth, non-degenerate, symmetric metrics on a differentiable manifold M, namely:

1. $\gamma_{\mu\nu}$ – the "premetric": vacuum metric corresponding to initial pure background without gravitational waves,
2. $g_{\mu\nu}$ – the "physical metric": full vacuum metric which describes both the background and the waves,
3. $q_{\mu\nu}$ – the "average metric": non-vacuum metric representing the background plus its perturbations with wavelength greater than L.

Next step is to define the Ricci and Einstein tensors for an arbitrary metric g as

$$\begin{aligned}
R_{\mu\nu}(g) &\equiv [\tfrac{1}{2}g^{\gamma\rho}(g_{\rho\nu,\mu}+g_{\rho\mu,\nu}-g_{\mu\nu,\rho})]_{,\gamma}-[\tfrac{1}{2}g^{\gamma\rho}(g_{\rho\gamma,\mu}+g_{\rho\mu,\gamma}-g_{\mu\gamma,\rho})]_{,\nu}\\
&\quad+[\tfrac{1}{2}g^{\gamma\delta}(g_{\rho\delta,\gamma}+g_{\rho\gamma,\delta}-g_{\gamma\delta,\rho})][\tfrac{1}{2}g^{\delta\rho}(g_{\rho\nu,\mu}+g_{\rho\mu,\nu}-g_{\mu\nu,\rho})] \qquad (1.1)\\
&\quad-[\tfrac{1}{2}g^{\gamma\rho}(g_{\rho\delta,\nu}+g_{\rho\nu,\delta}-g_{\nu\delta,\rho})][\tfrac{1}{2}g^{\delta\rho}(g_{\rho\gamma,\mu}+g_{\rho\mu,\gamma}-g_{\mu\gamma,\rho})]\ ,\\
G_{\mu\nu}(g) &\equiv R_{\mu\nu}(g)-\tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}(g)\ ,
\end{aligned}$$

where $g^{\rho\tau} = (g)^{-1}_{\rho\tau}$, the same expressions apply to γ and q. From the proposals (i)-(iii) it follows that $G_{\mu\nu}(\gamma) = 0 = G_{\mu\nu}(g)\, , G_{\mu\nu}(q) \neq 0$.

Now, the *differences* between the covariant components of the above metrics are introduced,

$$\begin{aligned} h_{\mu\nu} &\equiv g_{\mu\nu} - q_{\mu\nu} \ , \\ \eta_{\mu\nu} &\equiv q_{\mu\nu} - \gamma_{\mu\nu} \ . \end{aligned} \tag{1.2}$$

It is necessary to specify the semi-Riemann space: for raising or lowering indices and for covariant differentiation the *averaged non-vacuum metric* q will be used. Treating $h_{\mu\nu}$ as a perturbation of the metric $q_{\mu\nu}$ the Ricci tensor (1.1) can be expanded in a powers series

$$R_{\mu\nu}(g) = R^{(0)}_{\mu\nu}(q) + R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) + R^{(3)}_{\mu\nu}(q,h) + O(h^4) \ . \tag{1.3}$$

Analogously,

$$R_{\mu\nu}(\gamma) = R^{(0)}_{\mu\nu}(q) + R^{(1)}_{\mu\nu}(q,(-\eta)) + R^{(2)}_{\mu\nu}(q,(-\eta)) + O(\eta^3) \ . \tag{1.4}$$

It is obvious that $R^{(1)}_{\mu\nu}(q,(-\eta)) = -R^{(1)}_{\mu\nu}(q,\eta)$ and $R^{(2)}_{\mu\nu}(q,(-\eta)) = R^{(2)}_{\mu\nu}(q,\eta)$. According to assumptions that both g and γ are vacuum metrics the following relation holds

$$\begin{aligned} 0 &= R_{\mu\nu}(g) - R_{\mu\nu}(\gamma) \\ &= R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) + R^{(1)}_{\mu\nu}(q,\eta) + R^{(3)}_{\mu\nu}(q,h) + O(h^4) + O(\eta^2) \ . \end{aligned} \tag{1.5}$$

At this point Efroimsky sets three assumptions:

Assumption 1. The perturbations h and η are small in the sense that the terms of the orders $O(h^4)$ and $O(\eta^2)$ are negligible.

Assumption 2. The perturbations η and h^2 are of the same order.

Assumption 3. The tensor field h consists of modes with short wavelengths which do not exceed the given maximal value L.

Thus $h_{\mu\nu}$ characterizes *measurable gravitational waves* whereas $\eta_{\mu\nu}$ is a *shift of the background geometry* from vacuum premetric γ to nonvacuum effective average metric q due to the presence of gravitational waves. The equation (1.5) is the wave equation for perturbations h on the background $q = \gamma + \eta$. Using the Brill-Hartle averaging procedure [5] we obtain

$$R^{(1)}_{\mu\nu}(q,\eta) = -\langle R^{(2)}_{\mu\nu}(q,h)\rangle \ . \tag{1.6}$$

Using (1.4), the effective stress-energy tensor of gravitational waves is defined as

$$G_{\mu\nu}(q) = 8\pi T^{(gw)}_{\mu\nu} \equiv R^{(1)}_{\mu\nu}(q,\eta) - \tfrac{1}{2} q_{\mu\nu} q^{\alpha\beta} R^{(1)}_{\alpha\beta}(q,\eta) \ . \tag{1.7}$$

From (1.6) it follows that this tensor fully agrees with that of Isaacson [3].

The main advantage of the above Efroimsky's perturbation method is the possibility to consistently treat all low-frequency gravitational waves, and to explicitly derive effective stress-energy tensor (influencing the background) in this case. It can be extended to non-vacuum spacetimes with $T_{\mu\nu}$ of ideal fluid and/or with a possible cosmological constant Λ, see [1, 2].

1.2 Modification to include high-frequency waves

Let us start with observation that it is the nonvacuum background curved by the presence of gravitational waves — not the vacuum premetric γ — which is the basis of Isaacson's non-linear approach [3]. Therefore, the *nonvacuum average metric* q is considered as the background on which high-frequency gravitational waves h propagate.

We wish to use the Efroimsky formalism in the high-frequency regime such that the tensor field h contains high-frequency modes. We assume that they have short wavelengths λ, and a small amplitude $h = O(\varepsilon)$, where $\varepsilon = \lambda/S \ll 1$ is a small parameter because $\lambda \ll S$, S denoting a typical scale on which the background changes substantially, and $f = O(\varepsilon^n)$ if there exists a constant $C > 0$ such that $|f| < C\varepsilon^n$ as $\varepsilon \to 0$.

Since we can consider $S = O(1)$ it follows that $O(\varepsilon) = O(\lambda)$ and $\partial h \sim h/\lambda = O(1)$. This results in the orders of magnitude of the terms in the Ricci tensor expansion (1.3) as

$$R^{(0)}_{\mu\nu} = O(1), \quad R^{(1)}_{\mu\nu} = O(\varepsilon^{-1}), \quad R^{(2)}_{\mu\nu} = O(1), \quad R^{(3)}_{\mu\nu} = O(\varepsilon). \tag{1.8}$$

To apply the Efroimsky approach in this case we must consider the decomposition $q = \gamma + \eta$, where γ is the vacuum premetric and η represents (in this case) *substantial* shift of the background geometry due to the presence of high-frequency gravitational waves h.

Of course, the geometry shift η does not contain high-frequency perturbations. Considering the wave equation (1.5) and using the Brill-Hartle averaging to obtain the equation (1.6) we get in a conflict with the Assumption 1. and Assumption 2., since $\eta = O(1)$. In fact, it disables any consistent perturbation expansions in the powers of η.

Let us now suggest a modification of the Efroimsky formalism which will incorporate also the above case of a "substantial" change of the background geometry due to the presence of high-frequency waves. Instead of the perturbation expansion (1.4) we consider a formal decomposition of the Ricci tensor of the premetric $\gamma = q - \eta$, namely

$$0 = R_{\mu\nu}(\gamma) = R_{\mu\nu}(q) + \Delta R_{\mu\nu}(q, (-\eta)) \ , \tag{1.9}$$

by which equation the expression $\Delta R_{\mu\nu}$ is *defined*. Both terms on the right-hand side of (1.9) are of the same order $O(1)$.

The question concerning the gauge invariance of $\Delta R_{\mu\nu}$ with respect to generalized gauge transformations has been recently analyzed in detail by Anderson [16] in connection with possible definitions of the wave equation and stress-energy tensor for gravitational waves. Let us consider an arbitrary coordinate transformation of the type

$$\overline{x}^{\mu} = x^{\mu} + \xi^{\mu} \ , \tag{1.10}$$

that does not change the functional form of the background geometry q, i.e. $\overline{q}(\overline{x}) = q(\overline{x})$ so that $\gamma(x) \to \overline{\gamma}(\overline{x}) = q(\overline{x}) - \overline{\eta}(\overline{x})$. Performing the above coordinate transformation (1.10) of the Ricci tensor (1.9) we can derive

$$\Delta R_{\mu\nu}(q(x), (-\eta(x))) = \Delta R_{\mu\nu}(q(x), (-\overline{\eta}(x))) \ . \tag{1.11}$$

A generalized gauge transformation is defined in [16] as a transformation in which the quantity $\overline{\eta}(x)$ is substituted for $\eta(x)$ into the tensor expressions of interest. Obviously, the equation (1.11) expresses a generalized gauge invariance of $\Delta R_{\mu\nu}$.

After introducing the above decomposition (1.9) and demonstrating its invariance we can now present modification and generalization of the Efroimsky formalism expressed in the following relations,

$$R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) - \Delta R_{\mu\nu}(q,(-\eta)) + R^{(3)}_{\mu\nu}(q,h) + O(h^4) = 0 \ , \tag{1.12}$$

$$\Delta R_{\mu\nu}(q,(-\eta)) = \langle R^{(2)}_{\mu\nu}(q,h)\rangle_L \ , \tag{1.13}$$

$$G_{\mu\nu}(q) = 8\pi \tilde{T}^{(gw)}_{\mu\nu} \equiv -\Delta R_{\mu\nu}(q,(-\eta)) + \tfrac{1}{2} q_{\mu\nu} q^{\alpha\beta} \Delta R_{\alpha\beta}(q,(-\eta)) \ . \tag{1.14}$$

In case when gravitational waves do not have high-frequency modes it is still possible to employ the expansion of $-\Delta R_{\mu\nu}(q,(-\eta))$ in powers of η and use its dominant term $R^{(1)}_{\mu\nu}(q,\eta)$ instead. Thus we recover Efroimsky's previous results, cf. (1.5), (1.6), (1.7).

In general, however, expressing η in terms of h from the equation (1.13) becomes an extremely difficult task because it is no longer a linear equation for η. To overcome this problem we can use the equation (1.13) and substitute for $\Delta R_{\mu\nu}$ into the remaining equations (1.12) and (1.14). We obtain the relations

$$R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) - \langle R^{(2)}_{\mu\nu}(q,h)\rangle_L + R^{(3)}_{\mu\nu}(q,h) + O(h^4) = 0 \ , \tag{1.15}$$

$$-G_{\mu\nu}(q) = \langle R^{(2)}_{\mu\nu}(q,h)\rangle_L - \tfrac{1}{2} q_{\mu\nu} q^{\alpha\beta} \langle R^{(2)}_{\alpha\beta}(q,h)\rangle_L \equiv -8\pi T^{BH}_{\mu\nu} \ . \tag{1.16}$$

The equation (1.16) is obviously in perfect accordance with the Isaacson result [3]. In the highest order of high-frequency approximation the equation (1.15) clearly reduces to $R^{(1)}_{\mu\nu} = 0$ which also fully reproduces Isaacson's result. Additional terms in (1.15) can be used for study of nonlinear effects on the wave propagation.

Finally the equations (1.13) and (1.11) guarantee the gauge invariance of the stress-energy tensor $T^{BH}_{\mu\nu}$ defined in (1.16) (in the highest order). Proof of this property was presented already in the classic work [3], using however much more complicated method.

Chapter 2

Some high-frequency gravitational waves related to exact radiative spacetimes

In classic work [3] Isaacson presented a perturbation method which enables one to study properties of high-frequency gravitational waves, together with their influence on the cosmological background in which they propagate. It is this non-linear "back-reaction" effect on curvature of the background spacetime which distinguishes the high-frequency approximation scheme from other perturbation methods such as the standard Einstein's linearization of gravitational field in flat space [17, 18] or multipole expansions [19] that were developed to describe radiation from realistic astrophysical sources.

On the other hand, many *exact* solutions of Einstein's equations are known which represent gravitational radiation. Among the most important classes are planar *pp*-waves [22, 23] which belong to a large family of non-expanding radiative spacetimes [24,25], cylindrical Einstein-Rosen waves [26], expanding "spherical" waves of the Robinson-Trautman type [27,28], spacetimes with boost-rotation symmetry representing radiation generated by uniformly accelerated sources [29–31], cosmological models of the Gowdy type [32].

However, there are only several works in which *relation* between exact gravitational waves and those obtained by perturbations of non-flat backgrounds has been explicitly investigated and clarified, see e.g. [12, 14, 38]. The purpose of our contribution is to help to fill this "gap".

We first briefly summarize and generalize the Isaacson approach [3] to admit non-vacuum backgrounds, the cosmological constant Λ in particular. Modification of Isaacson's formalism allows us to incorporate also standard linearized gravitational waves into the common formalism.

2.1 High-frequency approximation versus standard linearization

Let us assume a formal decomposition of the vacuum spacetime metric $g_{\mu\nu}$ into the background metric $\gamma_{\mu\nu}$ and its perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon h_{\mu\nu} \ , \tag{2.1}$$

where, in a suitable coordinate system, $\gamma_{\mu\nu} = O(1)$ and $h_{\mu\nu} = O(\epsilon)$. The two distinct non-negative dimensionless parameters ε and ϵ have the following meaning: ε is the usual amplitude parameter of weak gravitational perturbations whereas the frequency parameter ϵ denotes the possible high-frequency character of radiation described by $h_{\mu\nu}$. The parameter $\epsilon = \lambda/L$ represents the ratio of a typical wavelength λ of gravitational waves and the scale L on which the background curvature changes significantly. Since L can be considered to have a finite value of order unity, we may write $O(\epsilon) = O(\lambda)$.

To derive the dynamical field equations we start with the order-of-magnitude estimates which indicate how fast the metric components vary. Symbolically, the derivatives are of the order $\partial\gamma \sim \gamma/L$, $\partial h \sim h/\lambda$. Next, we expand the Ricci tensor in powers of h,

$$R_{\mu\nu}(g) = R^{(0)}_{\mu\nu} + \varepsilon R^{(1)}_{\mu\nu} + \varepsilon^2 R^{(2)}_{\mu\nu} + \dots \ , \tag{2.2}$$

where

$$\begin{aligned}
R^{(0)}_{\mu\nu}(\gamma) &\equiv R_{\mu\nu}(\gamma) \ , \\
R^{(1)}_{\mu\nu}(\gamma, h) &\equiv \tfrac{1}{2}\gamma^{\rho\tau}\left(h_{\tau\mu;\nu\rho} + h_{\tau\nu;\mu\rho} - h_{\rho\tau;\mu\nu} - h_{\mu\nu;\rho\tau}\right) \ , \\
R^{(2)}_{\mu\nu}(\gamma, h) &\equiv \tfrac{1}{2}\Big[\tfrac{1}{2}h^{\rho\tau}{}_{;\nu}h_{\rho\tau;\mu} + h^{\rho\tau}(h_{\tau\rho;\mu\nu} + h_{\mu\nu;\tau\rho} - h_{\tau\mu;\nu\rho} \\
&\quad -h_{\tau\nu;\mu\rho}) + h^{\tau}{}_{\nu}{}^{;\rho}\left(h_{\tau\mu;\rho} - h_{\rho\mu;\tau}\right) \\
&\quad - \left(h^{\rho\tau}{}_{;\rho} - \tfrac{1}{2}h^{;\tau}\right)\left(h_{\tau\mu;\nu} + h_{\tau\nu;\mu} - h_{\mu\nu;\tau}\right)\Big] \ .
\end{aligned} \tag{2.3}$$

The semicolons denote covariant differentiation with respect to the *background* metric $\gamma_{\mu\nu}$, which is also used to raise or lower all indices. The orders of the terms (2.3) are

$$R^{(0)}_{\mu\nu} = O(1), \ \varepsilon R^{(1)}_{\mu\nu} = O(\epsilon^{-1}\varepsilon), \ \varepsilon^2 R^{(2)}_{\mu\nu} = O(\varepsilon^2), \ \varepsilon^3 R^{(3)}_{\mu\nu} = O(\epsilon\varepsilon^3). \tag{2.4}$$

Two limiting cases thus arise naturally. For the *standard linearization* ($\varepsilon \ll 1$, $\epsilon = 1$) the dominant term of $R_{\mu\nu}(g)$ is $R^{(0)}_{\mu\nu}$. Its first correction representing linearized (purely) gravitational waves is governed by

$$R^{(1)}_{\mu\nu}(\gamma, h) = 0 \ , \tag{2.5}$$

which is a dynamical equation for perturbations $h_{\mu\nu}$ on the fixed background $\gamma_{\mu\nu}$. The next term $R^{(2)}_{\mu\nu}(\gamma, h)$ can then be used to define energy-momentum tensor of these gravitational waves, but the background metric is *not* assumed to be influenced by it.

In the *high-frequency approximation* ($\epsilon \ll 1$, $\varepsilon = 1$) the dominant term is $R^{(1)}_{\mu\nu} = O(\epsilon^{-1})$ which gives the wave equation (2.5). The two terms of the order $O(1)$, namely $R^{(0)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$, are *both* used to give the Einstein equation for the background *non-vacuum* metric, which represents the essential influence of the high-frequency gravitational waves on the background. Of course, to obtain a consistent solution, one has to use both the wave equation *and* the Einstein equation for the background simultaneously.

2.1.1 Linear approximation

In analogy with the well-known theory of massless spin-2 fields in flat space [19] we wish to impose two TT gauge conditions,

$$h_{\mu\nu}{}^{;\nu} = 0 \ , \tag{2.6}$$

$$h^{\mu}{}_{\mu} = 0 \ . \tag{2.7}$$

In this gauge we arrive at the following wave equation

$$\Diamond h_{\mu\nu} \equiv h_{\mu\nu}{}^{;\beta}{}_{;\beta} - 2R^{(0)}_{\sigma\nu\mu\beta}\, h^{\beta\sigma} - R^{(0)}_{\mu\sigma}\, h^{\sigma}{}_{\nu} - R^{(0)}_{\nu\sigma}\, h^{\sigma}{}_{\mu} = 0 \ , \tag{2.8}$$

where the operator $\Diamond$ is the generalization of flat-space d'Alembertian.

In case of standard linearized waves ($\epsilon = 1$) there is an inconsistency between (2.8) and (2.6), except for backgrounds with a covariantly constant Ricci tensor (e.g., for the Einstein spaces). On the other hand, in the high-frequency limit ($\varepsilon = 1$), the inconsistency is negligible. Moreover, for all background metrics of *constant curvature* the equations are *fully consistent.*

2.1.2 Generalization to non-vacuum spacetimes

Before considering the second-order terms we now extend the formalism to be applicable to a larger class of spacetimes with (possibly) non-vanishing energy-momentum tensor $T_{\mu\nu}$. Namely, $g_{\mu\nu}$ satisfies Einstein's equations

$$R_{\mu\nu}(g) = 8\pi\, \tilde{T}_{\mu\nu}(g,\varphi) \ . \tag{2.9}$$

Here $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\beta}{}_{\beta}$, such that $T_{\mu\nu}(g,\varphi)$ depends on non-gravitational fields φ and on the full metric $g_{\mu\nu}$ but it *does not* contain the *derivatives* of $g_{\mu\nu}$. Note that this admits as particular cases a presence of electromagnetic field, and also Einstein spaces when $\tilde{T}_{\mu\nu} = \frac{1}{8\pi}\Lambda g_{\mu\nu}$. We expand both sides of the equation (2.9) as in (2.2). For ordinary linearization we thus get the equations $R^{(n)}_{\mu\nu} = 8\pi\, \tilde{T}^{(n)}_{\mu\nu}$ in each order $n = 0, 1, 2, \ldots$. For the high-frequency approximation we obtain, in the leading order, the equation (2.5) which is identical with the wave equation in the vacuum case. The second-order contributions, that are $O(1)$, represent an influence of the high-frequency gravitational waves and matter fields on the background, and can be rewritten in the form of Einstein's equation for the background as

$$G^{(0)}_{\mu\nu}(\gamma) - 8\pi\, T^{(0)}_{\mu\nu}(\gamma,\varphi) = -[R^{(2)}_{\mu\nu}(\gamma,h) - \tfrac{1}{2}\gamma_{\mu\nu} R^{(2)}(\gamma,h)] \equiv 8\pi\, T^{GW}_{\mu\nu} \ . \tag{2.10}$$

This defines the effective energy-momentum tensor $T^{GW}_{\mu\nu}$ of high-frequency gravitational waves.

2.1.3 The WKB approximation

In the following we shall restrict ourselves to the Isaacson approximation ($\varepsilon = 1$, $\epsilon \ll 1$), i.e. on study of high-frequency gravitational waves on curved backgrounds. Inspired by the plane-wave solution in flat space, the form $h_{\mu\nu} = \mathcal{A}\, e_{\mu\nu} \exp(i\phi)$ of the solution is assumed. The amplitude $\mathcal{A} = O(\epsilon)$ is a slowly changing real function of position, the phase ϕ is a real function with a large first derivative but no larger derivatives beyond, and $e_{\mu\nu}$ is a normalized polarisation tensor field. Substituting this into the conditions (2.6), (2.7), and the wave equation (2.8) we obtain, in the two highest orders which are gauge invariant,

$$\begin{aligned} &k^\mu k_\mu = 0\ , \quad k^\mu e_{\mu\nu} = 0\ , \quad k^\alpha e_{\mu\nu;\alpha} = 0\ , \\ &e^{\mu\nu} e_{\mu\nu} = 1\ , \quad \gamma^{\mu\nu} e_{\mu\nu} = 0\ , \quad \left(\mathcal{A}^2 k^\beta\right)_{;\beta} = 0\ . \end{aligned} \tag{2.11}$$

Moreover, using the WKB approximation of $T^{GW}_{\mu\nu}$ and the Brill-Hartle averaging procedure [5] (which guarantees the gauge invariance) Isaacson obtained the energy-momentum tensor [3]

$$T^{HF}_{\mu\nu} = \tfrac{1}{64\pi} \mathcal{A}^2 k_\mu k_\nu\ . \tag{2.12}$$

The energy-momentum tensor of high-frequency waves thus has the form of pure radiation.

2.2 Examples of high-frequency gravitational waves

Now we present some explicit classes of high-frequency gravitational waves. These are obtained by the above described WKB approximation method considering specific families of background spacetimes with a privileged geometry.

2.2.1 Non-expanding waves

As the background we first consider the Kundt class [24,33] of non-expanding, twist-free spacetimes in the form [40]

$$\mathrm{d}s^2 = F\,\mathrm{d}u^2 - 2\,\frac{Q^2}{P^2}\,\mathrm{d}u\,\mathrm{d}v + \frac{1}{P^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)\ , \tag{2.13}$$

with

$$\begin{aligned} P &= 1 + \frac{\alpha}{2}\left(x^2 + y^2\right)\ , \\ Q &= \left[1 + \frac{\beta}{2}\left(x^2 + y^2\right)\right] e + C_1\,x + C_2\,y\ , \\ F &= D\,\frac{Q^2}{P^2}\,v^2 - \frac{(Q^2)_{,u}}{P^2}\,v - \frac{Q}{P}\,H\ , \end{aligned} \tag{2.14}$$

where α, β, and e are constants (without loss of generality $e = 0$ or $e = 1$), C_1, C_2 and D are arbitrary functions of the retarded time u, and $H(x, y, u)$ is an arbitrary function of the spatial coordinates x, y, and of u. In particular, these are Petrov type N when $\alpha = -\beta = \frac{1}{6}\Lambda$ and $D = -2\beta e + C_1^2 + C_2^2$.

We consider the phase of high-frequency gravitational waves given by $\phi = \phi(u)$, and we seek solution in the WKB form, namely

$$h_{\mu\nu} = \mathcal{A}\, e_{\mu\nu} \exp\Big(i\phi(u)\Big) \,, \tag{2.15}$$

where the amplitude $\mathcal{A}$ and polarization tensor $e_{\mu\nu}$ are functions of the coordinates $\{u, v, x, y\}$. Applying now the equations (2.11) we obtain

$$\mathcal{A} = \mathcal{A}(u, x, y) \,, \tag{2.16}$$

The fact that the amplitude $\mathcal{A}$ is independent of the coordinate v expresses non-expanding character of the waves. The polarisation tensor is analogous to those used in the standard theory of linearized waves in flat space.

Using the Einstein tensor for the metric (2.13) with the cosmological term in equations (2.10) and (2.12), we determine the reaction of the background on the presence of the above high-frequency gravitational perturbations, namely

$$\frac{Q}{P}\left[P^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{2}{3}\Lambda\right] H(u, x, y) = \tfrac{1}{4}\mathcal{A}^2(u, x, y)\dot{\phi}^2 \,. \tag{2.17}$$

These *approximate* solutions can obviously be compared to specific *exact* radiative vacuum solutions which are given by H solving the field equation (2.17) with a vanishing right-hand side (when $\mathcal{A} = 0$, i.e. high-frequency perturbation waves are absent).

The above waves are non-expanding with the wave-fronts $u = const.$ being two-dimensional spaces of constant curvature given by $\alpha = \frac{1}{6}\Lambda$, cf. (2.13).

Another interesting subclass of the Kundt spacetimes of the form (2.13), (2.14) are explicit Petrov type *II* (or more special) metrics given by $\beta = \alpha$, $e = 1$, $C = 0$ and $D = 2(\Lambda - \alpha)$, namely

$$ds^2 = \Big[2(\Lambda - \alpha)\, v^2 - H\Big]\, \mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + \frac{1}{P^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \,. \tag{2.18}$$

For $H = 0$ these are electrovacuum solutions with the geometry of a direct product of two 2-spaces of constant curvature, in particular the Bertotti-Robinson, (anti-)Nariai or Plebański-Hacyan spaces [44–47]. Considering again (2.15) we obtain the results (2.16) as in the previous case. However, the reaction of high-frequency waves on the background is now different. It is determined by the equations (2.10) and (2.12) with the energy-momentum tensor consisting of a cosmological term plus that of a uniform non-null electromagnetic field described by the complex self-dual Maxwell tensor $F^{\mu\nu} = 4\Phi_1(m^{[\mu}\bar{m}^{\nu]} - k^{[\mu}l^{\nu]})$, where $\Phi_1 = \sqrt{\alpha - \frac{\Lambda}{2}}\, e^{i\,c}$, $c = const.$, and $\mathbf{m} = P\,\partial_{\bar{\zeta}}$, $\mathbf{k} = \partial_v$, $\mathbf{l} = \frac{1}{2}F\,\partial_v + \partial_u$ form the null tetrad. Straightforward calculation gives

$$P^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H = \tfrac{1}{4}\mathcal{A}^2(u, x, y)\dot{\phi}^2 \,. \tag{2.19}$$

Since the background spacetime is *not vacuum* but it contains electromagnetic field, we have to analyze the perturbation of the *complete* Einstein-Maxwell system, and its consistency.

The Einstein equations in the two highest orders (2.5) and (2.10) have already been solved. The Maxwell equations are also satisfied in the high-frequency limit, namely ${F^{\mu\nu}}_{|\nu} = O(\epsilon)$, where $|$ denotes the covariant derivative with respect to the full metric $g_{\mu\nu}$, because

$$ {F^{\mu\nu}}_{|\nu} = {F^{\mu\nu}}_{;\nu} - \tfrac{1}{2} h^{\alpha\beta} h_{\alpha\beta,\nu} F^{\mu\nu} + O(\epsilon^2) \ , \tag{2.20} $$

and considering that ${F^{\mu\nu}}_{;\nu} = 0$ (an electrovacuum background). In addition, the field equations are valid also in the next order $O(\epsilon)$ for the new electromagnetic field

$$ \mathcal{F}^{\mu\nu} = (1 + \tfrac{1}{4} h^{\alpha\beta} h_{\alpha\beta})\, F^{\mu\nu} \ , \tag{2.21} $$

since using (2.20) we obtain ${\mathcal{F}^{\mu\nu}}_{|\nu} = O(\epsilon^2)$. Both the Einstein and Maxwell equations are then satisfied in the two highest perturbative orders. Interestingly, these results hold for high-frequency perturbations of *any* "seed" electrovacuum background spacetimes.

2.2.2 Cylindrical waves

Next we consider the class of cylindrical Einstein-Rosen waves using the following metric in double null coordinates,

$$ ds^2 = e^{2\gamma - 2\psi}(-\mathrm{d}t^2 + \mathrm{d}\rho^2) + e^{2\psi}\mathrm{d}z^2 + \rho^2 e^{-2\psi}\mathrm{d}\varphi^2 \ . \tag{2.22} $$

These are exact radiative spacetimes of the Petrov type I (see, e.g. [26], [33], or equations (2.24)-(2.26) below).

We assume again $\phi = \phi(u)$ implying the wave vector $k_\mu = (\dot\phi, 0, 0, 0)$, i.e. the WKB perturbation of the form (2.15). By applying the conditions (2.11) we obtain

$$ \begin{aligned} \mathcal{A} &= \frac{\mathcal{U}(u)}{\sqrt{v-u}} \ , \\ e^{+}_{\mu\nu} &= \frac{1}{\sqrt{2}} e^{-2\psi} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(v-u)^2 & 0 \\ 0 & 0 & 0 & -e^{4\psi} \end{pmatrix} \ , \\ e^{\times}_{\mu\nu} &= \frac{1}{2}(v-u) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ ; \end{aligned} \tag{2.23} $$

notice that $v - u = \sqrt{2}\,\rho > 0$.

The back-reaction on the background (contained in a specific modification of the metric functions γ and ψ) is given by the following equations, cf. (2.12),

$$ (v-u)\,\psi_{,u}^2 + \gamma_{,u} = -\tfrac{1}{16}(v-u)\,\mathcal{A}^2\dot\phi^2 \ , \tag{2.24} $$

$$ (v-u)\,\psi_{,v}^2 - \gamma_{,v} = 0 \ , \tag{2.25} $$

$$ \psi_{,uv} - \frac{1}{2(v-u)}(\psi_{,v} - \psi_{,u}) = 0 \ . \tag{2.26} $$

This set of equations is *consistent* for the amplitude satisfying (2.23).

The above described perturbations depend on the null "retarded" coordinate u so that the high-frequency gravitational waves are *outgoing* (ρ is growing with t, on a fixed u). However, since the background metric (2.22) is invariant with respect to interchanging u with v, it is straightforward to consider also *ingoing* perturbations by assuming the phase to depend on the "advanced coordinate" v. This results in an interesting possibility to *introduce ingoing high-frequency gravitational cylindrical waves into the background of outgoing Einstein-Rosen waves or vice versa.*

Moreover, all the above results can further be extended to a class of generalized Einstein-Rosen (diagonal) metrics [34, 49] which describe G_2 *inhomogeneous cosmological models*,

$$ds^2 = e^{2\gamma-2\psi}(-\mathrm{d}t^2 + \mathrm{d}\rho^2) + e^{2\psi}\mathrm{d}z^2 + t^2 e^{-2\psi}\mathrm{d}\varphi^2 \ . \tag{2.27}$$

If the three-dimensional spacelike hypersurfaces are compact, the corresponding model is the famous Gowdy universe with the topology of three-torus [32, 34]. The only modification of the above results (in the double null coordinates) consists of replacing the factor $(v-u)$ with $(v+u)$, and each derivative with respect to u changing sign (e.g. $\gamma_{,u} \to -\gamma_{,u}$ or $\psi_{,uv} \to -\psi_{,uv}$).

2.2.3 Expanding waves

Finally, we assume that the background is an expanding Robinson-Trautman spacetime. The metric (generally of the Petrov type *II*) in the standard coordinates has the form, see e.g. [27, 28, 33, 43],

$$ds^2 = -\left(K - 2r(\ln\mathcal{P})_{,u} - 2\frac{m}{r} - \frac{\Lambda}{3}r^2\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \frac{r^2}{\mathcal{P}^2}(\mathrm{d}\eta^2 + \mathrm{d}\xi^2) \ , \tag{2.28}$$

where $K = \Delta(\ln\mathcal{P})$, $\Delta \equiv \mathcal{P}^2(\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2})$, and $m(u)$. When $\mathcal{P}(u,\eta,\xi)$ satisfies the Robinson-Trautman equation $\Delta K + 12\, m\,(\ln\mathcal{P})_{,u} - 4m_{,u} = 0$, the metric (2.28) is an exact vacuum solution of the Einstein equations.

In view of the existence of privileged congruence of null geodesics generated by ∂_r we introduce the phase $\phi = \phi(u)$. Applying the equations (2.11) we obtain

$$\mathcal{A} = \frac{1}{r}\, U(u,\eta,\xi) \ , \tag{2.29}$$

and the two polarization modes are easily inferred from (2.28).

The reaction of the waves on background is determined by the equations (2.10) and (2.12) with $T^{(0)}_{\mu\nu} = -\frac{1}{8\pi}\Lambda\gamma_{\mu\nu}$. From the only nontrivial component we immediately obtain the following equation

$$-\frac{\partial m}{\partial u} + 3\, m\,(\ln\mathcal{P})_{,u} + \tfrac{1}{4}\Delta K = \tfrac{1}{16}U^2\dot{\phi}^2 \ , \tag{2.30}$$

where $m(u)$, $\phi(u)$, whereas the remaining functions depend on coordinates $\{u,\eta,\xi\}$. Notice that this is *independent* of the cosmological constant Λ.

The expressions (2.29),(2.30) agree with results obtained by MacCallum and Taub [7] or recently by Hogan and Futamase [14] who used Burnett's technique [11].

Our results, which were derived by a straightforward approach, are slightly more general because they are not restricted to a constant frequency $\dot{\phi} = const$. Particular subcase of the Vaidya metric has already been studied before by Isaacson [3] and elsewhere [21].

2.3 General considerations

For construction of high-frequency gravitational perturbations we have employed the fact that all these spacetimes admit a non-twisting congruence of null geodesics. The corresponding tangent vectors k^μ are hypersurface orthogonal so that there exists a phase function ϕ which satisfies $\phi_{,\mu} = k_\mu$. The last equation in (2.11) can be put into the form $\frac{d}{dl}(\ln \mathcal{A}) = -\Theta$, where l is the affine parameter, and $\Theta = \frac{1}{2} k^\mu{}_{;\mu}$ is the expansion of the null congruence. This determines the behaviour of the amplitude $\mathcal{A}$ in the above spacetimes (2.16), (2.23), (2.29). The remaining equations (2.11) enables one to deduce the polarization tensors.

It has been also crucial that all the classes of spacetimes discussed admit *exact* solutions with the energy-momentum tensor of pure radiation, i.e., $G_{\mu\nu} - 8\pi\, T_{\mu\nu} = \frac{1}{8}\, \mathcal{A}^2 k_\mu k_\nu$, where $T_{\mu\nu}$ is either constant (representing the cosmological constant) or it describes an electromagnetic field. The relation between high-frequency perturbations and exact radiative solutions of Einstein's equations in each class is thus natural. In particular, it is possible to determine explicitly the reaction of the background on the presence of high-frequency gravitational waves.

Chapter 3

Spectra of high-frequency waves

In this part we will concentrate on solving the wave equation describing the propagation of high-frequency waves which was derived by Isaacson [3]. Although the complete Isaacson formalism incorporates also the reaction of the background to the wave, we will not consider this effect here. Rather, we will explicitly present spectra of high-frequency waves which may propagate in some fundamental cosmological models, in particular the Friedmann–Robertson–Walker spacetimes and in the anisotropic Kasner universe.

3.1 The Isaacson formalism

Isaacson's formalism [3] is based on the decomposition of the spacetime metric $g_{\mu\nu}$ into the background metric $\gamma_{\mu\nu}$ and its perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu} \ , \tag{3.1}$$

where, in a suitable coordinate system, $\gamma_{\mu\nu} = O(1)$ and $h_{\mu\nu} = O(\epsilon)$. By definition, $f = O(\epsilon^n)$ if there exists a constant $C > 0$ such that $|f| < C\epsilon^n$ as $\epsilon \to 0$. The quantity f need not necessarily be proportional to ϵ^n, it *can be even smaller* than $C\epsilon^n$ for $\epsilon \to 0$. Therefore, the assumption $h = O(\epsilon)$ does *not* automatically imply that $h \sim \epsilon$. The spectrum of possible high-frequency waves is thus not a priori restricted, it is only required that their amplitudes fall to zero at least linearly with ϵ, i.e. $|h(\epsilon)| < C\epsilon$.

The non-negative dimensionless parameter ϵ is the ratio of a typical wavelength λ of gravitational waves and the scale L on which the background curvature changes significantly. Isaacson's high-frequency approximation thus arises when $\lambda \ll L$, i.e. $\epsilon \ll 1$. Since L can be considered to have a finite value of order unity, we may write $O(\epsilon) = O(\lambda)$.

To derive the dynamical field equations we expand the Ricci tensor in powers of h,

$$R_{\mu\nu}(g) = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \dots \ . \tag{3.2}$$

Using the results from section 2.1 we obtain (in the high-frequency approximation ($\epsilon \ll 1$)) that the dominant term is $R^{(1)}_{\mu\nu} = O(\epsilon^{-1})$ which gives the wave equation $R^{(1)}_{\mu\nu}(\gamma, h) = 0$, i.e.

$$\gamma^{\rho\tau}\left(h_{\tau\mu;\nu\rho} + h_{\tau\nu;\mu\rho} - h_{\rho\tau;\mu\nu} - h_{\mu\nu;\rho\tau}\right) = 0 \ , \tag{3.3}$$

for the perturbations $h_{\mu\nu}$ on the curved background $\gamma_{\mu\nu}$ (considering the case of a vacuum full metric $g_{\mu\nu}$). The two terms of the order $O(1)$, namely $R^{(0)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$, can be used to give the equation for the background (non-vacuum) metric, which represents the essential influence of the high-frequency gravitational waves on the background.

Of course, to obtain a consistent solution, one has to use both the wave equation and the equation for the background simultaneously. We analyzed this problem explicitly in chapter 2 for spacetimes with preferred null directions, after simplification of the equations by the WKB approximation, see [15]. However, in this chapter we wish to concentrate on the equation (3.3). Our aim is to obtain spectra of high-frequency gravitational radiation propagating in an arbitrary direction in various cosmological models.

Now we impose the gauge conditions (2.6), (2.7). In this gauge the equation (3.3) reduces to the following wave equation,

$$\Diamond h_{\mu\nu} \equiv h_{\mu\nu}{}^{;\beta}{}_{;\beta} - 2R^{(0)}_{\sigma\nu\mu\beta}\, h^{\beta\sigma} - R^{(0)}_{\mu\sigma}\, h^{\sigma}{}_{\nu} - R^{(0)}_{\nu\sigma}\, h^{\sigma}{}_{\mu} = 0 \ , \tag{3.4}$$

where the operator $\Diamond$ is the generalization of flat-space d'Alembertian.

The gauge conditions (2.6),(2.7) still do not completely exhaust the gauge freedom, and we can thus demand the following additional condition,

$$h_{\mu 0} = 0 \ , \tag{3.5}$$

to simplify the calculations.

3.2 High-frequency waves in cosmological models

Now, we will investigate the solutions of the wave equation (3.4), subject to the gauge conditions (2.6) and (2.7), in some cosmologically relevant models with high degree of symmetry, namely the Friedmann–Robertson–Walker (FRW), anti–de Sitter, and anisotropic Kasner universes. As we shall see, the full spectrum of gravitational waves which propagate in an arbitrary direction is obtained explicitly for spacetimes with isotropic time slices (FRW models with $K = 0, 1, -1$) or with a spatial metric that is transformable to isotropic at each instant of time (Kasner), in contrast to anti–de Sitter universe which is globally only conformally isotropic in the metric form used below.

3.2.1 FRW models with spatial curvature $K = 0$

First, we will study spatially homogeneous and isotropic FRW spacetimes with a vanishing spatial curvature, and with the stress-energy tensor of an ideal fluid. As shown in [15], this tensor does not contain a derivative of the metric tensor, so that it satisfies the conditions of the Isaacson approximation generalized to the non-vacuum case, so that equations (2.6), (2.7), (3.4) are still valid.

In this special case of FRW spacetimes it is possible to write the metric using conformal time η in the usual form [19],

$$\mathrm{d}s^2 = a^2(\eta)\left(-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \ . \tag{3.6}$$

Next, we insert the covariant derivatives and the corresponding curvature tensor of this metric into the gauge conditions (2.6), (2.7) and in the wave equations (3.4). Using the additional freedom (3.5), the gauge condition $h^{\mu}{}_{\mu} = 0$ is simplified to $\bar{h} \equiv h^{i}{}_{i} = 0$ (considering latin indices to take the values $i, j, k = 1, 2, 3$ and using the (flat-space) summation convention over the same indices, but only when one of them is an upper and the other is a lower index), while the condition $h_{\mu\nu}{}^{;\nu} = 0$ implies

$$
\begin{aligned}
h^{i}{}_{i} &= 0 \quad (\text{for } \mu = 0) \ , \\
h^{i}{}_{j,i} &= 0 \quad (\text{for } \mu = j) \ .
\end{aligned} \tag{3.7}
$$

The only non-trivial components of the wave equation (3.4) can thus be put into the following form,

$$
a^2(-h_{ij,00} + h_{ij,k}{}^{k}) + 2a\dot{a}h_{ij,0} - 4\dot{a}^2 h_{ij} = 0 \ , \tag{3.8}
$$

and $\dot{a} = \frac{\partial a}{\partial \eta}$. The components $(0, 0)$ and $(0, i)$ of the wave equation are fulfilled identically due to the gauge conditions (3.7). Using (3.7) to also modify the dynamical equations (3.8), it is possible to transform equations for all the six non-zero components of the perturbation tensor into the *common form*,

$$
a^2\left(-\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2a\dot{a}\frac{\partial f}{\partial \eta} - 4\dot{a}^2 f = 0 \ , \tag{3.9}
$$

where $f(\eta, x, y, z)$ represents an arbitrary component h_{ij}. It is interesting to notice that the first gauge condition (3.7) restricts the number of independent components of the perturbation tensor to five which is in agreement with the number of independent components of a spin-2 field. Wave equation (3.9) can further be rewritten by introducing the covariant d'Alembertian operator,

$$
\Box f \equiv f^{;\mu}{}_{;\mu} = a^{-2}\left(-\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) - 2a^{-3}\dot{a}\frac{\partial f}{\partial \eta} \ ,
$$

into the form

$$
\Box f + \frac{4}{a^2}\left(\frac{f}{a}\right)^{\cdot} = 0 \ . \tag{3.10}
$$

By applying the Fourier transform in the coordinates $\vec{x} = (x, y, z)$,

$$
\tilde{f} \equiv \mathcal{F}[f](\eta, \vec{k}) = \int f(\eta, \vec{x}) \exp(i\vec{k} \cdot \vec{x}) \, d\vec{x} \ ,
$$

the equation (3.10) is converted to the form

$$
\left[\left(\frac{\partial^2}{\partial \eta^2} + |\vec{k}|^2\right) - 2\frac{\dot{a}}{a}\frac{\partial}{\partial \eta} + 4\frac{\dot{a}^2}{a^2}\right]\tilde{f} = 0 \ . \tag{3.11}
$$

An explicit solution of this second–order ordinary differential equation depends on the specific expansion function $a(\eta)$ which determines the background spacetime on which the waves propagate. The function $\tilde{f}(\eta, \vec{k})$, which is the solution of (3.11), represents a time dependent *spectrum of high-frequency gravitational perturbations*. Next, we will present the explicit solution for the particular case of the de Sitter spacetime.

Gravitational waves in the de Sitter spacetime

The de Sitter metric, which is the maximally symmetric spacetime with constant positive curvature $R = 4\Lambda$ when $\Lambda > 0$ is a cosmological constant, has in standard conformally flat coordinates the form (see e.g. [33])

$$\mathrm{d}s^2 = \frac{\alpha^2}{\eta^2}(-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \ , \tag{3.12}$$

where $\alpha = \sqrt{3/\Lambda}$. Therefore, in this case the expansion function is simply $a(\eta) = \alpha/\eta$.

Note, that the de Sitter manifold can be viewed as a four-dimensional hyperboloid embedded into five-dimensional flat spacetime. Depending on the choice of a specific spacelike section through this hyperboloid, one obtains all cases of FRW models of constant spatial curvature $K = 0, +1$ or -1, see [37, 50]. The metric (3.12) corresponds to the case $K = 0$.

Inserting this special form of the function $a(\eta)$ into equation (3.11) we obtain

$$\frac{\partial^2 \tilde{f}}{\partial \eta^2} + \frac{2}{\eta}\frac{\partial \tilde{f}}{\partial \eta} + \left(\frac{4}{\eta^2} + |\vec{k}|^2\right)\tilde{f} = 0 \ . \tag{3.13}$$

A general solution to this differential equation can be expressed using cylindric Bessel functions of the first kind J_ν and second kind Y_ν, with an imaginary index, namely

$$\tilde{f} = \frac{1}{\sqrt{\eta}}\left[A(\vec{k})J_{i\frac{\sqrt{15}}{2}}(|\vec{k}|\eta) + B(\vec{k})Y_{i\frac{\sqrt{15}}{2}}(|\vec{k}|\eta)\right] \ , \tag{3.14}$$

where A, B are arbitrary functions. This expression is in a complete agreement with the result obtained previously using synchronous coordinates for the de Sitter metric [51]. Note that synchronous coordinates cover only half ($\eta > 0$) of the de Sitter hyperboloid and therefore are not geodetically complete [50]. An inverse Fourier transform of equation (3.14) in the case of a monochromatic wave, $A(\vec{k}) = A_0\delta(\vec{k} - \vec{k_0})$, $B = B_0\delta(\vec{k} - \vec{k_0})$ leads to the following result

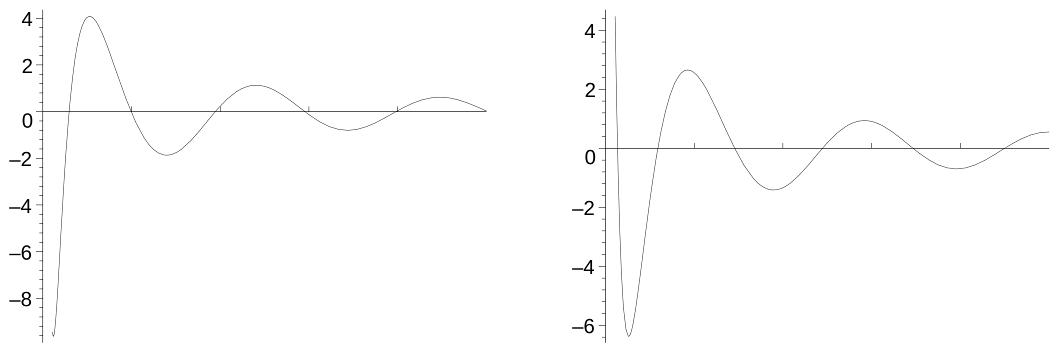
$$f = (2\pi)^{-3}\frac{1}{\sqrt{\eta}}\left[A_0 J_{i\frac{\sqrt{15}}{2}}(|\vec{k_0}|\eta) + B_0 Y_{i\frac{\sqrt{15}}{2}}(|\vec{k_0}|\eta)\right]e^{i\vec{k_0}\vec{x}} \ , \tag{3.15}$$

which represents a time evolution of the spectrum of high-frequency gravitational waves in the de Sitter "inflationary" universe. The typical plot for $|\vec{k_0}| = 1$ of the basic modes is given in figure 3.1.

3.2.2 Gravitational waves in the anti–de Sitter spacetime

Anti-de Sitter spacetime is a maximally symmetric spacetime with a constant negative curvature R=4Λ< 0. It may be viewed as a four-dimensional hyperboloid embedded into five-dimensional flat spacetime with metric signature (–,–,+,+,+), having thus two time axes [52]. We will use the conformally flat form of metric

$$ds^2 = \frac{\beta^2}{x^2}\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right) \ , \tag{3.16}$$



(a) (b)

Figure 3.1: The figure (a) is a plot of the function $Re\left\{\frac{1}{\sqrt{\eta}}J_{i\frac{\sqrt{15}}{2}}(\eta)\right\}$ and the figure (b) of the function $Re\left\{\frac{1}{\sqrt{\eta}}Y_{i\frac{\sqrt{15}}{2}}(\eta)\right\}$.

where $\beta = \sqrt{-3/\Lambda}$. These coordinates cover the whole manifold. It is easily seen that using the formal transformation $\hat{x} = i\eta$, $\hat{\eta} = ix$, $\hat{\alpha} = i\beta$ (i being the imaginary unit), and omitting the hats, we obtain the metric (3.12) of de Sitter spacetime. This offers the possibility to adopt the results obtained for the de Sitter spacetime, and to arrive at the spectrum of high-frequency perturbations for the anti–de Sitter spacetime. Unfortunately this would mean setting the components $h_{1\mu}$ of perturbation tensor to zero due to gauge condition $h_{0\mu} = 0$ applied in new coordinates. The general form of this condition is $h_{\mu\nu}v^{\nu} = 0$, where v^{ν} is the four–velocity of an observer. Therefore, the condition $h_{1\mu} = 0$ implies that the observer moves faster than the speed of light in the direction of $\frac{\partial}{\partial x}$ (in the coordinates of metric (3.16)). Moreover, the new coordinate $\hat{x}$ is purely imaginary and it would thus be impossible to use the Fourier transform.

Hence we will attempt to solve the problem directly using the metric (3.16) and assuming $h_{\mu 0} = 0$. The gauge condition $h^{\mu}{}_{\mu} = 0$ simplifies to the form (using the summation convention introduced in section 3.2.1)

$$h^{i}{}_{i} = 0 \ .$$

Non-trivial components of the gauge condition $h_{\mu\nu}{}^{;\nu}$ are the following

$$x\, h^{i}{}_{j,i} - 2h_{1j} = 0 \ .$$

Using the gauge conditions to simplify the dynamical equations (3.4) for perturbations $h_{\mu\nu}$ we obtain the following system

$$h_{11} = 0 \ , \quad h_{22} = -h_{33} \ , \quad h_{21,0} = 0 \ , \quad h_{31,0} = 0 \ ,$$

$$h_{21,2} + h_{31,3} = 0 \ ,$$
$$x^2\, h_{21,k}{}^{k} + 4h_{21} = 0 \ ,$$

$$
\begin{aligned}
x^2\, h_{31,k}{}^{k} + 4h_{31} &= 0\ , \\
x^2(-h_{22,00} + h_{22,k}{}^{k}) + 2xh_{22,1} - 4xh_{21,2} + 4h_{22} &= 0\ , \\
x^2(-h_{23,00} + h_{23,k}{}^{k}) + 2xh_{23,1} - 2x(h_{31,2} + h_{21,3}) + 4h_{23} &= 0\ .
\end{aligned} \tag{3.17}
$$

From equations (3.17) it is obvious that there are only two dynamical degrees of freedom corresponding to $h_{22} = -h_{33}$ and h_{23}. The residual non-trivial components h_{21} and h_{31} are independent of conformal time and therefore play the role of (supplementary) boundary conditions. The most natural choice is to put $h_{21} = 0 = h_{31}$. The solution of the set of equations (3.17) can be interpreted as a wave propagating in the direction $\frac{\partial}{\partial x}$ which is purely transversal and has two polarizations. In contradistinction to the de Sitter case, we do not obtain the same results for perturbations propagating in a general direction different from $\frac{\partial}{\partial x}$. This is a consequence of "anisotropy" of the anti–de Sitter spacetime in these coordinates.

Using the above choice of the boundary conditions we can write the following unified form of equation for both degrees of freedom $h_{22} = -h_{33}$ and h_{23},

$$
-\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{2}{x}\frac{\partial f}{\partial x} + \frac{4}{x^2} f = 0\ ,
$$

where f stands for h_{22} or h_{23}. Performing the following separation of variables $f(\eta, x, y, z) = g(x) \exp i(-k_0\eta + k_2 y + k_3 z)$, we obtain

$$
\frac{\partial^2 g}{\partial x^2} + \frac{2}{x}\frac{\partial g}{\partial x} + \left(\frac{4}{x^2} + k_1^2\right) g = 0\ , \tag{3.18}
$$

where $k_1^2 = k_0^2 - k_2^2 - k_3^2$. The equation (3.18) is formally equivalent to (3.13) (when replacing x with η, and k_1^2 with $|\vec{k}|^2$). Therefore, the solution is a monochromatic high-frequency gravitational wave

$$
f = \frac{1}{\sqrt{x}} \left[A\, J_{i\frac{\sqrt{15}}{2}}(k_1 x) + B\, Y_{i\frac{\sqrt{15}}{2}}(k_1 x) \right] \exp i(-k_0\eta + k_2 y + k_3 z)\ ,
$$

which is analogous to the wave (3.15) in the de Sitter spacetime.

Let us finally mention an interesting connection of the above result to exact gravitational waves in the anti–de Sitter spacetime described by the Defrise solution [33, 53]. The metric was investigated in [54] using the form

$$
\begin{aligned}
ds^2 \;=\;& \beta^2(d\theta^2 + \sinh^2\theta d\phi^2) + 8\beta^2(\cosh\theta + \sinh\theta\cos\phi)^2 du dv \\
& 16\beta^2(\cosh\theta + \sinh\theta\cos\phi)^4 d(u) du^2\ ,
\end{aligned} \tag{3.19}
$$

where $\theta \in [0, \infty)$, $\phi \in [0, 2\pi)$, u,$v \in (-\infty, +\infty)$. The wavefronts $u = const.$ are two-dimensional hyperbolic surfaces with constant negative curvature $-\beta$ parameterized by θ and ϕ. The solution (3.19) can be interpreted also in the perturbative sense. The background is represented by the metric (3.19) with $d(u) = 0$, and the component γ_{uu} of the metric proportional to $d(u)$ corresponds to high-frequency perturbations with small but rapidly varying function $d(u){=}O(\epsilon)$. The gauge conditions (2.6),(2.7) are fulfilled identically. The wave equation (3.4) is satisfied to the order $O(\epsilon)$, since each non-trivial component has the form $d(u)f(\theta, \phi)$. This is a satisfactory result implying that the exact solution (3.19) is consistent with the high-frequency Isaacson approximation.

3.2.3 FRW models with spatial curvatures $K = \pm 1$

FRW metrics with a positive or negative constant curvature of spatial sections may be written in the standard form [55]

$$\mathrm{d}s^2 = a^2(\eta)(-\mathrm{d}\eta^2 + {}^3\gamma_{ij}\mathrm{d}x_i\mathrm{d}x_j) \ , \tag{3.20}$$

where the tensor ${}^3\gamma_{ij}$ is the metric of homogeneous and isotropic three-space of uniform spatial curvature K, and the usual choice of coordinates leads to

$${}^3\gamma_{ij}\mathrm{d}x_i\mathrm{d}x_j = \mathrm{d}\chi^2 + f^2(\chi)[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2] \ , \tag{3.21}$$

with $f = \sin\chi$ for $K = 1$, and $f = \sinh\chi$ for $K = -1$. Let $\gamma_{\alpha\beta}$ and ; denote the FRW metric and the corresponding covariant derivative, respectively. The covariant derivative with respect to ${}^3\gamma_{ij}$ will be denoted by $|$.

To look for the solution of the wave equation (3.4) in the way similar to the flat-space FRW $K = 0$ models is complicated. Therefore, we consider a somewhat simplified form of the metric perturbations which is widely used in literature (see, e.g. [56]). The conformal time and spatial dependence of the perturbations are separated in the following way

$$h_{\mu\nu} = f(\eta)\,Q_{\mu\nu} \ , \tag{3.22}$$

where $Q_{\mu\nu}$ satisfies $Q_{\mu 0} = 0$, in accordance with the additional gauge (3.5). The spatial components of Q_{ij} form a traceless, divergenceless tensor (thus ensuring that $h_{\mu\nu}$ satisfies the gauge conditions (2.6),(2.7)) which is a solution of

$$Q_{ij|l}{}^{|l} + k^2\,Q_{ij} = 0 \ . \tag{3.23}$$

Such Q_{ij} are called a tensor harmonics, and the equation (3.23) is a generalized Helmholtz equation with k representing the wave number which sets the scale of the perturbations relative to the background coordinates. The expansion of perturbations into tensor harmonics was investigated from the mathematical point of view e.g. in [57].

To simplify the form of the curvature terms in the wave equation (3.4) one can use the well-known decomposition of the Riemann tensor [33]

$$R_{\sigma\mu\nu\beta} = C_{\sigma\mu\nu\beta} + \gamma_{\sigma[\nu}R_{\beta]\mu} - \gamma_{\mu[\nu}R_{\beta]\sigma} - \frac{1}{3}\gamma_{\sigma[\nu}\gamma_{\beta]\mu}R \ , \tag{3.24}$$

where $C_{\sigma\mu\nu\beta}$ is the traceless Weyl tensor. Using (3.24) and the fact that the spatial part R_{ij} of the Ricci tensor is a multiple of γ_{ij}, we derive that

$$R_{\sigma\mu\nu\beta}h^{\sigma\beta} = \frac{1}{2}\left(R^{\nu}{}_{\nu} + R^{\mu}{}_{\mu} - \frac{1}{3}R\right)h_{\mu\nu} \tag{3.25}$$

(no summation over μ, ν here). From the FRW metric (3.20) we obtain

$$R_{ij} = a^{-4}(\ddot{a}a + \dot{a}^2 + 2Ka^2)\gamma_{ij} \quad \text{and} \quad R = 6a^{-3}(\ddot{a} + Ka) \ ,$$

where $R = \gamma^{\mu\nu}R_{\mu\nu}$. Using the form (3.22) of the perturbation tensor $h_{\mu\nu}$ we derive the following form of the curvature terms in the wave equation (3.4),

$$2R_{\sigma\mu\nu\beta}h^{\sigma\beta} + R_{\mu\sigma}h^{\sigma}{}_{\nu} + R_{\nu\sigma}h^{\sigma}{}_{\mu} = (2a^{-3}\ddot{a} + 4a^{-4}\dot{a}^2 + 6Ka^{-2})fQ_{\mu\nu} \ . \tag{3.26}$$

For the covariant d'Alembertian of the perturbation tensor we can write

$$h_{\mu\nu}{}^{;\beta}{}_{;\beta} = f^{;\beta}{}_{;\beta} Q_{\mu\nu} + 2\dot{f} Q_{\mu\nu;0}\gamma^{00} + f(Q_{\mu\nu;00}\gamma^{00} + Q_{\mu\nu;ii}\gamma^{ii}) \,, \tag{3.27}$$

where

$$\begin{aligned} f^{;\alpha}{}_{;\alpha} &= -a^{-2}\ddot{f} - 2\dot{a}a^{-3}\dot{f} \quad , \quad Q_{\mu\nu;0} = -2\dot{a}a^{-1}Q_{\mu\nu} \,, \\ Q_{\mu\nu;00} &= (6\dot{a}^2a^{-2} - 2\ddot{a}a^{-1})Q_{\mu\nu} \quad , \quad Q_{\mu\nu;ii} = Q_{\mu\nu|ii} + 2\dot{a}^2a^{-4}\gamma_{ii}Q_{\mu\nu} \,. \end{aligned}$$

Combining equations (3.26), (3.27) and using the Helmholtz equation (3.23), the wave equation (3.4) is reduced to the second order ordinary differential equation for the amplitude of the perturbations depending on the conformal time,

$$\left[\left(\frac{\partial^2}{\partial\eta^2} + k^2\right) - 2\frac{\dot{a}}{a}\frac{\partial}{\partial\eta} + 4\frac{\dot{a}^2}{a^2} + 6K\right] f = 0 \,. \tag{3.28}$$

Notice that when $K = 0$, the equation (3.28) exactly reduces to (3.11), and we observe that in this case the parameter k defined in the Helmholtz equation (3.23) to the norm of the wavevector $\vec{k}$.

Example: waves in the (anti–) de Sitter spacetime

Now we will solve the equation (3.28) explicitly in three special cases of non-flat FRW spacetimes with the cosmological term.

We start with the de Sitter spacetime whose metric could be given in the FRW form with any value of spatial curvature K, see the beginning of section 3.2.1 and [50]. The value $K = 1$ for metric in the form (3.20) corresponds to spatial sections of the de Sitter hyperboloid being spheres S^3. The expansion function then takes the form

$$a(\eta) = \frac{\alpha}{\sin\eta} \,, \tag{3.29}$$

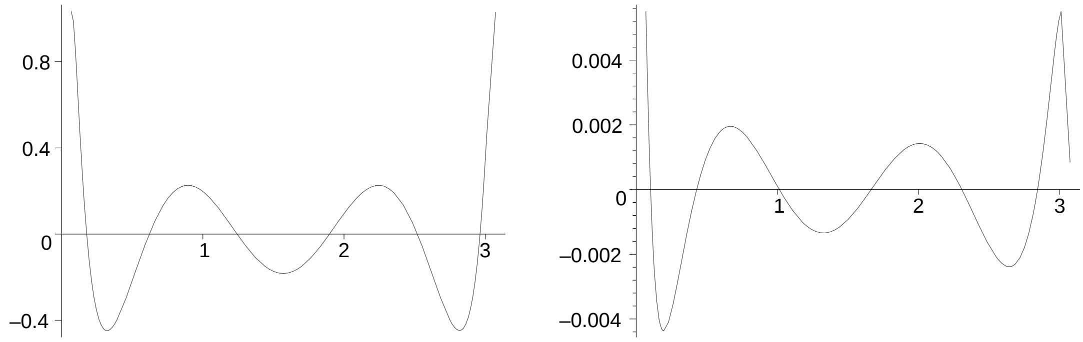
and the coordinates (3.21) cover the whole hyperboloid. Solution of the equation (3.28) with the expansion function (3.29) takes the form

$$\begin{aligned} f(\eta) \quad = \quad \frac{e^{i\pi/4}}{\sqrt{\sin\eta}} \Bigg[& C_1\, \mathrm{P}_l\left(\sqrt{3+k^2} - \frac{1}{2}, i\frac{\sqrt{15}}{2}; \cos\eta\right) \\ & + C_2\, \mathrm{Q}_l\left(\sqrt{3+k^2} - \frac{1}{2}, i\frac{\sqrt{15}}{2}; \cos\eta\right)\Bigg] \,, \end{aligned} \tag{3.30}$$

where $\mathrm{P}_l(u, v; z)$, resp. $\mathrm{Q}_l(u, v; z)$ are Legendre functions of the first, or of the second kind, respectively, which satisfy the differential equation

$$(1 - z^2)\, y'' - 2z\, y' + \left(v(v+1) - \frac{u^2}{1 - z^2}\right) y = 0 \,, \tag{3.31}$$

for $y(z)$. The points $z = 1, -1, \infty$ are singularities of this equation (except in special cases) and ordinary branch points of the Legendre functions in the complex domain. When we take the branch cuts to be $(-\infty, -1)$ and $(1, \infty)$, and if we compose Legendre functions with cosine function, as in (3.30), we obtain standard spherical



(a) (b)

Figure 3.2: The figure (a) is a plot of the function $Re\left\{\frac{1}{\sqrt{\sin\eta}}\,\mathrm{P}_l(\sqrt{19}-\frac{1}{2}, i\frac{\sqrt{15}}{2}; \cos\eta)\right\}$, and the figure (b) of the same expression, only with Q_l instead of P_l.

harmonics. The sample plot of the basic modes of the solution for $k = 4$ is presented in figure 3.2.

When $K = -1$, the spatial sections are hyperbolic and the expansion parameter is

$$a(\eta) = \frac{\alpha}{\sinh\eta} \;. \tag{3.32}$$

These coordinates cover only part of the hyperboloid. Solution of the equation (3.28) with the expansion function (3.32) is the following,

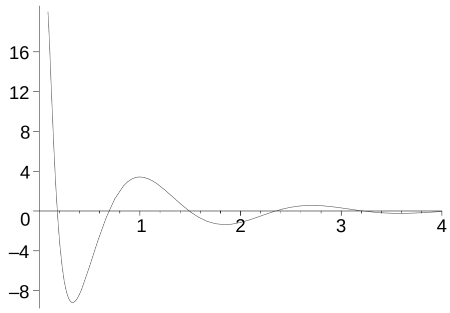
$$\begin{aligned} f(\eta) \;\; = \;\; & \frac{1}{\sqrt{\sinh\eta}}\left[C_1\,\mathrm{P}_l\left(\sqrt{3-k^2}-\frac{1}{2}, i\frac{\sqrt{15}}{2}; \cosh\eta\right)\right. \\ & \left. + C_2\,\mathrm{Q}_l\left(\sqrt{3-k^2}-\frac{1}{2}, i\frac{\sqrt{15}}{2}; \cosh\eta\right)\right] \;. \end{aligned} \tag{3.33}$$

When we take the branch cuts to be $(-\infty, -1)$ and $(-1, 1)$, and compose Legendre functions with hyperbolic cosine, as in (3.33), we obtain so called toroidal functions. The sample plot of the basic modes of the solution for $k = 4$ is given in figure 3.3.

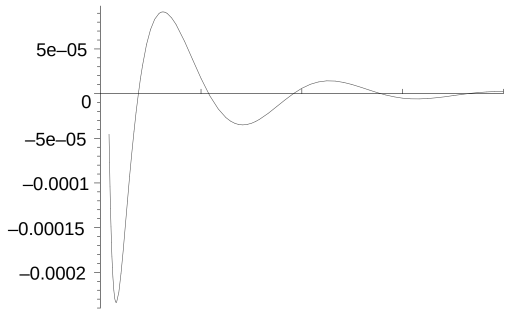
Finally, we give the solution for the anti–de Sitter spacetime represented by the FRW metric with $K = -1$ and the following expansion parameter [50],

$$a(\eta) = \frac{\beta}{\cosh\eta} \;. \tag{3.34}$$

The corresponding coordinates cover only part of the hyperboloid mentioned in the section 3.2.2 . Solution of the equation (3.28) with the expansion function (3.34) has



(a)



(b)

Figure 3.3: The figure (a) is a plot of the function $Re\left\{\frac{1}{\sqrt{\sinh\eta}}\,\mathrm{P}_l(\sqrt{-13}-\frac{1}{2}, i\frac{\sqrt{15}}{2}; \cosh\eta)\right\}$, and the figure (b) of the same expression, only with P_l replaced by Q_l.

the form

$$f(\eta) = \frac{1}{\sqrt{\cosh\eta}}\left[C_1\,\mathrm{P}_l\left(\sqrt{3-k^2}-\frac{1}{2}, i\frac{\sqrt{15}}{2}; i\sinh\eta\right)\right.$$
$$\left. +C_2\,\mathrm{Q}_l\left(\sqrt{3-k^2}-\frac{1}{2}, i\frac{\sqrt{15}}{2}; i\sinh\eta\right)\right] . \tag{3.35}$$

The sample plot of the basic modes for $k = 4$ is presented in 3.4.

3.2.4 Waves in the anisotropic Kasner universe

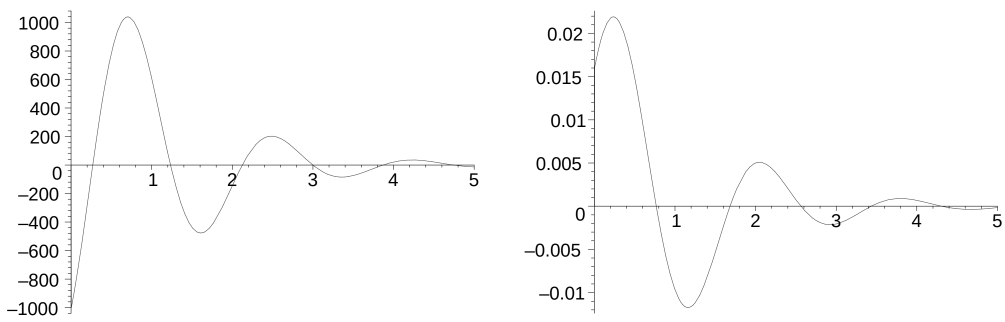
The Kasner universe is a special case of the Bianchi type I class of homogeneous but anisotropic spacetimes. Its metric in synchronous coordinates has the form [55]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p_1}\mathrm{d}x^2 + t^{2p_2}\mathrm{d}y^2 + t^{2p_3}\mathrm{d}z^2 , \tag{3.36}$$

where p_1, p_2, p_3 are constants. This metric represents a solution of vacuum Einstein's equations if the following relations hold:

$$p_1 + p_2 + p_3 = 1 , \quad {p_1}^2 + {p_2}^2 + {p_3}^3 = 1 . \tag{3.37}$$

However, in fact we need not assume these relations. We may consider the matter content of the universe described by the energy-momentum tensor which does not contain a derivative of the metric. This fulfills the conditions of a generalization of the Isaacson approximation to non-vacuum spacetimes, as described in [15]. Let us however mention that it has recently been shown [58, 59] that it is impossible to retain anisotropy when the Kasner universe is filled with a viscous fluid, dominant energy condition holds, and entropy is nondecreasing. However the anisotropy is permitted when it is filled with an ideal fluid satisfying the Zel'dovic equation of state.



(a) (b)

Figure 3.4: The figure (a) is a plot of the function $Re\left\{\frac{1}{\sqrt{\cosh\eta}}\,\mathrm{P}_l(\sqrt{-13}-\frac{1}{2}, i\frac{\sqrt{15}}{2}; i\sinh\eta)\right\}$, and the figure (b) of the same expression, only with P_l replaced by Q_l.

As in the previous calculations we will use $h_{0\mu} = 0$ as an additional condition. The traceless gauge condition has the form (using the summation convention defined in section 3.2.1)

$$t^{-2p_i} h^i{}_i = 0 \ . \tag{3.38}$$

The gauge condition $h_{\mu\nu}{}^{;\nu} = 0$ results in the equations

$$p_i\, t^{-2p_i} h^i{}_i = 0 \ , \quad t^{-2p_i} h^i{}_{j,i} = 0 \ . \tag{3.39}$$

Using these gauge conditions we can simplify the dynamical equations (3.4) to the following form (no summation over i, j in the second equation)

$$p_i\, t^{-2p_i} h^i{}_{j,i} = 0 \ , \tag{3.40}$$

$$-h_{ij,00}\, t^2 + \left(-\sum_k p_k + 2p_i + 2p_j\right) t\, h_{ij,0} + t^{(-2p_k+2)} h_{ij,k}{}^k - 4p_i p_j\, h_{ij} = 0 \ . \tag{3.41}$$

Using the covariant d'Alembertian, the differential equation (3.41) can be rewritten as

$$\Box\, h_{ij} + 2\frac{(p_i + p_j)}{t}\, h_{ij,0} - 4\frac{p_i p_j}{t^2}\, h_{ij} = 0 \ .$$

Let us denote an arbitrary component h_{ij} of the perturbation tensor simply as f (even though the wave equation (3.41) is different for different indices i, j) and let us define

$$A = 2(p_i + p_j) - \sum_k p_k \ , \qquad B = p_i p_j \ .$$

We will look for the solutions of (3.41) in the following special form,

$$f(t, x, y, z) = X(t, x) + Y(t, y) + Z(t, z) \ . \tag{3.42}$$

When we insert (3.42) into the wave equation (3.41), its left-hand side splits into three parts with each depending only on one spatial coordinate. The simplest possible solution is to equate each of these parts to zero, satisfying thus the equation. For example, in spatial coordinate x we obtain

$$\frac{\partial^2 X}{\partial t^2} - \frac{A}{t}\frac{\partial X}{\partial t} - t^{-2p_1}\frac{\partial^2 X}{\partial x^2} + \frac{4B}{t^2}X = 0 \; . \tag{3.43}$$

Applying now the one-dimensional Fourier transform in the coordinate x on the equation (3.43) we arrive at the ordinary differential equation

$$\frac{\mathrm{d}^2 \tilde{X}}{\mathrm{d}t^2} - \frac{A}{t}\frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} + t^{-2p_1}k_1^2\,\tilde{X} + \frac{4B}{t^2}\tilde{X} = 0 \; , \tag{3.44}$$

where $\tilde{X} = \mathcal{F}[X]$. Making an ansatz $\tilde{X} = t^{\frac{1}{2}(A+1)}F(t)$ and using the coordinate transformation $s = \frac{1}{1-p_1}k_1\,t^{1-p_1}$ we obtain the standard form of the Bessel equation

$$s^2\,\ddot{G} + s\,\ddot{G} + \left(\frac{16B-(A+1)^2}{4(1-p_1)^2} + s^2\right)G = 0 \; , \tag{3.45}$$

where $G(s) = F(t)$, and the dot denotes differentiation with respect to s. Solving (3.45) and transforming this back to $\tilde{X}$ and t, the solution of equation (3.43) takes the following form

$$\begin{aligned}\tilde{X} &= t^{\frac{1}{2}(A+1)}\\ &\times\left[C_1^+(k_1)\,J\left(\frac{\sqrt{(A+1)^2-16B}}{2(1-p_1)}, \frac{k_1 t^{(1-p_1)}}{1-p_1}\right) + C_1^-(k_1)\,Y\left(\frac{\sqrt{(A+1)^2-16B}}{2(1-p_1)}, \frac{k_1 t^{(1-p_1)}}{1-p_1}\right)\right] ,\end{aligned} \tag{3.46}$$

where $J(\nu, z)$, and $Y(\nu, z)$, is the Bessel function of the first kind, and of the second kind, respectively. For the *vacuum* Kasner universe (for which the relations (3.37) hold) we obtain $(A+1)^2 - 16B = 4(p_i - p_j)^2$, and thus the index of the Bessel functions is a real number. Generally, assuming that all p_i are positive, it turns out that for $\sum_k p_k < 1$ the index is always real, but for $\sum_k p_k > 1$ it might be imaginary.

Proceeding in the same way for the functions Y, and Z, the form of the solution (3.46) is reproduced except for the replacement of k_1, p_1, C_1^+, C_1^- with k_2, p_2, C_2^+, C_2^-, and k_3, p_3, C_3^+, C_3^-, respectively. The complete solution may thus be composed in the following way

$$f(t,x,y,z) = \mathcal{F}_3^{-1}\left[\tilde{X}(t,k_1)\,\delta(k_2)\delta(k_3) + \tilde{Y}(t,k_2)\,\delta(k_1)\delta(k_3) + \tilde{Z}(t,k_3)\,\delta(k_1)\delta(k_2)\right] \; , \tag{3.47}$$

where $\mathcal{F}_3^{-1}$ denotes the inverse Fourier transform in three dimensions, and δ denotes the Dirac delta function. The spectrum is then determined by the three functions $C_j^{\pm}(k_j)$, where $j = 1, 2, 3$. The monochromatic wave with the wavevector (k_1^0, k_2^0, k_3^0) is obtained by setting

$$C_j^{\pm}(k_j) = c_j^{\pm}\delta(k_j - k_j^0) \; ,$$

and has the form

$$\begin{aligned}f(t,x_1,x_2,x_3) &= t^{\frac{1}{2}(A+1)}\\ \times\sum_{j=1,2,3}&\left[c_j^+\,J\left(\frac{\sqrt{(A+1)^2-16B}}{2(1-p_j)}, \frac{k_j^0 t^{(1-p_j)}}{1-p_j}\right) + c_j^-\,Y\left(\frac{\sqrt{(A+1)^2-16B}}{2(1-p_j)}, \frac{k_j^0 t^{(1-p_j)}}{1-p_j}\right)\right] e^{ik_j^0 x_j} \; ,\end{aligned}$$

where $x_1 = x, x_2 = y, x_3 = z$.

Chapter 4

Radiative spacetimes approaching the Vaidya metric

The classic Vaidya metric [33,60–62] is a spherically symmetric type D solution of the Einstein equations in the presence of pure radiation matter field which propagates at the speed of light. In various contexts this "null dust" may be interpreted as high-frequency electromagnetic or gravitational waves, incoherent superposition of aligned waves with random phases and polarisations, or as massless scalar particles or neutrinos. The Vaidya solution depends on an arbitrary "mass function" $m(u)$ of the retarded time u which characterises the profile of the pure radiation (it is a "retarded mass" measured at conformal infinity).

In fact, the Vaidya spacetime belongs to a large Robinson–Trautman class of expanding nontwisting solutions [27, 28, 33]. Various aspects of this family have been studied in the last two decades. In particular, the existence, asymptotic behaviour and global structure of *vacuum* Robinson–Trautman spacetimes of type II with spherical topology were investigated, most recently in the works of Chruściel and Singleton [85–87]. In these rigorous studies, which were based on the analysis of solutions to the nonlinear Robinson–Trautman equation for generic, arbitrarily strong smooth initial data, the spacetimes were shown to exist globally for all positive retarded times, and to converge asymptotically to a corresponding Schwarzschild metric. Interestingly, extension across the "Schwarzschild-like" event horizon can only be made with a finite order of smoothness. Subsequently, these results were generalized in [88, 89] to the Robinson–Trautman vacuum spacetimes which admit a nonvanishing *cosmological constant* Λ. It was demonstrated that these cosmological solutions settle down exponentially fast to a Schwarzschild–(anti-)de Sitter solution at large times u.

Our aim here is to further extend the Chruściel–Singleton analysis of the Robinson–Trautman vacuum equation by including matter, namely *pure radiation*. It was argued already by Bičák and Perjés [90] that with $\Lambda = 0$ such spacetimes should generically approach the Vaidya metric asymptotically. We will analyze this problem in more detail, including also the possibility of $\Lambda \neq 0$.

4.1 The metric and field equations

In standard coordinates the Robinson–Trautman metric has the form [28, 33, 43]

$$\mathrm{d}s^2 = -\left(K - 2r(\ln P)_{,u} - 2\frac{m}{r} - \frac{\Lambda}{3}r^2\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2\frac{r^2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}\ , \tag{4.1}$$

where $K = \Delta(\ln P)$ with $\Delta \equiv 2P^2\partial_\zeta\partial_{\bar{\zeta}}$ being the Gaussian curvature of the 2-surfaces $2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}$, $m(u)$ is the mass function, and Λ is the cosmological constant. When the function $P(u,\zeta,\bar{\zeta})$ satisfies the fourth-order Robinson–Trautman field equation

$$\Delta K + 12\, m\,(\ln P)_{,u} - 4m_{,u} = 2\kappa\, n^2\ , \tag{4.2}$$

the metric describes a spacetime (generally of the Petrov type *II*) filled with pure radiation field $T_{\mu\nu} = n^2(u,\zeta,\bar{\zeta})\, r^{-2}\, k_\mu k_\nu$, where $\mathbf{k} = \partial_r$ is aligned along the degenerate principal null direction (we use the convention $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa\, T_{\mu\nu}$). In particular, vacuum Robinson–Trautman spacetimes are given by $n = 0$, in which case m can be set to a constant by a suitable coordinate transformation [33].

Here we will restrict ourselves to nonvacuum cases for which the dependence of the mass function $m(u)$ on the null coordinate u is only caused by a *homogeneous* pure radiation with the density $n^2(u)\, r^{-2}$. When the mass function $m(u)$ is decreasing, the field equation (4.2) can be naturally split into the following pair,

$$\begin{aligned} \Delta K + 12\, m(u)\,(\ln P)_{,u} &= 0\ , && (4.3)\\ -2\, m(u)_{,u} &= \kappa\, n^2(u)\ . && (4.4) \end{aligned}$$

In fact, it was demonstrated in [90] that such a separation can always be achieved using the coordinate freedom. It is then possible to reformulate equation (4.3) using $g_{ab} = f(u,\zeta,\bar{\zeta})^{-2} g^0_{ab}$, where $g^0_{ab}(\zeta,\bar{\zeta})$ is the metric on a 2-dimensional sphere S^2, and

$$P = f P_0\ , \quad P_0 = 1 + \tfrac{1}{2}\zeta\bar{\zeta}\ . \tag{4.5}$$

Then the equation (4.3) becomes

$$\frac{\partial f}{\partial u} = -\frac{1}{12m(u)}\, f\, \Delta K\ . \tag{4.6}$$

4.2 Linear mass function

Let us first consider the simplest choice of $m(u)$ which, in fact, has been widely used in literature (see e.g. [65, 67, 91]): we will assume that the mass function is a *linearly* decreasing positive function

$$m(u) = -\mu\, u, \qquad \mu = \mathrm{const} > 0\ , \tag{4.7}$$

on the interval $[u_0, 0]$. The constant value $u_0 < 0$ localises an initial null hypersurface on which an arbitrary sufficiently smooth *initial data* given by the function

$$f_0(\zeta,\bar{\zeta}) = f(u = u_0, \zeta, \bar{\zeta})\ , \tag{4.8}$$

are prescribed, see figure 4.1.

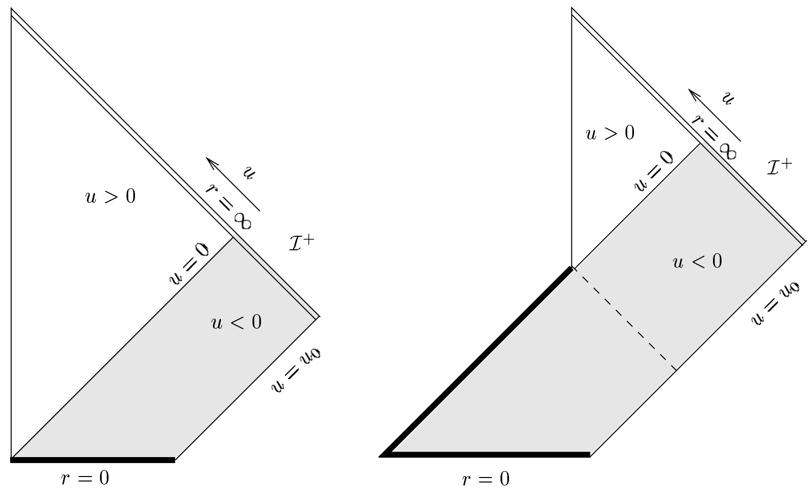


Figure 4.1: Schematic conformal diagrams of the Robinson–Trautman exact spacetimes which exist for any smooth initial data prescribed on u_0. Pure radiation field is present in the shaded region $u < 0$. Near $u = 0$ the solutions approach the Vaidya metric, and can be extended to flat Minkowski region $u > 0$. Thick line indicates the curvature singularity at $r = 0$ whereas double line represents future conformal infinity $\mathcal{I}^+$ at $r = \infty$ ($\Lambda = 0$ is assumed). The global structure depends on the value of the parameter μ of the linear mass function (4.7): left diagram corresponds to $\mu > 1/16$, the right one applies when $\mu \leq 1/16$.

4.2.1 Existence of the solutions

Now, the idea is to employ the Chruściel–Singleton results [85–87] concerning the analysis of the Robinson–Trautman *vacuum* equation, in particular the existence and asymptotic behaviour of its solutions. In the vacuum case m in equation (4.3) is constant, and the solution $f(u, \zeta, \bar{\zeta})$ of the characteristic initial value problem (4.8) exists and is unique (in spite of the singularity at $r = 0$). In the presence of pure radiation given by (4.7) it is possible to "eliminate" the variable mass function from the Robinson–Trautman field equation (4.6) mathematically by a simple reparametrisation

$$\tilde{u} = -\mu^{-1} \ln(-u) \ , \tag{4.9}$$

cf. [90]. Indeed, equation (4.6) is then converted to

$$\frac{\partial \tilde{f}}{\partial \tilde{u}} = -\frac{1}{12} \tilde{f} \, \tilde{\Delta} \tilde{K} \ . \tag{4.10}$$

Notice that the transformation (4.9) moves the hypersurface $u = 0$, on which the mass function $m(u)$ reaches zero, to $\tilde{u} = +\infty$.

Chruściel [86] derived the asymptotic expansion (as $\tilde{u} \to \infty$) for the function $\tilde{f}$ satisfying the evolution equation (4.10) for any smooth initial data $\tilde{f}_0 = f_0$ on $\tilde{u}_0 = -\mu^{-1} \ln(-u_0)$. In our case of pure radiation field (4.7) we employ the transformation (4.9) on Chruściel's original results to obtain the following asymptotic expansion of f as $u \to 0_-$,

$$\begin{aligned} f &= 1 + f_{1,0}\,(-u)^{2/\mu} + f_{2,0}\,(-u)^{4/\mu} + \cdots + f_{14,0}\,(-u)^{28/\mu} \\ &\quad -\mu^{-1} f_{15,1}\,\ln(-u)\,(-u)^{30/\mu} + f_{15,0}\,(-u)^{30/\mu} + \cdots \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{i,j}\,[-\mu^{-1}\ln(-u)]^j\,(-u)^{2i/\mu}\,, \end{aligned} \tag{4.11}$$

where $f_{i,j}$ are smooth functions on S^2 such that $f_{i,j} = 0$ for $j > 0$, $i \le 14$. As a result, for the initial data (4.8) the Robinson–Trautman type II spacetimes which contain uniform pure radiation field with the linear mass function (4.7) do exist in the whole region $u_0 \le u < 0$. It is also obvious that the function f approaches 1 as $u \to 0_-$ (where also $m(u) \to 0$) according to (4.11). In other words, these spacetimes *approach the spherically symmetric Vaidya–(anti-)de Sitter metric* near $u = 0$.

At $u = 0$ all of the mass $m(u)$ is radiated away, and we can attach Minkowski space (de Sitter space when $\Lambda > 0$, anti-de Sitter when $\Lambda < 0$; the presence of the cosmological constant would change the character of conformal infinity $\mathcal{I}$ which would become spacelike or timelike, respectively) in the region $u > 0$ along the hypersurface $u = 0$.

4.2.2 Extension of the metric across $u = 0$

It follows from (4.11) that the smoothness of f on $u = 0$ is only finite. Depending on the value of μ two different cases have to be discussed separately: $2/\mu$ is an integer, and $2/\mu$ is a real non-integer positive number.

When $2/\mu$ is an integer then due to the presence of the $\ln(-u)$ term associated with $f_{15,1} \neq 0$ the function f is of the class $C^{(30/\mu)-1}$. Note, that it is always at least C^{14} because $\mu \le 2$ in this case.

In the generic case when $2/\mu$ is not an integer the function f is only of the class $C^{\{2/\mu\}}$, where the symbol $\{x\}$ denotes the largest integer smaller than x. For $\mu > 2$ it is not even C^1 but it remains continuous.

To investigate further the smoothness of the metric when approaching the hypersurface $u = 0_-$ which is the analogue of the Schmidt–Tod boundary of vacuum Robinson–Trautman spacetimes [81,86] we should consider the conformal picture using suitable double-null coordinates. Such Kruskal-type coordinates for the Vaidya solution with linear mass function (4.7) were introduced by Hiscock [65–67]. Using his results, we put the Robinson–Trautman metric with linear mass function into the form

$$\begin{aligned} \mathrm{d}s^2 = &-\left(K - 1 - 2\frac{f_{,u}}{f}\,r\right)\mathrm{d}u^2 \\ &-\left(2r + u + 2\mu\frac{u^2}{r}\right)\mathrm{d}u\mathrm{d}w + 2\frac{r^2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}\,, \end{aligned} \tag{4.12}$$

where $r(u, w)$.

The general Robinson–Trautman metric (4.12) is evidently one order less smooth than f due to the presence of the function $f_{,u}/f$. Consequently, for $2/\mu$ being integer or non-integer number, the metric (4.12) is of the class $C^{(30/\mu)-2}$ or $C^{\{2/\mu\}-1}$, respectively.

We would like to obtain analogous results concerning smoothness of the extension also for a non-zero value of the cosmological constant Λ. Unfortunately, as far as we know, there is no *explicit* transformation of the Vaidya–de Sitter metric to the Kruskal-type coordinates even for the linear mass function. However, we can use a general argumentation: the coordinate u is already suitably compactified and we are only determining the complementary null coordinate w to obtain the Vaidya–de Sitter metric in the Kruskal-type coordinates (which is smooth on $u = 0$). Moreover $r(u, w)$ is finite and smooth when approaching the hypersurface $u = 0$. The smoothness is thus not affected by the specific transformation to the Kruskal-type coordinates and it is the same as for the vanishing cosmological constant. This is different from vacuum spacetimes with $m = \text{const} \neq 0$ studied in [88, 89] because in the present case $m \to 0$ near $u = 0$, and the influence of Λ on the smoothness becomes negligible.

4.3 General mass function

The results obtained above can be considerably generalized. Inspired by a similar idea outlined in [90] we may consider a reparametrisation on the null coordinate u by

$$\tilde{u} = \gamma(u) \ , \tag{4.13}$$

where γ is an arbitrary continuous strictly monotonous function. Now, by applying the substitution (4.13) in equation (4.10) we obtain

$$\frac{\partial f}{\partial u} = -\frac{\dot{\gamma}}{12} f \, \Delta K \ , \tag{4.14}$$

(where the dot denotes a differentiation) which is the evolution equation for the function $f(u, \zeta, \bar{\zeta})$. This is exactly the Robinson–Trautman equation (4.6) for the mass function

$$m(u) = \frac{1}{\dot{\gamma}(u)} \ . \tag{4.15}$$

To obtain a positive mass we assume a growing function $\gamma(u)$. Considering (4.4) this corresponds to a universe filled with homogeneous pure radiation

$$n^2(u) = \frac{2}{\kappa} \frac{\ddot{\gamma}}{\dot{\gamma}^2} \ . \tag{4.16}$$

For consistency the function γ must be convex.

In particular, the linear mass function (4.7) discussed above is a special case of (4.15) for the transformation (4.13) of the form (4.9). More general explicit solutions can be obtained, e.g., by considering the power function

$$\gamma(u) = (-u)^{-p} \ , \qquad p > 0 \ . \tag{4.17}$$

The asymptotic behaviour of such solutions is determined by expression (4.5) with

$$f = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{N_i} f_{i,j} \, (-u)^{-jp} \exp\left[-2i(-u)^{-p}\right] \ ,$$

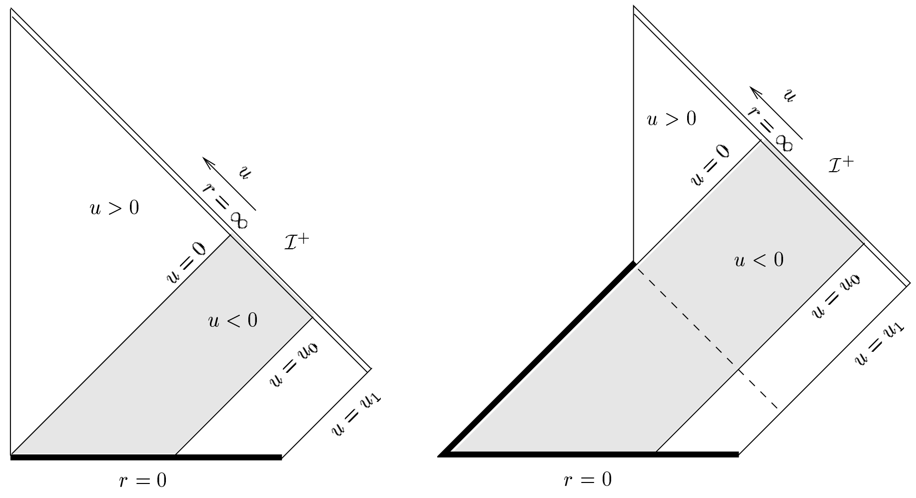


Figure 4.2: Possible extensions of the Robinson–Trautman radiative spacetimes into the region $u < u_0$. Pure radiation is present only in the shaded region, everywhere else it is a vacuum solution. For $u \in (u_1, u_0)$ the mass function is constant, $m(u_0) = -\mu\, u_0$, but the spacetime is not spherically symmetric — it is *not* the Schwarzschild solution ($\mu > 1/16$ on the left, $\mu \leq 1/16$ on the right).

where $f_{i,j} = 0$ for $j > 0$ if $i \leq 14$. Interestingly, the function f is now smooth on $u = 0$ for any power coefficient p.

Another simple explicit choice is

$$\gamma(u) = -M^{-1} \ln [\sinh(-u)] \ , \qquad M > 0 \ , \tag{4.18}$$

which implies the following expansion near $u = 0_-$

$$f = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{N_i} f_{i,j} \left(-M^{-1} \ln [\sinh(-u)]\right)^j \ \sinh^{2i/M}(-u) \ .$$

If $2/M$ is an integer then the function f belongs to the class $C^{(30/M)-1}$, otherwise it is of the class $C^{\{2/M\}}$.

4.4 Possible modifications and applications

The Robinson–Trautman pure radiation solutions in the region $u_0 \leq u \leq 0$ approaching the Vaidya metric near $u = 0$, which can be extended (albeit non-smoothly) to flat Minkowski space in the region $u \geq 0$ as in figure 4.1, may be used for construction of various models of radiative spacetimes. For example, it is natural to further extend the solution "backwards" into the region $u_1 < u \leq u_0$ by the Robinson–Trautman *vacuum* solution with a constant mass $m_0 = m(u_0)$, such that the function f is continuous on u_0. This is shown in figure 4.2.

In the presence of the cosmological constant Λ the schematic conformal diagram on figure 4.2 has to be modified in such a way that for all values of u the conformal infinity $\mathcal{I}^+$ becomes timelike (for $\Lambda > 0$) or spacelike (for $\Lambda < 0$).

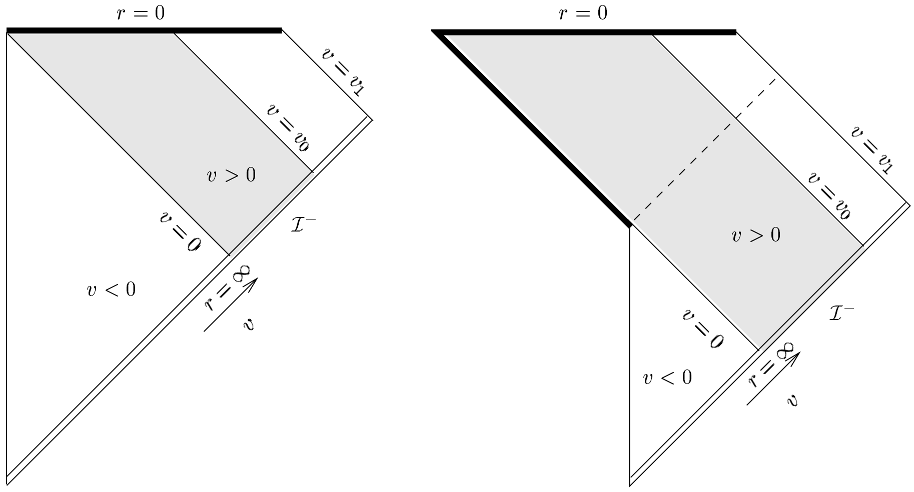


Figure 4.3: Time-reversed version of figure 4.2 represents the "advanced" form of the Robinson–Trautman spacetimes which describes an ingoing flow of radiation.

Another possible modification is to consider the "advanced" form of the spacetimes (which describes an ingoing flow) rather than the "retarded" form (corresponding to outgoing flow) employed above (see, e.g., [71] for more details). This time-reversed form is obtained formally by a simple substitution $u \to -v$ in the metrics and corresponding functions. In this case $m(v)$ is an increasing mass function in $v \in [0, v_0]$. This is joined with flat Minkowskian region $v < 0$, and extended to the region $v \geq v_0$ by the corresponding Robinson–Trautman–(anti-)de Sitter black hole vacuum solution, see figure 4.3. In analogy with (4.11), we obtain

$$f = \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} f_{i,j} \, \left(-\mu^{-1} \ln v\right)^j \, v^{2i/\mu} \ , \tag{4.19}$$

so that the smoothness of the metric on the boundary $v = 0$ depends on the parameter μ. For $v \in (v_0, v_1)$ the spacetime is vacuum but not spherically symmetric. The metric diverges as $v \to \infty$. Our results can thus be interpreted in such a way that — at least within the Robinson–Trautman family of solutions — the model [67] of collapse to a naked shell-focusing singularity which is based on the spherically symmetric Vaidya metric *is not stable* against perturbations.

Conclusion

In the first chapter, we have compared the Efroimsky [1,2] and the Isaacson [3] self-consistent perturbation schemes which describe propagation of weak gravitational waves on a cosmological background. In both these approaches the background is influenced by the waves, i.e. the non-linear effects are taken into account. The classical Isaacson method applies to high-frequency waves. On the other hand, the Efroimsky formalism is applicable to low-frequency gravitational waves but does not admit the high-frequency limit. We have suggested a modification of the Efroimsky formalism by employing the gauge-invariant decomposition (1.9) of the Ricci tensor, introduced recently by Anderson [16]. The resulting generalized system of equations (1.12)-(1.14) fully recovers the Efroimsky results in the absence of high-frequency modes, in the high-frequency limit it reproduces Isaacson's formulae.

In the second chapter, the Isaacson approach [3] to study high-frequency perturbations of Einstein's equations was briefly reviewed and compared with the standard weak-field limit. In our contribution we generalized Isaacson's method to include non-vacuum spacetimes, in particular an electromagnetic field and/or a non-vanishing value of the cosmological constant Λ. Then we explicitly analyzed possible high-frequency gravitational waves in three large families of background universes, namely non-expanding spacetimes of the Kundt type, cylindrical Einstein-Rosen waves and related inhomogeneous cosmological models (such as the Gowdy universe), and the Robinson-Trautman expanding spacetimes. These backgrounds are of various Petrov types. For example, high-frequency gravitational waves can be introduced into electrovacuum conformally flat Bertotti-Robinson space, type D Nariai and Plebański-Hacyan spaces, their type N and type II generalizations, or into algebraically general Einstein-Rosen universes.

In the third chapter we have investigated the spectra of high-frequency waves propagating on several important cosmological models using the wave equation derived by Isaacson [3]. It was demonstrated that the application of tensor harmonics, inspired by Bardeen [56], considerably simplifies the solution of the wave equation for the non–flat FRW models. It turns out that the explicit solutions of the wave equation are expressed using special (Bessel and Legendre) functions containing purely imaginary indices.

In the last chapter we have analyzed exact solutions of the Robinson–Trautman class which contain homogeneous pure radiation and a cosmological constant. We have demonstrated that these solutions exist for any smooth initial data, and that they approach the spherically symmetric Vaidya–(anti-)de Sitter metric. It generalizes previous results according to which vacuum Robinson–Trautman spacetimes approach asymptotically the spherically symmetric Schwarzschild–(anti-)de Sitter metric. We have investigated extensions of these solutions into Minkowski region, and we have shown that its order of smoothness is in general only finite. Finally, we suggested some applications of the results. For example, it follows that the model of gravitational collapse of a shell of null dust diverges as $v \to \infty$ which indicates that investigations of such process based on the spherically symmetric Vaidya metric are, in fact, not stable against "non-linear perturbations", at least within the Robinson–Trautman family of exact solutions.

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Appendices

Svítek O and Podolský J 2004 *The Efroimsky formalism for weak gravitational waves adapted to high-frequency perturbations* Class. Quantum Grav. **21** 3579-3585

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The Efroimsky formalism for weak gravitational waves adapted to high-frequency perturbations

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Abstract

The Efroimsky perturbation scheme for consistent treatment of gravitational waves and their influence on the background is summarized and compared with the classical Isaacson high-frequency approach. We demonstrate that the Efroimsky method in its present form is not compatible with the Isaacson limit of high-frequency gravitational waves, and we propose its natural generalization to resolve this drawback.

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1. Introduction

Recently, Efroimsky introduced and developed a new formalism for the consistent treatment of weak gravitational waves [1, 2]. This interesting mathematical framework is remarkable, mainly due to the possibility of ascribing the stress–energy tensor even to low-frequency gravitational waves influencing the background, which is in contrast to the standard linearization approach where the background is kept fixed. This is achieved by introducing a natural low-frequency cut-off, employing three different metrics (the premetric, the complete physical metric and the average metric) and careful analysis of their mutual relations.

On the other hand, in a now classic paper [3] Isaacson (inspired by previous works [4, 5]) presented a perturbation method which can be used for studies of high-frequency gravitational waves. Such waves also influence the cosmological background in which they propagate. Isaacson's work stimulated further contributions in which his method was reformulated using various formalisms, and explicitly applied to particular spacetimes (see, e.g., [6–15]).

In our present work we first briefly summarize and compare the two above-mentioned perturbation schemes. In particular, it is shown that the Efroimsky method is not consistent if high-frequency gravitational waves are considered. Next (in section 3), we propose a possible modification of the Efroimsky formalism which may resolve this drawback.

2. The formalism

Efroimsky's approach [1, 2] is based on introducing three different smooth, non-degenerate, symmetric metrics on a differentiable manifold M, namely:

(i) $\gamma_{\mu\nu}$—the 'premetric', a vacuum metric corresponding to initial pure background without gravitational waves;
(ii) $g_{\mu\nu}$—the 'physical metric', a full vacuum metric which describes both the background and the waves;
(iii) $q_{\mu\nu}$—the 'average metric', a nonvacuum metric representing the background plus its perturbations with wavelength greater than L. In fact, it is the averaged full metric $g_{\mu\nu}$, where the cut-off value L depends on the observer's experimental abilities. Since no detector can measure gravitational waves of arbitrarily long wavelengths, the existence of such a low-frequency cut-off is a natural assumption.

One motivation for using these three distinct metrics is to resolve a (slight) discrepancy in the standard linearization approach which considers only the metrics $\gamma_{\mu\nu}$, $g_{\mu\nu}$, and decomposition $g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small perturbation. The contravariant components obtained as an inverse of $g_{\mu\nu}$ are $g^{\mu\nu} = \gamma^{\mu\nu} - h^{\mu\nu} + O(h^2)$, but $\gamma_{\mu\nu}$ is commonly used for raising and lowering indices. It is thus not clear which semi-Riemannian manifold this equality relates to. Such inconsistency can be ignored in the lowest order because it leads to the correct linear approximation of the wave equation. To extend the weak-field formalism to higher-order terms, the distinction between the premetric γ and the average metric q is necessary as it exhibits the back-reaction of the waves on the background geometry. (Here and hereafter, indices of the metric tensors are sometimes suppressed for notational simplicity.)

The next step is to define the Ricci and Einstein tensors for an arbitrary metric g as

$$\begin{aligned} R_{\mu\nu}(g) \equiv & \left[\tfrac{1}{2}g^{\gamma\rho}(g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho})\right]_{,\gamma} - \left[\tfrac{1}{2}g^{\gamma\rho}(g_{\rho\gamma,\mu} + g_{\rho\mu,\gamma} - g_{\mu\gamma,\rho})\right]_{,\nu} \\ & + \left[\tfrac{1}{2}g^{\gamma\delta}(g_{\rho\delta,\gamma} + g_{\rho\gamma,\delta} - g_{\gamma\delta,\rho})\right]\left[\tfrac{1}{2}g^{\delta\rho}(g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho})\right] \\ & - \left[\tfrac{1}{2}g^{\gamma\rho}(g_{\rho\delta,\nu} + g_{\rho\nu,\delta} - g_{\nu\delta,\rho})\right]\left[\tfrac{1}{2}g^{\delta\rho}(g_{\rho\gamma,\mu} + g_{\rho\mu,\gamma} - g_{\mu\gamma,\rho})\right], \\ G_{\mu\nu}(g) \equiv & R_{\mu\nu}(g) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}(g), \end{aligned} \tag{1}$$

where $g^{\rho\tau} = (g)^{-1}_{\rho\tau}$; the same expressions apply to γ and q. These equations remain a tensor even if we transfer to another semi-Riemann space (the reason is that covariant tensors are defined on a metric space rather than on some particular semi-Riemann one). From proposals (i)–(iii) it follows that $G_{\mu\nu}(\gamma) = 0 = G_{\mu\nu}(g)$, $G_{\mu\nu}(q) \neq 0$.

Now, the *differences* between the covariant components of the above metrics are introduced,

$$h_{\mu\nu} \equiv g_{\mu\nu} - q_{\mu\nu}, \qquad \eta_{\mu\nu} \equiv q_{\mu\nu} - \gamma_{\mu\nu}. \tag{2}$$

It is necessary to specify the semi-Riemann space: for raising or lowering indices and for covariant differentiation, the averaged nonvacuum metric q will be used. Consequently, h and η are tensor fields on the semi-Riemann manifold (M, q), i.e.

$$h^{\mu\nu} \equiv q^{\mu\alpha}q^{\nu\beta}h_{\alpha\beta}, \qquad \eta^{\mu\nu} \equiv q^{\mu\alpha}q^{\nu\beta}\eta_{\alpha\beta}. \tag{3}$$

Treating $h_{\mu\nu}$ as a perturbation of the metric $q_{\mu\nu}$ the Ricci tensor (1) can be expanded in a power series

$$R_{\mu\nu}(g) = R^{(0)}_{\mu\nu}(q) + R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) + R^{(3)}_{\mu\nu}(q,h) + O(h^4), \tag{4}$$

where

$$
\begin{aligned}
&R^{(0)}_{\mu\nu}(q) \equiv R_{\mu\nu}(q),\\
&R^{(1)}_{\mu\nu}(q,h) \equiv \tfrac{1}{2}q^{\rho\tau}(h_{\tau\mu;\nu\rho} + h_{\tau\nu;\mu\rho} - h_{\rho\tau;\mu\nu} - h_{\mu\nu;\rho\tau}),\\
&R^{(2)}_{\mu\nu}(q,h) \equiv \tfrac{1}{2}\left[\tfrac{1}{2}h^{\rho\tau}{}_{;\mu}h_{\rho\tau;\nu} + h^{\rho\tau}(h_{\rho\tau;\mu\nu} + h_{\mu\nu;\rho\tau} - h_{\tau\mu;\nu\rho} - h_{\tau\nu;\mu\rho})\right.\\
&\qquad\left. + h^{\tau}{}_{\nu}{}^{;\rho}(h_{\tau\mu;\rho} - h_{\rho\mu;\tau}) - \left(h^{\rho\tau}{}_{;\rho} - \tfrac{1}{2}h_{\rho}^{\rho;\tau}\right)(h_{\tau\mu;\nu} + h_{\tau\nu;\mu} - h_{\mu\nu;\tau})\right].
\end{aligned}
\tag{5}
$$

Analogously,

$$R_{\mu\nu}(\gamma) = R^{(0)}_{\mu\nu}(q) + R^{(1)}_{\mu\nu}(q,(-\eta)) + R^{(2)}_{\mu\nu}(q,(-\eta)) + O(\eta^3). \tag{6}$$

It is obvious that $R^{(1)}_{\mu\nu}(q,(-\eta)) = -R^{(1)}_{\mu\nu}(q,\eta)$ and $R^{(2)}_{\mu\nu}(q,(-\eta)) = R^{(2)}_{\mu\nu}(q,\eta)$. According to assumptions that both g and γ are vacuum metrics the following relation holds:

$$
\begin{aligned}
0 &= R_{\mu\nu}(g) - R_{\mu\nu}(\gamma)\\
&= R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) + R^{(1)}_{\mu\nu}(q,\eta) + R^{(3)}_{\mu\nu}(q,h) + O(h^4) + O(\eta^2).
\end{aligned}
\tag{7}
$$

At this point Efroimsky sets three assumptions:

Assumption 1. The perturbations h and η are small in the sense that the terms of the orders $O(h^4)$ and $O(\eta^2)$ are negligible.

Assumption 2. The perturbations η and h^2 are of the same order.

Assumption 3. The tensor field h consists of modes with short wavelengths which do not exceed the given maximal value L.

A physical interpretation of the perturbations given by (2) is thus the following: $h_{\mu\nu}$ characterizes measurable gravitational waves whereas $\eta_{\mu\nu}$ is a shift of the background geometry from vacuum premetric γ to nonvacuum effective average metric q due to the presence of gravitational waves. This enables us to interpret equation (7) as the wave equation for perturbations h on the background $q = \gamma + \eta$. To make this wave equation applicable, one has to express η in terms of h. Using the Brill–Hartle averaging procedure [5] over a spacetime volume of size L for (7) (Efroimsky considers only space averaging but when the measurement lasts much longer than the period of waves one can employ a spacetime average) we obtain

$$R^{(1)}_{\mu\nu}(q,\eta) = -\left\langle R^{(2)}_{\mu\nu}(q,h)\right\rangle_L. \tag{8}$$

The averaging brackets on the left-hand side are omitted because the term contains only the modes with wavelength greater than L. It is thus clear from (8) and (5) that assumption 2 is natural since the left-hand side is linear in η whereas the right-hand side is quadratic in h.

Let us finally recall the derivation of the stress–energy tensor of gravitational waves. By analogy with the Ricci tensor expansion (4) the Einstein tensor of the vacuum premetric γ is represented as a series

$$0 = G_{\mu\nu}(\gamma) = G_{\mu\nu}(q) + G^{(1)}_{\mu\nu}(q,(-\eta)) + O(\eta^2), \tag{9}$$

and the effective stress–energy tensor of gravitational waves is defined as

$$G_{\mu\nu}(q) = 8\pi T^{(gw)}_{\mu\nu} \equiv R^{(1)}_{\mu\nu}(q,\eta) - \tfrac{1}{2}q_{\mu\nu}q^{\alpha\beta}R^{(1)}_{\alpha\beta}(q,\eta). \tag{10}$$

From (8) it follows (considering the Brill–Hartle averaging) that this tensor fully agrees with that of Isaacson [3].

The main advantage of Efroimsky's perturbation method above is the possibility of consistently treating all low-frequency gravitational waves, and of explicitly deriving an effective stress–energy tensor (influencing the background) in this case. It can be extended to nonvacuum spacetimes with $T_{\mu\nu}$ of ideal fluid and/or with a possible cosmological constant Λ, see [1, 2]. However, there are some problems concerning high-frequency gravitational waves which will now be discussed.

3. Modification to include high-frequency waves

In this section we first explicitly demonstrate that one cannot consistently apply Efroimsky's treatment on Isaacson's high-frequency waves [3] because assumption 2 is not fulfilled in such a case. Then we will present a possible solution to this problem.

Let us start with the observation that it is the nonvacuum background curved by the presence of gravitational waves—not the vacuum premetric γ—which is the basis of Isaacson's nonlinear approach. Therefore, the nonvacuum average metric q is considered as the background on which high-frequency gravitational waves h propagate.

We wish to use the Efroimsky formalism in the high-frequency regime such that the tensor field h contains high-frequency modes. We assume that they have short wavelengths λ, and a small amplitude $h = O(\varepsilon)$, where $\varepsilon = \lambda/S \ll 1$ is a small parameter because $\lambda \ll S$, S denoting a typical scale on which the background changes substantially.

Let us emphasize that we follow here the same definition of the symbol $O(\varepsilon^n)$ as in [3], namely $f = O(\varepsilon^n)$ if there exists a constant $C > 0$ such that $|f| < C\varepsilon^n$ as $\varepsilon \to 0$. The quantity f need not necessarily be proportional to ε^n, it *can be even smaller* than $C\varepsilon^n$ for $\varepsilon \to 0$. Therefore, the assumption $h = O(\varepsilon)$ does *not* automatically imply that $h \sim \varepsilon$. The spectrum of possible high-frequency waves is thus not *a priori* restricted, it is only required that their amplitudes fall to zero at least linearly with ε, i.e. $|h(\varepsilon)| < C\varepsilon$.

Since we can consider $S = O(1)$ it follows that $O(\varepsilon) = O(\lambda)$ and $\partial h \sim h/\lambda = O(1)$. In accordance with Isaacson's approach (note that the decomposition now reads $g = q + h$, instead of the notation $g = \gamma + h$ used in [3]) we obtain the following orders of magnitude for the derivatives of the background q and the perturbation h:

$$\begin{aligned} q_{\mu\nu} &= O(1), & h_{\mu\nu} &= O(\varepsilon), \\ q_{\mu\nu,\alpha} &= O(1), & h_{\mu\nu,\alpha} &= O(1), \\ q_{\mu\nu,\alpha\beta} &= O(1), & h_{\mu\nu,\alpha\beta} &= O(\varepsilon^{-1}). \end{aligned} \tag{11}$$

This results in the orders of magnitude of the terms in the Ricci tensor expansion (4), (5) as

$$R^{(0)}_{\mu\nu} = O(1), \qquad R^{(1)}_{\mu\nu} = O(\varepsilon^{-1}), \qquad R^{(2)}_{\mu\nu} = O(1), \qquad R^{(3)}_{\mu\nu} = O(\varepsilon). \tag{12}$$

To apply the Efroimsky approach in this case we must consider the decomposition $q = \gamma + \eta$, where γ is the vacuum premetric and η represents (in this case) a substantial shift of the background geometry due to the presence of high-frequency gravitational waves h. We also introduce the scale L, such that $\lambda \ll L \ll S$. This enables us simultaneously to consider an averaging procedure in accordance with the Isaacson approach, and also to introduce a meaningful cut-off scale L even if the wavelengths of high-frequency waves are not assumed to reach this value.

Of course, the geometry shift η does not contain high-frequency perturbations. Considering the wave equation (7) and using the Brill–Hartle averaging over a spacetime volume L to obtain equation (8) we get into a conflict with assumption 2 which prescribes $O(\eta) = O(h^2)$. Indeed, if $h = O(\varepsilon)$ there should be $\eta = O(\varepsilon^2)$. But the right-hand side of (8) is now of the order of $O(1)$, see (12), and the same magnitude should also have the left-hand side. Since η does not contain high-frequency waves, it is essential that $\eta = O(1)$. This is obviously in contradiction with both assumptions 1 and 2. In fact, it disables any consistent perturbation expansions in the powers of η.

Let us now suggest a modification of the Efroimsky formalism which will incorporate also the above case of a 'substantial' change of the background geometry due to the presence of high-frequency waves. Instead of the perturbation expansion (6) we consider a formal decomposition of the Ricci tensor of the premetric $\gamma = q - \eta$, namely

$$0 = R_{\mu\nu}(\gamma) = R_{\mu\nu}(q) + \Delta R_{\mu\nu}(q, (-\eta)), \tag{13}$$

by which the expression $\Delta R_{\mu\nu}$ is *defined*. Both terms on the right-hand side of (13) are of the same order $O(1)$. Moreover, the quantity $\Delta R_{\mu\nu}$ is conserved with respect to the background geometry q which is easily seen from equation (13) and the relation $(R_{\mu\nu}(q))^{;\nu} = 0$ (the differentiation relates to the background metric q).

The question concerning the gauge invariance of $\Delta R_{\mu\nu}$ with respect to generalized gauge transformations has recently been analysed in detail by Anderson [16] in connection with possible definitions of the wave equation and stress–energy tensor for gravitational waves. Let us consider an arbitrary coordinate transformation of the type

$$\overline{x}^{\mu} = x^{\mu} + \xi^{\mu}, \tag{14}$$

which does not change the functional form of the background geometry q, i.e. $\overline{q}(\overline{x}) = q(\overline{x})$ so that $\gamma(x) \to \overline{\gamma}(\overline{x}) = q(\overline{x}) - \overline{\eta}(\overline{x})$. Now, to prove the invariance of $\Delta R_{\mu\nu}$ we adopt (slightly modified) Anderson's argumentation. Performing the above coordinate transformation (14) of the Ricci tensor decomposition (13) we obtain

$$\overline{R}_{\mu\nu}(q(\overline{x})) + \overline{\Delta R}_{\mu\nu}(q(\overline{x}), (-\overline{\eta}(\overline{x}))) = \overline{R}_{\mu\nu}(\overline{\gamma}(\overline{x})) = 0. \tag{15}$$

Here $\overline{R}_{\mu\nu}$ and $\overline{\Delta R}_{\mu\nu}$ are the same as $R_{\mu\nu}$ and $\Delta R_{\mu\nu}$, respectively, because definition (1) is maintained in any coordinate. Evaluating relation (15) at $\overline{x} = x$ we thus get $R_{\mu\nu}(q(x)) = -\Delta R_{\mu\nu}(q(x), (-\overline{\eta}(x)))$, and using (13) we obtain

$$\Delta R_{\mu\nu}(q(x), (-\eta(x))) = \Delta R_{\mu\nu}(q(x), (-\overline{\eta}(x))). \tag{16}$$

A generalized gauge transformation is defined in [16] as a transformation in which the quantity $\overline{\eta}(x)$ is substituted for $\eta(x)$ into the tensor expressions of interest. This incorporates, as a particular case, the well-known infinitesimal gauge transformation

$$\overline{\eta}_{\mu\nu}(x) = \eta_{\mu\nu}(x) + \xi_{\mu;\nu} + \xi_{\nu;\mu}, \tag{17}$$

where η, ξ and their derivatives are small. Obviously, equation (16) expresses a generalized gauge invariance of $\Delta R_{\mu\nu}$.

After introducing the above decomposition (13) and demonstrating its invariance we can now present modification and generalization of the Efroimsky formalism. Replacing the term $R^{(1)}_{\mu\nu}(q, \eta)$ by $-\Delta R_{\mu\nu}(q, (-\eta))$ in equations (7), (8), (10), and omitting the terms $O(\eta^2)$ we obtain relations

$$R^{(1)}_{\mu\nu}(q, h) + R^{(2)}_{\mu\nu}(q, h) - \Delta R_{\mu\nu}(q, (-\eta)) + R^{(3)}_{\mu\nu}(q, h) + O(h^4) = 0, \tag{18}$$

$$\Delta R_{\mu\nu}(q, (-\eta)) = \left\langle R^{(2)}_{\mu\nu}(q, h)\right\rangle_L, \tag{19}$$

$$G_{\mu\nu}(q) = 8\pi \tilde{T}^{(gw)}_{\mu\nu} \equiv -\Delta R_{\mu\nu}(q, (-\eta)) + \tfrac{1}{2} q_{\mu\nu} q^{\alpha\beta} \Delta R_{\alpha\beta}(q, (-\eta)). \tag{20}$$

In the case where the gravitational waves do not have high-frequency modes it is still possible to employ the expansion of $-\Delta R_{\mu\nu}(q, (-\eta))$ in powers of η and use its dominant term $R^{(1)}_{\mu\nu}(q, \eta)$ instead. Thus we recover Efroimsky's previous results (cf (7), (8), (10)).

In general, however, expressing η in terms of h from equation (19) becomes an extremely difficult task because it is no longer a linear equation for η. To overcome this problem we can use equation (19) and substitute for $\Delta R_{\mu\nu}$ into the remaining equations (18) and (20). We obtain the relations

$$R^{(1)}_{\mu\nu}(q,h) + R^{(2)}_{\mu\nu}(q,h) - \left\langle R^{(2)}_{\mu\nu}(q,h)\right\rangle_L + R^{(3)}_{\mu\nu}(q,h) + O(h^4) = 0, \tag{21}$$

$$-G_{\mu\nu}(q) = \left\langle R^{(2)}_{\mu\nu}(q,h)\right\rangle_L - \tfrac{1}{2}q_{\mu\nu}q^{\alpha\beta}\left\langle R^{(2)}_{\alpha\beta}(q,h)\right\rangle_L \equiv -8\pi T^{BH}_{\mu\nu}. \tag{22}$$

Equation (22) is obviously in perfect agreement with the Isaacson result [3] which represents the background response to the presence of high-frequency gravitational waves, using the Brill–Hartle averaging to introduce the effective stress–energy tensor $T^{BH}_{\mu\nu}$ for high-frequency gravitational waves. Equation (21) is the wave equation for perturbations h on the average metric q. In the highest order of high-frequency approximation this clearly reduces to $R^{(1)}_{\mu\nu} = 0$ which also fully reproduces Isaacson's result. Additional terms in (21) can be used for study of nonlinear effects on the wave propagation.

Note finally another interesting consequence of equation (19) and the gauge invariance (16) of $\Delta R_{\mu\nu}$. This directly guarantees gauge invariance of the stress–energy tensor $T^{BH}_{\mu\nu}$ defined in (22) (in the highest order). Proof of this property was presented already in the classic work [3], using, however, a much more complicated method.

4. Concluding remarks

In our contribution we have compared the Efroimsky [1, 2] and the Isaacson [3] self-consistent perturbation schemes which describe propagation of weak gravitational waves on a cosmological background. In both these approaches the background is influenced by the waves, i.e. the nonlinear effects are taken into account. The classical Isaacson method applies to high-frequency waves. On the other hand, the Efroimsky formalism is applicable to low-frequency gravitational waves but does not admit the high-frequency limit. We have suggested a modification of the Efroimsky formalism by employing the gauge-invariant decomposition (13) of the Ricci tensor, introduced recently by Anderson [16]. The resulting generalized system of equations (18)–(20) fully recovers the Efroimsky results in the absence of high-frequency modes, in the high-frequency limit it reproduces Isaacson's formulae.

Although we have considered here for simplicity only vacuum metrics $\gamma_{\mu\nu}$ and $g_{\mu\nu}$, possible generalization to nonvacuum spacetimes is straightforward. In fact, Efroimsky has already generalized his formalism to spacetimes with ideal-fluid-like matter and a cosmological term [1, 2]; in the case of the Isaacson high-frequency approach, this was done recently in [15].

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Some High-Frequency Gravitational Waves Related to Exact Radiative Spacetimes

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A formalism is introduced which may describe both standard linearized waves and gravitational waves in Isaacson's high-frequency limit. After emphasizing main differences between the two approximation techniques we generalize the Isaacson method to non-vacuum spacetimes. Then we present three large explicit classes of solutions for high-frequency gravitational waves in particular backgrounds. These involve non-expanding (plane, spherical or hyperbolical), cylindrical, and expanding (spherical) waves propagating in various universes which may contain a cosmological constant and electromagnetic field. Relations of high-frequency gravitational perturbations of these types to corresponding exact radiative spacetimes are described.

KEY WORDS: gravitational waves; high-frequency limit; exact solutions.

1. INTRODUCTION

In classic work [1] Isaacson presented a perturbation method which enables one to study properties of high-frequency gravitational waves, together with their influence on the cosmological background in which they propagate. It is this non-linear "back-reaction" effect on curvature of the background spacetime which distinguishes the high-frequency approximation scheme from other perturbation methods such as the standard Einstein's linearization of gravitational field in flat space [2, 3] or multipole expansions [4] that were developed to describe radiation from realistic astrophysical sources.

The high-frequency perturbations were originally considered by Wheeler [5] and then applied to investigation of gravitational geons by Brill and Hartle [6].

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Isaacson's systematic study [1] stimulated further works in which his treatment was developed and also re-formulated in various formalisms. Choquet-Bruhat [7, 8] analyzed high-frequency gravitational radiation using a generalized WKB "two-timing" method. Averaged Lagrangian technique which leads to Isaacson's results with less calculation was introduced by MacCallum and Taub [9, 10]. Comparison of these approaches, and clarification of assumptions that have to be made in order to provide a consistent high-frequency approximation limit was also given by Araujo [11, 12]. Elster [13] proposed an alternative method that is based on expanding null-tetrad components of the Weyl tensor. Recently, Burnett developed a weak limit approach [14] in which the high-frequency limit can be introduced and studied in a mathematically rigorous way. These general methods have been, of course, applied to study explicit particular examples of high-frequency gravitational waves, see e.g. [1, 8, 9, 15, 16].

On the other hand, many *exact* solutions of Einstein's equations are known which represent gravitational radiation. Among the most important classes are planar *pp*-waves [17, 18] which belong to a large family of non-expanding radiative spacetimes [19, 20], cylindrical Einstein-Rosen waves [21], expanding "spherical" waves of the Robinson-Trautman type [22, 23], spacetimes with boost-rotation symmetry representing radiation generated by uniformly accelerated sources [24–26], cosmological models of the Gowdy type [27], and others — for comprehensive reviews containing also a number of references see, e.g., [28–32].

However, there are only several works in which *relation* between exact gravitational waves and those obtained by perturbations of non-flat backgrounds has been explicitly investigated and clarified, see e.g. [10, 33, 16]. The purpose of our contribution is to help to fill this "gap".

We first briefly summarize and generalize the Isaacson approach [1] to admit non-vacuum backgrounds, the cosmological constant Λ in particular. Modification of Isaacson's formalism allows us to incorporate also standard linearized gravitational waves into the common formalism. Then, in section 3 we study properties of high-frequency gravitational waves in specific classes of spacetimes with special algebraic or geometric structure. In particular, we focus on waves which propagate in backgrounds with $\Lambda \neq 0$. This is motivated not only theoretically but also by recent observations [34] which seem to indicate that (effective) positive cosmological constant played a fundamental role in the early universe, but it is also important for its present and future dynamics.

2. HIGH-FREQUENCY APPROXIMATION VERSUS STANDARD LINEARIZATION

Let us assume a formal decomposition of the spacetime metric $g_{\mu\nu}$ into the background metric $\gamma_{\mu\nu}$ and its perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon h_{\mu\nu}, \tag{1}$$

where, in a suitable coordinate system, $\gamma_{\mu\nu} = O(1)$ and $h_{\mu\nu} = O(\epsilon)$ [by definition, $f = O(\epsilon^n)$ if there exists a constant $C > 0$ such that $|f| < C\epsilon^n$ as $\epsilon \to 0$]. The two distinct non-negative dimensionless parameters ε and ϵ have the following meaning: ε is the usual amplitude parameter of weak gravitational perturbations whereas the frequency parameter ϵ denotes the possible high-frequency character of radiation described by $h_{\mu\nu}$. To be more specific, the parameter $\varepsilon \ll 1$ characterizes (for $\epsilon = 1$) the amplitude of linearized gravitational waves in the ordinary weak field limit of Einstein's equations. The second independent parameter $\epsilon = \lambda/L$ represents, on the other hand, the ratio of a typical wavelength λ of gravitational waves and the scale L on which the background curvature changes significantly. Isaacson's high-frequency approximation [1] arises when $\lambda \ll L$, i.e. $\epsilon \ll 1$ (and $\varepsilon = 1$). Since L can be considered to have a finite value of order unity, we may write $O(\epsilon) = O(\lambda)$.

To derive the dynamical field equations we start with the order-of-magnitude estimates which indicate how fast the metric components vary. Symbolically, the derivatives are of the order $\partial\gamma \sim \gamma/L$, $\partial h \sim h/\lambda$, so that the following formulas

$$\begin{aligned}
\gamma_{\mu\nu} &= O(1), & h_{\mu\nu} &= O(\epsilon),\\
\gamma_{\mu\nu,\alpha} &= O(1), & h_{\mu\nu,\alpha} &= O(1),\\
\gamma_{\mu\nu,\alpha\beta} &= O(1), & h_{\mu\nu,\alpha\beta} &= O(\epsilon^{-1}),
\end{aligned} \tag{2}$$

are valid. Next, we expand the Ricci tensor in powers of h,

$$R_{\mu\nu}(g) = R^{(0)}_{\mu\nu} + \varepsilon R^{(1)}_{\mu\nu} + \varepsilon^2 R^{(2)}_{\mu\nu} + \varepsilon^3 R^{(3)}_{\mu\nu} + \dots, \tag{3}$$

where

$$\begin{aligned}
R^{(0)}_{\mu\nu}(\gamma) &\equiv R_{\mu\nu}(\gamma),\\
R^{(1)}_{\mu\nu}(\gamma, h) &\equiv \tfrac{1}{2}\gamma^{\rho\tau}(h_{\tau\mu;\nu\rho} + h_{\tau\nu;\mu\rho} - h_{\rho\tau;\mu\nu} - h_{\mu\nu;\rho\tau}),\\
R^{(2)}_{\mu\nu}(\gamma, h) &\equiv \tfrac{1}{2}\big[\tfrac{1}{2}h^{\rho\tau}{}_{;\nu}h_{\rho\tau;\mu} + h^{\rho\tau}(h_{\tau\rho;\mu\nu} + h_{\mu\nu;\tau\rho} - h_{\tau\mu;\nu\rho}\\
&\quad - h_{\tau\nu;\mu\rho}) + h^{\tau}{}_{\nu}{}^{;\rho}(h_{\tau\mu;\rho} - h_{\rho\mu;\tau})\\
&\quad - \left(h^{\rho\tau}{}_{;\rho} - \tfrac{1}{2}h^{;\tau}\right)(h_{\tau\mu;\nu} + h_{\tau\nu;\mu} - h_{\mu\nu;\tau})\big].\\
R^{(3)}_{\mu\nu}(\gamma, h) &\equiv \tfrac{1}{4}h^{\sigma\tau}h_{\sigma\rho;\mu}h^{\rho}{}_{\tau;\nu} + \dots.
\end{aligned} \tag{4}$$

The semicolons denote covariant differentiation with respect to the *background* metric $\gamma_{\mu\nu}$, which is also used to raise or lower all indices. Considering relations (2), the orders of the terms (4) are

$$R^{(0)}_{\mu\nu} = O(1),\ \varepsilon R^{(1)}_{\mu\nu} = O(\epsilon^{-1}\varepsilon),\ \varepsilon^2 R^{(2)}_{\mu\nu} = O(\varepsilon^2),\ \varepsilon^3 R^{(3)}_{\mu\nu} = O(\epsilon\varepsilon^3). \tag{5}$$

Two limiting cases thus arise naturally. For the *standard linearization* ($\varepsilon \ll 1$, $\epsilon = 1$) the dominant term of $R_{\mu\nu}(g)$ is $R^{(0)}_{\mu\nu} = O(1)$ which corresponds to the background $\gamma_{\mu\nu}$ [to find, e.g., a vacuum spacetime metric $g_{\mu\nu}$ we solve $R^{(0)}_{\mu\nu}(\gamma) = 0$]. Its first correction representing linearized (purely) gravitational waves is governed by

$$R^{(1)}_{\mu\nu}(\gamma, h) = 0, \tag{6}$$

which is a dynamical equation for perturbations $h_{\mu\nu}$ on the fixed background $\gamma_{\mu\nu}$. The next term $R^{(2)}_{\mu\nu}(\gamma, h)$ can then be used to define energy-momentum tensor of these gravitational waves, but the background metric is *not* assumed to be influenced by it. Improvements to this inconsistency can be obtained by iteration procedure. More rigorous but somewhat complicated solution to this problem was recently proposed by Efroimsky [35].

In the *high-frequency approximation* ($\epsilon \ll 1$, $\varepsilon = 1$) the dominant term is $R^{(1)}_{\mu\nu} = O(\epsilon^{-1})$ which gives the wave equation (6) for the perturbations $h_{\mu\nu}$ on the curved background $\gamma_{\mu\nu}$ (considering a vacuum full metric $g_{\mu\nu}$). The two terms of the order $O(1)$, namely $R^{(0)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$, are *both* used to give the Einstein equation for the background *non-vacuum* metric, which represents the essential influence of the high-frequency gravitational waves on the background. Of course, to obtain a consistent solution, one has to use both the wave equation *and* the Einstein equation for the background simultaneously.

2.1. Linear Approximation

Interestingly, it follows that the wave equation for $h_{\mu\nu}$, which arises from the linear perturbation of the Ricci tensor in vacuum for *both* the above limiting cases $\varepsilon \ll 1$, $\epsilon = 1$, and $\epsilon \ll 1$, $\varepsilon = 1$, is the *same* equation (6). In analogy with the well-known theory of massless spin-2 fields in flat space [4] we wish to impose two TT gauge conditions,

$$h_{\mu\nu}{}^{;\nu} = 0, \tag{7}$$

$$h^{\mu}{}_{\mu} = 0. \tag{8}$$

In this gauge we arrive at the following wave equation

$$\Diamond h_{\mu\nu} \equiv h_{\mu\nu}{}^{;\beta}{}_{;\beta} - 2R^{(0)}_{\sigma\nu\mu\beta}\, h^{\beta\sigma} - R^{(0)}_{\mu\sigma}\, h^{\sigma}{}_{\nu} - R^{(0)}_{\nu\sigma}\, h^{\sigma}{}_{\mu} = 0, \tag{9}$$

where the operator $\Diamond$ is the generalization of flat-space d'Alembertian. Contracting (9) we obtain $(h^{\mu}{}_{\mu})^{;\beta}{}_{;\beta} = 0$, so that the condition (8) is always consistent with (9). However, if we differentiate $\Diamond h_{\mu\nu}$ and use equations (7), (2), we find that

$$(\Diamond h_{\mu\nu})^{;\nu} = \big(R^{(0)}_{\nu\beta;\mu} - 2R^{(0)}_{\mu\nu;\beta}\big) h^{\nu\beta}, \quad \text{where} \tag{10}$$

$$(\Diamond h_{\mu\nu})^{;\nu} = O(\epsilon^{-2})\,, \quad \big(R^{(0)}_{\nu\beta;\mu} - 2R^{(0)}_{\mu\nu;\beta}\big) h^{\nu\beta} = O(\epsilon). \tag{11}$$

Thus, in case of standard linearized waves ($\epsilon = 1$) there is an obvious inconsistency, except for backgrounds with a covariantly constant Ricci tensor (e.g., for the Einstein spaces). On the other hand, in the high-frequency limit ($\varepsilon = 1$), the inconsistency between (9) and (7) is extremely small (the left and the right sides of (10) differ by ϵ^3 where $\epsilon \ll 1$). Moreover, for all background metrics of *constant curvature* the equations are *fully consistent*. This is an important advantage of the equation (9) containing also terms of non-dominant order (namely those proportional to the Riemann or Ricci tensors), if compared to other "simpler" wave equations (e.g., $h_{\mu\nu}{}^{;\beta}{}_{;\beta} = 0$) for which the left and right sides of (10) generally differ by only two orders of magnitude.

2.2. Nonlinear Terms and the Effective Energy-Momentum Tensor

Before considering the second-order terms we now extend the formalism to be applicable to a larger class of spacetimes with (possibly) non-vanishing energy-momentum tensor $T_{\mu\nu}$. Namely, $g_{\mu\nu}$ need not be a vacuum metric (as only considered in [1]) but it satisfies Einstein's equations

$$R_{\mu\nu}(g) = 8\pi\, \tilde{T}_{\mu\nu}(g, \varphi). \tag{12}$$

Here $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\beta}{}_{\beta}$, such that $T_{\mu\nu}(g, \varphi)$ depends on non-gravitational fields φ and on the full metric $g_{\mu\nu}$ but it *does not* contain the *derivatives* of $g_{\mu\nu}$. Note that this admits as particular cases a presence of electromagnetic field, and also Einstein spaces when $\tilde{T}_{\mu\nu} = \frac{1}{8\pi}\, \Lambda g_{\mu\nu}$. Under the assumptions (2) valid for the decomposition (1) we expand the equation (12) as

$$\begin{aligned} R^{(0)}_{\mu\nu}(\gamma) + \varepsilon R^{(1)}_{\mu\nu}(\gamma, h) + \varepsilon^2 R^{(2)}_{\mu\nu}(\gamma, h) + \ldots = \\ 8\pi \left[\, \tilde{T}^{(0)}_{\mu\nu}(\gamma, \varphi) + \varepsilon\, \tilde{T}^{(1)}_{\mu\nu}(\gamma, h, \varphi) + \varepsilon^2\, \tilde{T}^{(2)}_{\mu\nu}(\gamma, h, \varphi) + \ldots \,\right], \end{aligned} \tag{13}$$

where $\tilde{T}^{(0)}_{\mu\nu}(\gamma, \varphi) \equiv \tilde{T}_{\mu\nu}(\gamma, \varphi)$, and the remaining terms on the right-hand side are linear and quadratic in h, respectively. The orders of magnitude of the terms in the expansion of the Ricci tensor have been described above, cf. (5). For the energy-momentum tensor one obtains

$$\tilde{T}^{(0)}_{\mu\nu} = O(1), \quad \tilde{T}^{(1)}_{\mu\nu} = O(\epsilon), \quad \tilde{T}^{(2)}_{\mu\nu} = O(\epsilon^2). \tag{14}$$

For ordinary linearization we thus get the equations $R^{(n)}_{\mu\nu} = 8\pi\, \tilde{T}^{(n)}_{\mu\nu}$ in each order $n = 0, 1, 2, \ldots$. For the high-frequency approximation we obtain from (13) in the leading order $O(\epsilon^{-1})$ the equation (6) which is identical with the wave equation in the vacuum case. The second-order contributions, that are $O(1)$, represent an influence of the high-frequency gravitational waves and matter fields on the background,

$$R^{(0)}_{\mu\nu}(\gamma) - 8\pi\, \tilde{T}^{(0)}_{\mu\nu}(\gamma, \varphi) = -R^{(2)}_{\mu\nu}(\gamma, h). \tag{15}$$

This equation (which in case of a vacuum spacetime reduces to the Isaacson result) can be rewritten in the form of Einstein's equation for the background as

$$G^{(0)}_{\mu\nu}(\gamma) - 8\pi\, T^{(0)}_{\mu\nu}(\gamma,\varphi) = -\big[R^{(2)}_{\mu\nu}(\gamma,h) - \tfrac{1}{2}\gamma_{\mu\nu}R^{(2)}(\gamma,h)\big] \equiv 8\pi\, T^{GW}_{\mu\nu}. \tag{16}$$

This defines the effective energy-momentum tensor $T^{GW}_{\mu\nu}$ of high-frequency gravitational waves.

2.3. Gravitational Waves in the WKB Approximation

In the following we shall restrict ourselves to the Isaacson approximation ($\varepsilon = 1$, $\epsilon \ll 1$), i.e. on study of high-frequency gravitational waves on curved backgrounds. Inspired by the plane-wave solution in flat space, the form $h_{\mu\nu} = \mathcal{A}\, e_{\mu\nu}\exp(i\phi)$ of the solution is assumed. The amplitude $\mathcal{A} = O(\epsilon)$ is a slowly changing real function of position, the phase ϕ is a real function with a large first derivative but no larger derivatives beyond, and $e_{\mu\nu}$ is a normalized polarisation tensor field. The above assumption, introduced in [1], is called the WKB approximation, or the geometric optics limit [4]. The wave vector normal to surfaces of constant phase is $k_\mu \equiv \phi_{,\mu}$ and the orders of various relevant quantities are $R^{(0)}_{\mu\nu\gamma\delta} = O(1)$, $\mathcal{A}_{,\mu} = O(\epsilon)$, $k_\mu = O(\epsilon^{-1})$, and $k_{\mu;\nu} = O(\epsilon^{-1})$. Substituting this into the conditions (7), (8), and the wave equation (9) we obtain, in the two highest orders which are gauge invariant,

$$\begin{aligned} k^\mu k_\mu = 0, \quad & k^\mu e_{\mu\nu} = 0, \quad k^\alpha e_{\mu\nu;\alpha} = 0,\\ e^{\mu\nu}e_{\mu\nu} = 1, \quad & \gamma^{\mu\nu}e_{\mu\nu} = 0, \quad (\mathcal{A}^2 k^\beta)_{;\beta} = 0. \end{aligned} \tag{17}$$

These express that a beam of high-frequency gravitational waves propagate along rays which are null geodesics with tangent k^μ, with parallelly transported polarization orthogonal to the rays. Moreover, using the WKB approximation of $T^{GW}_{\mu\nu}$ and the Brill-Hartle averaging procedure [6] (which guarantees the gauge invariance) Isaacson obtained for gravitational waves in the geometric optics limit the energy-momentum tensor [1]

$$T^{HF}_{\mu\nu} = \tfrac{1}{64\pi}\mathcal{A}^2 k_\mu k_\nu. \tag{18}$$

The energy-momentum tensor of high-frequency waves thus has the form of pure radiation. This fully agrees with results obtained by alternative techniques [8, 9, 14].

3. EXAMPLES OF HIGH-FREQUENCY GRAVITATIONAL WAVES

Now we present some explicit classes of high-frequency gravitational waves. These are obtained by the above described WKB approximation method considering specific families of background spacetimes with a privileged geometry.

3.1. Non-Expanding Waves

As the background we first consider the Kundt class [19, 28] of non-expanding, twist-free spacetimes in the form [36]

$$\mathrm{d}s^2 = F\,\mathrm{d}u^2 - 2\,\frac{Q^2}{P^2}\,\mathrm{d}u\,\mathrm{d}v + \frac{1}{P^2}\,(\mathrm{d}x^2 + \mathrm{d}y^2), \tag{19}$$

with

$$\begin{aligned} P &= 1 + \frac{\alpha}{2}\,(x^2 + y^2), \\ Q &= \left[1 + \frac{\beta}{2}\,(x^2 + y^2)\right] e + C_1\,x + C_2\,y, \\ F &= D\,\frac{Q^2}{P^2}\,v^2 - \frac{(Q^2)_{,u}}{P^2}\,v - \frac{Q}{P}\,H, \end{aligned} \tag{20}$$

where α, β, and e are constants (without loss of generality $e = 0$ or $e = 1$), C_1, C_2 and D are arbitrary functions of the retarded time u, and $H(x, y, u)$ is an arbitrary function of the spatial coordinates x, y, and of u.

In particular, these are Petrov type N (or conformally flat) solutions of Einstein's equations with cosmological constant Λ when $\alpha = -\beta = \frac{1}{6}\Lambda$ and $D = -2\beta e + C_1^2 + C_2^2$, see e.g. [36–39]. Such metrics represent exact pure gravitational waves propagating along principal null direction ∂_v if H satisfies the equation $P^2(H_{,xx} + H_{,yy}) + \frac{2}{3}\Lambda\,H = 0$. However, in our treatment here the function H does *not* describe exact gravitational waves but rather it characterizes the *influence* of high-frequency perturbations on the background metric, which is assumed to be initially given by (19), (20) with $H = 0$.

We consider the phase of high-frequency gravitational waves given by $\phi = \phi(u)$, and we seek solution in the WKB form, namely

$$h_{\mu\nu} = \mathcal{A}\,e_{\mu\nu}\,\exp(i\phi(u)), \tag{21}$$

where the amplitude $\mathcal{A}$ and polarization tensor $e_{\mu\nu}$ are functions of the coordinates $\{u, v, x, y\}$. The corresponding wave vector is $k_\mu = (\dot\phi, 0, 0, 0)$, where the dot denotes differentiation with respect to u. Applying now all the equations (17) we obtain

$$\begin{aligned} \mathcal{A} &= \mathcal{A}(u, x, y), \\ e^{+}_{\mu\nu} &= \frac{1}{\sqrt{2}\,P^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \end{aligned}$$

$$e^{\times}_{\mu\nu} = \frac{1}{\sqrt{2}\,P^2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{22}$$

The fact that the amplitude $\mathcal{A}$ is independent of the coordinate v expresses non-expanding character of the waves. The special polarisation tensors, denoted as $+$ and $\times$, are analogous to those used in the standard theory of linearized waves in flat space. A general polarisation is easily obtained by considering $e_{\mu\nu} = a\, e^{+}_{\mu\nu} + b\, e^{\times}_{\mu\nu}$, where $a^2(u,x,y) + b^2(u,x,y) = 1$.

Using the Einstein tensor for the metric (19) with the cosmological term in equations (16) and (18), we determine the reaction of the background on the presence of the above high-frequency gravitational perturbations, namely

$$\frac{Q}{P}\left[P^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{2}{3}\,\Lambda\right] H(u,x,y) = \tfrac{1}{4}\mathcal{A}^2(u,x,y)\dot{\phi}^2. \tag{23}$$

Notice that $\mathcal{A} = O(\epsilon)$ and $\dot{\phi} = O(\epsilon^{-1})$. Therefore, the influence of high-frequency gravitational waves on the background, represented by the function H, is of the order $O(1)$. These *approximate* solutions can obviously be compared to specific *exact* radiative vacuum solutions which are given by H solving the field equation (23) with a vanishing right-hand side (when $\mathcal{A} = 0$, i.e. high-frequency perturbation waves are absent).

The above waves are non-expanding with the wave-fronts $u = const.$ being two-dimensional spaces of constant curvature given by $\alpha = \frac{1}{6}\Lambda$, cf. (19). For $\Lambda = 0$ these are plane-fronted waves, for $\Lambda > 0$ they are spheres, and for $\Lambda < 0$ hyperbolical surfaces.

Another interesting subclass of the Kundt spacetimes of the form (19), (20) are explicit Petrov type II (or more special) metrics given by $\beta = \alpha, e = 1, C = 0$ and $D = 2(\Lambda - \alpha)$, namely

$$ds^2 = [\,2(\Lambda - \alpha)\,v^2 - H\,]\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + \frac{1}{P^2}\,(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{24}$$

For $H = 0$ these are electrovacuum solutions with the geometry of a direct product of two 2-spaces of constant curvature, in particular the Bertotti-Robinson, (anti-)Nariai or Plebański-Hacyan spaces [40–43], see e.g. [44, 36]. Considering again (21) we obtain the results (22) as in the previous case. However, the reaction of high-frequency waves on the background is now different. It is determined by the equations (16) and (18) with the energy-momentum tensor consisting of a cosmological term plus that of a uniform non-null electromagnetic field described by the complex self-dual Maxwell tensor $F^{\mu\nu} = 4\Phi_1(m^{[\mu}\bar{m}^{\nu]} - k^{[\mu}l^{\nu]})$, where $\Phi_1 = \sqrt{\alpha - \frac{\Lambda}{2}}\,e^{i\,c}$, $c = const.$, and $\mathbf{m} = P\,\partial_{\bar{\zeta}}$, $\mathbf{k} = \partial_v$, $\mathbf{l} = \frac{1}{2}F\,\partial_v + \partial_u$ form the null tetrad. Straightforward calculation gives

$$P^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) H = \tfrac{1}{4}\mathcal{A}^2(u,x,y)\dot{\phi}^2. \tag{25}$$

This result is analogous to the equation (23), but the present situation is now more complicated since the background spacetime is *not vacuum* but it contains electromagnetic field. (In fact, the term with the cosmological constant Λ in (23) has been entirely compensated by this.) Therefore, we have to analyze the perturbation of the *complete* Einstein-Maxwell system, and its consistency.

The Einstein equations in the two highest orders (6) and (16) have already been solved. We will now demonstrate that the Maxwell equations are also satisfied in the high-frequency limit, namely ${F^{\mu\nu}}_{|\nu} = O(\epsilon)$, where $|$ denotes the covariant derivative with respect to the full metric $g_{\mu\nu}$. Indeed, using antisymmetry of $F^{\mu\nu}$ we can write ${F^{\mu\nu}}_{|\nu} = {F^{\mu\nu}}_{,\,\nu} + \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\nu} F^{\mu\nu}$. Considering (2) and the gauge condition (8) we obtain $g^{\alpha\beta} g_{\alpha\beta,\nu} = \gamma^{\alpha\beta}\gamma_{\alpha\beta,\nu} - h^{\alpha\beta} h_{\alpha\beta,\nu} + O(\epsilon^2)$ because $\gamma^{\alpha\beta} h_{\alpha\beta,\nu} - h^{\alpha\beta}\gamma_{\alpha\beta,\nu} = ({h^{\beta}}_{\beta})_{;\nu} - 2\, h^{\alpha\beta}\gamma_{\alpha\beta;\nu} = 0$, so that

$${F^{\mu\nu}}_{|\nu} = {F^{\mu\nu}}_{;\nu} - \tfrac{1}{2} h^{\alpha\beta} h_{\alpha\beta,\nu} F^{\mu\nu} + O(\epsilon^2). \tag{26}$$

Consequently, if the original background represents an electrovacuum spacetime, ${F^{\mu\nu}}_{;\nu} = 0$, the Maxwell equations ${F^{\mu\nu}}_{|\nu} = O(\epsilon)$ for the full metric are satisfied in the dominant order $O(1)$ in the high-frequency limit $\epsilon \ll 1$. In addition, the field equations are valid also in the next order $O(\epsilon)$ for the new electromagnetic field

$$\mathcal{F}^{\mu\nu} = \left(1 + \tfrac{1}{4} h^{\alpha\beta} h_{\alpha\beta}\right) F^{\mu\nu}, \tag{27}$$

since using (26) we obtain ${\mathcal{F}^{\mu\nu}}_{|\nu} = O(\epsilon^2)$. Starting from an electromagnetic field $F^{\mu\nu}$ satisfying ${F^{\mu\nu}}_{;\nu} = 0$ with respect to the background metric $\gamma_{\mu\nu}$, we can thus construct the electromagnetic field $\mathcal{F}^{\mu\nu}$ which satisfies the Maxwell equations ${\mathcal{F}^{\mu\nu}}_{|\nu} = O(\epsilon^2)$ with respect to the full metric $g_{\mu\nu}$ in the presence of high-frequency gravitational waves. Both the Einstein and Maxwell equations are then satisfied in the two highest perturbative orders. Interestingly, these results hold for high-frequency perturbations of *any* “seed” electrovacuum background spacetimes.

In particular, if the backgrounds are direct product spacetimes (24) for $H = 0$ with uniform non-null electromagnetic field $\Phi_1 = const.$ then high-frequency gravitational waves (21), (22) introduce H which is given by equation (25). According to (27), the electromagnetic field is perturbed by the term proportional to $h^{\alpha\beta} h_{\alpha\beta} = \mathcal{A}^2 e^{2i\phi} = O(\epsilon^2)$, see (17), namely

$$\Phi_1^g = \Phi_1 \left[1 + \tfrac{1}{2}\mathcal{A}^2(u,x,y)\, e^{2i\phi(u)}\right]. \tag{28}$$

This remains non-null but it is no longer uniform. The full spacetime thus describes non-uniform, non-null electromagnetic field plus the null field of high-frequency gravitational waves.

3.2. Cylindrical Waves

Next we consider the class of cylindrical Einstein-Rosen waves,

$$ds^2 = e^{2\gamma-2\psi}(-\mathrm{d}t^2+\mathrm{d}\rho^2)+e^{2\psi}\mathrm{d}z^2+\rho^2 e^{-2\psi}\mathrm{d}\varphi^2. \tag{29}$$

If the functions $\psi(t,\rho)$ and $\gamma(t,\rho)$ satisfy the corresponding field equations (see, e.g. [21],[28], or equations (33)-(35) below) these are exact radiative spacetimes of the Petrov type I. We conveniently define double null coordinates $u=\frac{1}{\sqrt{2}}(t-\rho)$ and $v=\frac{1}{\sqrt{2}}(t+\rho)$; in these coordinates $\{u,v,\varphi,z\}$ the metric takes the form

$$ds^2 = -2\,e^{2\gamma-2\psi}\mathrm{d}u\,\mathrm{d}v+e^{2\psi}\mathrm{d}z^2+\tfrac{1}{2}(v-u)^2e^{-2\psi}\mathrm{d}\varphi^2. \tag{30}$$

We assume this to be the class of background universes into which we wish to introduce high-frequency gravitational waves. We assume again $\phi=\phi(u)$ implying the wave vector $k_\mu=(\dot\phi,0,0,0)$, i.e. the WKB perturbation of the form (21). By applying all the conditions (17) we obtain

$$\begin{aligned}
\mathcal{A} &= \frac{\mathcal{U}(u)}{\sqrt{v-u}}\,,\\
e^{+}_{\mu\nu} &= \frac{1}{\sqrt{2}}\,e^{-2\psi}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&\frac{1}{2}(v-u)^2&0\\0&0&0&-e^{4\psi}\end{pmatrix},\\
e^{\times}_{\mu\nu} &= \frac{1}{2}\,(v-u)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix};
\end{aligned}\tag{31}$$

notice that $v-u=\sqrt{2}\,\rho$. Thus the perturbative solution is given by

$$h_{\mu\nu} = \frac{\mathcal{U}(u)}{\sqrt{v-u}}\,e_{\mu\nu}\exp(i\phi(u))\,. \tag{32}$$

The back-reaction on the background (contained in a specific modification of the metric functions γ and ψ) is given by the following equations, cf. (18),

$$(v-u)\,\psi_{,u}^2+\gamma_{,u} = -\tfrac{1}{16}(v-u)\,\mathcal{A}^2\dot\phi^2, \tag{33}$$

$$(v-u)\,\psi_{,v}^2-\gamma_{,v} = 0, \tag{34}$$

$$\psi_{,uv}-\frac{1}{2v-u}(\psi_{,v}-\psi_{,u}) = 0. \tag{35}$$

Interestingly, this set of equations is *consistent*: by differentiating equation (33) with respect to v, equation (34) with respect to u, and combining them, one obtains

(35) provided the amplitude $\mathcal{A}(u, v)$ satisfies the equation

$$((v-u)\,\mathcal{A}^2)_{,v} = 0. \tag{36}$$

However, this is automatically satisfied for the amplitude (31). It is thus quite simple to introduce gravitational waves in the WKB approximation into the cylindrical spacetimes (30). If the functions γ and ψ representing the background are solutions of the vacuum equations [i.e. (33)-(35) with a vanishing right-hand side of (33)] then for introducing high-frequency gravitational waves it is sufficient *just to alter the function* γ as

$$\gamma(u, v) \to \gamma(u, v) + \tilde{\gamma}(u), \tag{37}$$

where

$$\frac{\partial\tilde{\gamma}(u)}{\partial u} = -\tfrac{1}{16}\,\mathcal{U}^2\dot{\phi}^2. \tag{38}$$

In particular, when $\psi = 0 = \gamma$ the background (29) is a flat Minkowski space. By assuming non-trivial $\tilde{\gamma}$ we obtain Petrov type N spacetime with high-frequency gravitational waves which have cylindrical wave-fronts. In a general case this perturbation is propagating in the background which is the Einstein-Rosen cylindrical wave of Petrov type I. The effect on background is given by the relation (38) where $\mathcal{U}(u) = O(\epsilon)$ is an arbitrary amplitude function.

The above described perturbations depend on the null "retarded" coordinate u so that the high-frequency gravitational waves are *outgoing* (ρ is growing with t, on a fixed u). However, since the background metric (30) is invariant with respect to interchanging u with v, it is straightforward to consider also *ingoing* perturbations by assuming the phase to depend on the "advanced coordinate" v, namely

$$h_{\mu\nu} = \frac{\mathcal{V}(v)}{\sqrt{u-v}}\, e_{\mu\nu}\, \exp(i\phi(v)). \tag{39}$$

Then the term proportional to $\mathcal{A}^2\dot{\phi}^2$ will appear on the right-hand side of equation (34) instead of (33). This results in an interesting possibility to *introduce ingoing high-frequency gravitational cylindrical waves into the background of outgoing Einstein-Rosen waves* just by assuming $\tilde{\gamma}(v)$ in (37) such that

$$\frac{\partial\tilde{\gamma}(v)}{\partial v} = +\tfrac{1}{16}\,\mathcal{V}^2\dot{\phi}^2, \tag{40}$$

or vice versa.

Moreover, all the above results can further be extended to a class of generalized Einstein-Rosen (diagonal) metrics [29, 45] which describe G_2 *inhomogeneous cosmological models,*

$$ds^2 = e^{2\gamma-2\psi}(-\mathrm{d}t^2 + \mathrm{d}\rho^2) + e^{2\psi}\mathrm{d}z^2 + t^2 e^{-2\psi}\mathrm{d}\varphi^2. \tag{41}$$

If the three-dimensional spacelike hypersurfaces are compact, the corresponding model is the famous Gowdy universe with the topology of three-torus [27, 29]. In the double null coordinates just one component of the metric is now different from (30), namely $g_{\varphi\varphi} = \frac{1}{2}(v+u)^2 e^{-2\psi(u,v)}$. The only modification of the above results (in the double null coordinates) consists of replacing the factor $(v-u)$ with $(v+u)$, and each derivative with respect to u changing sign (e.g. $\gamma_{,u} \to -\gamma_{,u}$ or $\psi_{,uv} \to -\psi_{,uv}$). High-frequency gravitational waves in inhomogeneous cosmologies of the form (41) can thus easily be constructed.

3.3. Expanding Waves

Finally, we assume that the background is an expanding Robinson-Trautman spacetime. The metric (generally of the Petrov type II) in the standard coordinates has the form, see e.g. [22, 23, 28, 39],

$$ds^2 = -\left(K - 2r(\ln\mathcal{P})_{,u} - 2\frac{m}{r} - \frac{\Lambda}{3}r^2\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \frac{r^2}{\mathcal{P}^2}(\mathrm{d}\eta^2 + \mathrm{d}\xi^2), \tag{42}$$

where $K = \Delta(\ln\mathcal{P})$, $\Delta \equiv \mathcal{P}^2(\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2})$, and $m(u)$. When $\mathcal{P}(u,\eta,\xi)$ satisfies the Robinson-Trautman equation $\Delta K + 12\,m\,(\ln\mathcal{P})_{,u} - 4m_{,u} = 0$, the metric (42) is an exact vacuum solution of the Einstein equations.

In view of the existence of privileged congruence of null geodesics generated by ∂_r we introduce the phase $\phi = \phi(u)$ and the wave vector $k_\mu = (\dot{\phi}, 0, 0, 0)$ of high-frequency gravitational waves. We again assume the WKB form (21) of the solution. Applying the equations (17) we obtain

$$\mathcal{A} = \frac{1}{r}\,U(u,\eta,\xi),$$

$$e^{+}_{\mu\nu} = \frac{1}{\sqrt{2}}\frac{r^2}{\mathcal{P}^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$e^{\times}_{\mu\nu} = \frac{1}{\sqrt{2}}\frac{r^2}{\mathcal{P}^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{43}$$

A general solution has the form $h_{\mu\nu} = r^{-1}U(u,\eta,\xi)\,e_{\mu\nu}\exp(i\phi(u))$, where $U(u,\eta,\xi)$ and $\phi(u)$ are arbitrary functions, and $e_{\mu\nu} = a\,e^{+}_{\mu\nu} + b\,e^{\times}_{\mu\nu}$ with $a^2(u,\eta,\xi) + b^2(u,\eta,\xi) = 1$. Introducing the amplitudes $U^{+} = a\,U$, $U^{\times} = b\,U$

for both polarizations, we can write the solution as

$$h_{\mu\nu} = \frac{1}{r}\left[U^{+}e^{+}_{\mu\nu} + U^{\times}e^{\times}_{\mu\nu}\right]\exp(i\phi(u)). \qquad (44)$$

If the wave-surfaces $r = const., u = const.$ with the metric $dl^2 = \mathcal{P}^{-2}(\mathrm{d}\eta^2 + \mathrm{d}\xi^2)$ are homeomorfic to $\mathcal{S}^2$, the waves can be interpreted as "spherical". In the asymptotic region $r \to \infty$ such solutions locally approach plane waves [16].

The reaction of the waves on background is determined by the equations (16) and (18) with $T^{(0)}_{\mu\nu} = -\frac{1}{8\pi}\Lambda\gamma_{\mu\nu}$. From the only nontrivial component we immediately obtain the following equation

$$-\frac{\partial m}{\partial u} + 3\,m\,(\ln\mathcal{P})_{,u} + \tfrac{1}{4}\Delta K = \tfrac{1}{16}[(U^{+})^2 + (U^{\times})^2]\dot{\phi}^2, \qquad (45)$$

where $m(u)$, $\phi(u)$, whereas the remaining functions depend on coordinates $\{u, \eta, \xi\}$. Notice that this is *independent* of the cosmological constant Λ.

The expressions (44),(45) agree with results obtained by MacCallum and Taub [9] or recently by Hogan and Futamase [16] who used Burnett's technique [14]. Our results, which were derived by a straightforward approach, are slightly more general because they are not restricted to a constant frequency $\dot{\phi} = const.$ Particular subcase of the Vaidya metric has already been studied before by Isaacson [1] and elsewhere [8].

4. CONCLUSIONS

The Isaacson approach to study high-frequency perturbations of Einstein's equations was briefly reviewed and compared with the standard weak-field limit. In our contribution we generalized Isaacson's method to include non-vacuum space-times, in particular an electromagnetic field and/or a non-vanishing value of the cosmological constant Λ. Then we explicitly analyzed possible high-frequency gravitational waves in three large families of background universes, namely non-expanding spacetimes of the Kundt type, cylindrical Einstein-Rosen waves and related inhomogeneous cosmological models (such as the Gowdy universe), and the Robinson-Trautman expanding spacetimes. These backgrounds are of various Petrov types. For example, high-frequency gravitational waves can be introduced into electrovacuum conformally flat Bertotti-Robinson space, type D Nariai and Plebański-Hacyan spaces, their type N and type II generalizations, or into algebraically general Einstein-Rosen universes.

For construction of high-frequency gravitational perturbations we have employed the fact that all these spacetimes admit a non-twisting congruence of null geodesics. The corresponding tangent vectors k^{μ} are hypersurface orthogonal so that there exists a phase function ϕ which satisfies $\phi_{,\mu} = k_{\mu}$. The last equation in (17) can be put into the form $\frac{d}{dl}(\ln\mathcal{A}) = -\Theta$, where l is the affine parameter, and $\Theta = \frac{1}{2} k^{\mu}_{;\mu}$ is the expansion of the null congruence. This determines the behaviour of the amplitude $\mathcal{A}$ in the above spacetimes (22), (31), (43). The remaining equations (17) enables one to deduce the polarization tensors.

It has been also crucial that all the classes of spacetimes discussed admit *exact* solutions with the energy-momentum tensor of pure radiation, i.e., $G_{\mu\nu} - 8\pi\, T_{\mu\nu} = \frac{1}{8}\, \mathcal{A}^2 k_\mu k_\nu$, where $T_{\mu\nu}$ is either constant (representing the cosmological constant) or it describes an electromagnetic field. The relation between high-frequency perturbations and exact radiative solutions of Einstein's equations in each class is thus natural. In particular, it is possible to determine explicitly the reaction of the background on the presence of high-frequency gravitational waves.

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Radiative spacetimes approaching the Vaidya metric

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Abstract

We analyze a class of exact type II solutions of the Robinson–Trautman family which contain pure radiation and (possibly) a cosmological constant. It is shown that these spacetimes exist for any sufficiently smooth initial data, and that they approach the spherically symmetric Vaidya–(anti-)de Sitter metric. We also investigate extensions of the metric, and we demonstrate that their order of smoothness is in general only finite. Some applications of the results are outlined.

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1 Introduction

The classic Vaidya metric [1–4] (see also [5, 6] followed by reprints of the original Vaidya papers) is a spherically symmetric type D solution of the Einstein equations in the presence of pure radiation matter field which propagates at the speed of light. In various contexts this "null dust" may be interpreted as high-frequency electromagnetic or gravitational waves, incoherent superposition of aligned waves with random phases and polarisations, or as massless scalar particles or neutrinos. The Vaidya solution depends on an arbitrary "mass function" $m(u)$ of the retarded time u which characterises the profile of the pure radiation (it is a "retarded mass" measured at conformal infinity). Various sandwiches and shells of null matter can thus be constructed that are bounded either by flat ($m = 0$) or Schwarzschild-like ($m = \text{const} \neq 0$) vacuum regions. Due to this property such solutions have been extensively used as models of spherically symmetric gravitational collapse of a star, as an exterior solution describing objects consisting of heat-conducting matter, as an interesting toy model for investigation of singularities and their possible removal by quantum effects, for studies of various formulations of the cosmic censorship conjecture on both classical and quantum level, process of black-hole evaporation, and for other purposes (see, e.g., [7–16] for more details and related references).

In fact, the Vaidya spacetime belongs to a large Robinson–Trautman class of expanding nontwisting solutions [4,17,18]. Various aspects of this family have been studied in the last two decades. In particular, the existence, asymptotic behaviour and global structure of *vacuum* Robinson–Trautman spacetimes of type II with spherical topology were investigated [19–28], most recently in the works of Chruściel and Singleton [29–31]. In these rigorous studies, which were based on the analysis of solutions to the nonlinear Robinson–Trautman equation for generic, arbitrarily strong smooth initial data, the spacetimes were shown to exist globally for all positive retarded times, and to converge asymptotically to a corresponding Schwarzschild metric. Interestingly, extension across the "Schwarzschild-like" event horizon can only be made with a finite order of smoothness. Subsequently, these results were generalized in [32, 33] to the Robinson–Trautman vacuum spacetimes which admit a nonvanishing *cosmological constant* Λ. It was demonstrated that these cosmological solutions settle down exponentially fast to a Schwarzschild–(anti-)de Sitter solution at large times u. In certain cases the interior of a Schwarzschild–de Sitter black hole can be joined to an "external" cosmological Robinson–Trautman region across the horizon with a higher order of smoothness than in the corresponding case with $\Lambda = 0$. For the extreme value $9\Lambda m^2 = 1$, the extension is smooth but not analytic (and not unique). The models with $\Lambda > 0$ also exhibit explicitly the cosmic no-hair conjecture under the presence of gravitational waves. On the other hand, when $\Lambda < 0$ the smoothness of such an extension is lower.

Our aim here is to further extend the Chruściel–Singleton analysis of the Robinson-Trautman vacuum equation by including matter, namely *pure radiation*. It was argued already by Bičák and Perjés [34] that with $\Lambda = 0$ such spacetimes should generically approach the Vaidya metric asymptotically. We will analyze this problem in more detail, including also the possibility of $\Lambda \neq 0$ in which case the Robinson–Trautman spacetimes containing pure radiation can be shown to approach the radiating Vaidya–(anti-)de Sitter metric.

2 The metric and field equations

In standard coordinates the Robinson–Trautman metric has the form [4, 18, 35]

$$\mathrm{d}s^2 = -\Big(K - 2r(\ln P)_{,u} - 2\frac{m}{r} - \frac{\Lambda}{3}r^2\Big)\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2\frac{r^2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar\zeta\ , \tag{1}$$

where $K = \Delta(\ln P)$ with $\Delta \equiv 2P^2\partial_\zeta\partial_{\bar\zeta}$ being the Gaussian curvature of the 2-surfaces $2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar\zeta$, $m(u)$ is the mass function, and Λ is the cosmological constant. When the function $P(u, \zeta, \bar\zeta)$ satisfies the fourth-order Robinson–Trautman field equation

$$\Delta K + 12\,m\,(\ln P)_{,u} - 4m_{,u} = 2\kappa\, n^2\ , \tag{2}$$

the metric describes a spacetime (generally of the Petrov type II) filled with pure radiation field $T_{\mu\nu} = n^2(u, \zeta, \bar\zeta)\, r^{-2}\, k_\mu k_\nu$, where $\mathbf{k} = \partial_r$ is aligned along the degenerate principal null direction (we use the convention $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa\, T_{\mu\nu}$). In particular, vacuum Robinson–Trautman spacetimes are given by $n = 0$, in which case m can be set to a constant by a suitable coordinate transformation [4]. Vacuum spacetimes (1) — possibly with a nonvanishing Λ — thus satisfy the equation $12\,m\,(\ln P)_{,u} = -\Delta K$. These include the spherically symmetric Schwarzschild–(anti-)de Sitter solution which corresponds to $P_0 = 1 + \frac{1}{2}\zeta\bar\zeta$. Indeed, replacing the complex stereographic coordinate ζ with angular coordinates by $\zeta = \sqrt{2}\,e^{i\phi}\tan(\theta/2)$, we obtain $2P_0^{-2}\mathrm{d}\zeta\mathrm{d}\bar\zeta = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$, and $K_0 = \Delta_0 \ln(P_0) = 1$.

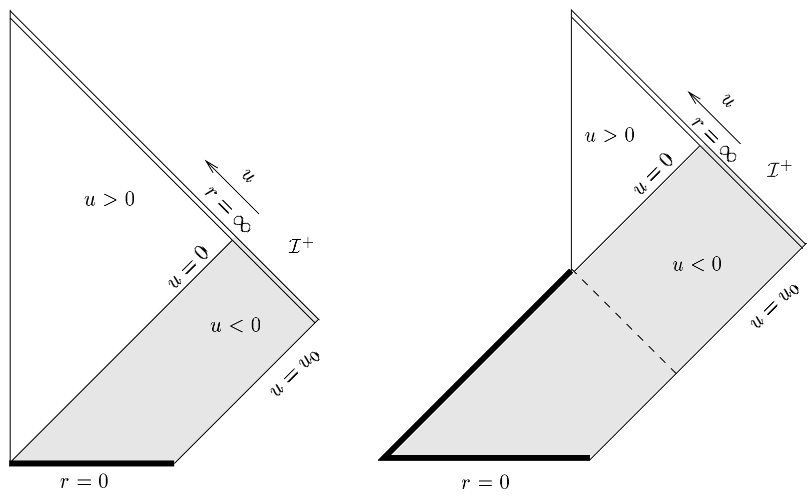

Figure 1: Schematic conformal diagrams of the Robinson–Trautman exact spacetimes which exist for any smooth initial data prescribed on u_0. Pure radiation field is present in the shaded region $u<0$. Near $u=0$ the solutions approach the Vaidya metric, and can be extended to flat Minkowski region $u>0$. Thick line indicates the curvature singularity at $r=0$ whereas double line represents future conformal infinity $\mathcal{I}^+$ at $r=\infty$ ($\Lambda=0$ is assumed). The global structure depends on the value of the parameter μ of the linear mass function (8): left diagram corresponds to $\mu>1/16$, the right one applies when $\mu\le 1/16$.

Here we will restrict ourselves to nonvacuum cases for which the dependence of the mass function $m(u)$ on the null coordinate u is only caused by a *homogeneous* pure radiation with the density $n^2(u)\,r^{-2}$. When the mass function $m(u)$ is decreasing, the field equation (2) can be naturally split into the following pair,

$$\begin{aligned} \Delta K + 12\,m(u)\,(\ln P)_{,u} &= 0\ , && (3)\\ -2\,m(u)_{,u} &= \kappa\, n^2(u)\ . && (4) \end{aligned}$$

In fact, it was demonstrated in [34] that such a separation can always be achieved using the coordinate freedom. It is then possible to reformulate equation (3) by introducing a u-dependent family of smooth 2-metrics g_{ab} on the submanifold $r=\text{const}$, $u=\text{const}$, such that $g_{ab}=f(u,\zeta,\bar\zeta)^{-2}g^0_{ab}$, where $g^0_{ab}(\zeta,\bar\zeta)$ is the metric on a 2-dimensional sphere S^2. Since g_{ab} is of the form $2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar\zeta$ in our case, we can write

$$P = fP_0\ ,\quad P_0 = 1+\tfrac{1}{2}\zeta\bar\zeta\ , \tag{5}$$

and equation (3) becomes

$$\frac{\partial f}{\partial u} = -\frac{1}{12m(u)}\, f\,\Delta K\ , \tag{6}$$

where Δ is the Laplace operator associated with the metric g_{ab}. Denoting Δ_0 and $K_0=1$ as the corresponding quantities related to g^0_{ab}, we obtain

$$\Delta = f^2\Delta_0\ ,\qquad K = f^2(1+\Delta_0(\ln f))\ . \tag{7}$$

3 Linear mass function

Let us first consider the simplest choice of $m(u)$ which, in fact, has been widely used in literature (see e.g. [7,9,36]): we will assume that the mass function is a *linearly* decreasing positive function

$$m(u) = -\mu\, u, \qquad \mu = \text{const} > 0 \;, \tag{8}$$

on the interval $[u_0, 0]$. Notice that for (8) the pure radiation field is uniform because equation (4) implies $n = \sqrt{2\mu/\kappa} = \text{const}$, independent of the retarded time u. The constant value $u_0 < 0$ localises an initial null hypersurface (that extends between the curvature singularity at $r = 0$ and the conformal infinity $r = \infty$) on which an arbitrary sufficiently smooth *initial data* given by the function

$$f_0(\zeta, \bar\zeta) = f(u = u_0, \zeta, \bar\zeta) \;, \tag{9}$$

are prescribed, see Fig. 1.

3.1 Existence of the solutions

Now, the idea is to employ the Chruściel–Singleton results [29–31] concerning the analysis of the Robinson–Trautman *vacuum* equation, in particular the existence and asymptotic behaviour of its solutions. In the vacuum case m in equation (3) is constant, and the solution $f(u, \zeta, \bar\zeta)$ of the characteristic initial value problem (9) exists and is unique (in spite of the singularity at $r = 0$). In the presence of pure radiation given by (8) it is possible to "eliminate" the variable mass function from the Robinson–Trautman field equation (6) mathematically by a simple reparametrisation

$$\tilde u = -\mu^{-1} \ln(-u) \;, \tag{10}$$

cf. [34]. Indeed, equation (6) is then converted to

$$\frac{\partial \tilde f}{\partial \tilde u} = -\frac{1}{12}\, \tilde f\, \tilde\Delta \tilde K \;, \tag{11}$$

where $\tilde f(\tilde u, \zeta, \bar\zeta) = f(u(\tilde u), \zeta, \bar\zeta)$, $\tilde K = \tilde f^2(1 + \Delta_0 \ln(\tilde f))$, and $\tilde\Delta = \tilde f^2 \Delta_0$. Notice that the transformation (10) moves the hypersurface $u = 0$, on which the mass function $m(u)$ reaches zero, to $\tilde u = +\infty$.

Chruściel [30] derived the following asymptotic expansion (as $\tilde u \to \infty$) for the function $\tilde f$ satisfying the evolution equation (11) for any smooth initial data $\tilde f_0 = f_0$ on $\tilde u_0 = -\mu^{-1} \ln(-u_0)$, namely

$$\begin{aligned} \tilde f \;&=\; 1 + f_{1,0}\, e^{-2\tilde u} + f_{2,0}\, e^{-4\tilde u} + \cdots + f_{14,0}\, e^{-28\tilde u} \\ &\qquad + f_{15,1}\, \tilde u\, e^{-30\tilde u} + f_{15,0}\, e^{-30\tilde u} + \cdots \\ &=\; \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} f_{i,j}\, \tilde u^j\, e^{-2i\tilde u} \;, \end{aligned} \tag{12}$$

where $f_{i,j}$ are smooth functions on S^2 such that $f_{i,j} = 0$ for $j > 0$, $i \le 14$. The function $\tilde f$ thus exists and converges exponentially fast to 1, which means physically that the radiative Robinson–Trautman vacuum spacetimes approach asymptotically the Schwarzschild–(anti-)de Sitter solution as $\tilde u \to \infty$, see relation (5). In our case of pure radiation field (8) we employ the transformation (10) on expression (12) to obtain the following asymptotic expansion of f as $u \to 0_-$,

$$
\begin{aligned}
f &= 1 + f_{1,0}\,(-u)^{2/\mu} + f_{2,0}\,(-u)^{4/\mu} + \cdots + f_{14,0}\,(-u)^{28/\mu} \\
&\qquad -\mu^{-1} f_{15,1}\,\ln(-u)\,(-u)^{30/\mu} + f_{15,0}\,(-u)^{30/\mu} + \cdots \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{i,j}\,[-\mu^{-1}\ln(-u)]^j\,(-u)^{2i/\mu}\ .
\end{aligned}
\tag{13}
$$

As a result, for the initial data (9) the Robinson–Trautman type II spacetimes which contain uniform pure radiation field with the linear mass function (8) do exist in the whole region $u_0 \le u < 0$. It is also obvious that the function f approaches 1 as $u \to 0_-$ (where also $m(u) \to 0$) according to (13). In other words, these spacetimes *approach the spherically symmetric Vaidya–(anti-)de Sitter metric* near $u = 0$.

The global structure of such spacetimes is schematically indicated on Fig. 1. In fact, there are two possibly different conformal diagrams depending on the value of μ: for $\mu > 1/16$ there is a white hole singularity at $r = 0$, for $\mu \le 1/16$ there is also a naked singularity, see e.g. [9,13,16,36] for more details. At $u = 0$ all of the mass $m(u)$ is radiated away, and we can attach Minkowski space (de Sitter space when $\Lambda > 0$, anti-de Sitter when $\Lambda < 0$; the presence of the cosmological constant would change the character of conformal infinity $\mathcal{I}$ which would become spacelike or timelike, respectively) in the region $u > 0$ along the hypersurface $u = 0$. We will now investigate the smoothness of such an extension.

3.2 Extension of the metric across $u = 0$

It follows from (13) that the smoothness of f on $u = 0$ is only finite. Depending on the value of μ two different cases have to be discussed separately: $2/\mu$ is an integer, and $2/\mu$ is a real non-integer positive number.

When $2/\mu$ is an integer then due to the presence of the $\ln(-u)$ term associated with $f_{15,1} \ne 0$ the function f is of the class $C^{(30/\mu)-1}$. For μ very small, the integer number $(30/\mu) - 1$ is large so that f becomes smoothly extendable to 1 across $u = 0$ as $\mu \to 0$. This represents a naked-singularity Robinson–Trautman spacetime (see the right part of Fig. 1) unless $\mu = 0$ which gives flat space everywhere. In the limiting case $\mu = 1/16$ the function f is of the class C^{479}. For the (white hole) Robinson–Trautman spacetimes given by $\mu > 1/16$ the smoothness is lower. However, it is always at least C^{14} because $\mu \le 2$ in this case.

In the generic case when $2/\mu$ is not an integer the function f is only of the class $C^{\{2/\mu\}}$, where the symbol $\{x\}$ denotes the largest integer smaller than x. Again, with $\mu \to 0$ the function f becomes smoothly extendable. For $\mu < 1/16$ the function f is at least of the class C^{32}, for $\mu > 2$ it is not even C^1 but it remains continuous.

To investigate further the smoothness of the metric when approaching the hypersurface $u = 0_-$ which is the analogue of the Schmidt–Tod boundary of vacuum Robinson–Trautman spacetimes [25, 30] we should consider the conformal picture using suitable double-null coordinates. Such Kruskal-type coordinates for the Vaidya solution with linear mass function (8) were introduced by Hiscock [7–9], see also [16,36], and we will use this transformation only to replace the coordinate r since the null coordinate u is already appropriate. Introducing a new coordinate w by

$$
\mathrm{d}w = \frac{\mathrm{d}u}{u} - \frac{2\mathrm{d}z}{z(2\mu\, z^2 - z + 2)}\ , \quad \text{where} \quad z = -\frac{u}{r}\ , \tag{14}
$$

we put the Robinson–Trautman metric with linear mass function into the form

$$\mathrm{d}s^2 = -\left(K - 1 - 2\frac{f_{,u}}{f}\,r\right)\mathrm{d}u^2 \\ -\left(2r + u + 2\mu\frac{u^2}{r}\right)\mathrm{d}u\mathrm{d}w + 2\frac{r^2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}\ , \tag{15}$$

where $r(u,w)$. For the pure Vaidya metric characterized by $f = 1$ and $K_0 = 1$ the first term vanishes identically so that the coordinates of (15) are indeed the Kruskal-type coordinates for the Vaidya spacetime with a linear mass function.

The smoothness of a general Robinson–Trautman metric (15) depends only on the smoothness of the metric coefficients g_{uu} and $g_{\zeta\bar{\zeta}}$ (containing the function f) since the coefficient g_{uw} tends to $-r$ as $u \to 0$. The smoothness of $g_{\zeta\bar{\zeta}}$ (for any finite r) and of K is the same as of f, see (7). The function $f_{,u}/f$ is evidently one order less smooth than f. Consequently, for $2/\mu$ being integer or non-integer number, the metric (15) is of the class $C^{(30/\mu)-2}$ or $C^{\{2/\mu\}-1}$, respectively. We again observe that the spacetimes approaching the linear Vaidya metric with naked singularity (i.e., for small values of the parameter μ) possess higher order of smoothness at $u = 0$.

One might be worried about the invariance of our results, namely with respect to a rescaling of the null coordinate $u(\hat{u})$ leading to a different smoothness of the function f and of the metric. In order to change the smoothness on the hypersurface $u = 0$ such rescaling must have a singular character there. But this would lead to a degeneracy of the metric coefficient $g_{\hat{u}w}$ of the Vaidya metric, which is forbidden. Consequently, the above results are in this sense unique.

We would like to obtain analogous results concerning smoothness of the extension also for a non-zero value of the cosmological constant Λ. Unfortunately, as far as we know, there is no *explicit* transformation of the Vaidya–de Sitter metric to the Kruskal-type coordinates even for the linear mass function (contrary to the Schwarzschild–de Sitter case [33]). However, it is possible to start with the Vaidya–de Sitter metric

$$\mathrm{d}s^2 = -h(u,r)\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2\ , \tag{16}$$

where $h(u,r) = 1 + 2\mu\,u\,r^{-1} - \frac{\Lambda}{3}r^2$, and perform a coordinate transformation

$$\mathrm{d}w = g\,\mathrm{d}u + 2\frac{g}{h}\,\mathrm{d}r\ , \tag{17}$$

where $g(u,r)$ is some function. We arrive at the double-null form for the metric

$$\mathrm{d}s^2 = -\frac{h}{g}\,\mathrm{d}u\mathrm{d}w + r^2(u,w)\,\mathrm{d}\Omega^2\ . \tag{18}$$

Of course, we have to ensure that $\mathrm{d}w$ in (17) is a differential of the coordinate w. The integrability condition ($\mathrm{d}^2w = 0$) gives the following quasilinear PDE,

$$h^2\frac{\partial g}{\partial r} - 2h\frac{\partial g}{\partial u} + 4\frac{\mu}{r}g = 0\ , \tag{19}$$

for the undetermined function g, which is difficult to solve analytically. The method of characteristic curves leads to the first-order ODE of the Abel type which has not yet been solved, but the existence of its solution is guaranteed. [It is possible to apply the perturbative approach starting from the solvable case of the de Sitter metric ($\mu = 0$) and then linearise the PDE in the parameter μ. The result, however, can not be presented in a useful closed form.] For our purposes it suffices to use a general argumentation: the coordinate u is already suitably compactified and we are only determining the complementary null coordinate w to obtain the Vaidya–de Sitter metric in the Kruskal-type coordinates (which is smooth on $u = 0$). The corresponding Robinson–Trautman metric in these coordinates differs only by the term $g_{uu}(u, r, \zeta, \bar{\zeta})\,\mathrm{d}u^2$ (which is absent in the Vaidya–de Sitter case in the double null coordinates), and by a different metric coefficient $g_{\zeta\bar{\zeta}} = r^2 f^{-2} P_0^{-2}$, where $r(u, w)$ is finite and smooth when approaching the hypersurface $u = 0$. The smoothness is thus not affected by the specific transformation (17) and it is the same as for the vanishing cosmological constant. This is different from vacuum spacetimes with $m = \text{const} \neq 0$ studied in [32,33] because in the present case $m \to 0$ near $u = 0$, and the influence of Λ on the smoothness becomes negligible.

4 General mass function

The results obtained above can be considerably generalized. Inspired by a similar idea outlined in [34] we may consider a reparametrisation on the null coordinate u by

$$\tilde{u} = \gamma(u) \ , \tag{20}$$

where γ is an arbitrary continuous strictly monotonous function. We start with the evolution equation (11) for which the existence and uniqueness of solutions has been proven, and their general asymptotic behaviour (12) has been demonstrated. Now, by applying the substitution (20) in equation (11) we obtain

$$\frac{\partial f}{\partial u} = -\frac{\dot{\gamma}}{12} f \,\Delta K \ , \tag{21}$$

(where the dot denotes a differentiation) which is the evolution equation for the function $f(u, \zeta, \bar{\zeta})$. This is exactly the Robinson–Trautman equation (6) for the mass function

$$m(u) = \frac{1}{\dot{\gamma}(u)} \ . \tag{22}$$

For a given smooth initial data on u_0 there thus exists the Robinson–Trautman spacetime (1), including the cosmological constant Λ, with the mass function (22). To obtain a positive mass we consider a growing function $\gamma(u)$. Considering (4) this corresponds to a universe filled with homogeneous pure radiation

$$n^2(u) = \frac{2}{\kappa} \frac{\ddot{\gamma}}{\dot{\gamma}^2} \ . \tag{23}$$

For consistency the function γ must be convex. An asymptotic behaviour of the function f as $\gamma(u) \to \infty$ is easily obtained from the expansion (12) by substituting relation (20).

In particular, the linear mass function (8) discussed above is a special case of (22) for the transformation (20) of the form (10). More general explicit solutions can be obtained, e.g., by considering the power function

$$\gamma(u) = (-u)^{-p} \ , \qquad p > 0 \ , \tag{24}$$

which gives

$$m(u) = \frac{1}{p} (-u)^{1+p} \ , \qquad n^2(u) = \frac{2(p+1)}{\kappa p} (-u)^p \ . \tag{25}$$

Both functions m and n approach zero as $u \to 0$. Due to the theorems mentioned above, there exist Robinson–Trautman type II spacetimes in the region $u < 0$ which approach the spherically symmetric Vaidya–(anti-)de Sitter metric as $u \to 0_-$ with the mass function and pure radiation given by (25). The asymptotic behaviour of such solutions is determined by expression (5) with

$$f = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{N_i} f_{i,j} \, (-u)^{-jp} \, \exp\left[-2i(-u)^{-p}\right] , \tag{26}$$

where $f_{i,j} = 0$ for $j > 0$ if $i \leq 14$. Interestingly, the function f is now smooth on $u = 0$ for any power coefficient p, but this still does not guarantee that the extension into flat region $u > 0$ is analytic (see [33] for a similar situation).

Another simple explicit choice is

$$\gamma(u) = -M^{-1} \ln\left[\sinh(-u)\right] , \qquad M > 0 , \tag{27}$$

which implies (see also [16])

$$m(u) = M \, \tanh(-u) , \qquad n^2(u) = \frac{2M}{\kappa \, \cosh^2 u} . \tag{28}$$

In the region $u < 0$ the mass function monotonically decreases from M to zero, while the pure radiation field grows from zero to the value $2M/\kappa$ as $u \to 0$. Let us note that in this case the integrated radiation density is finite on the interval $(-\infty, 0)$, $\int_{-\infty}^{0} n^2(u) = 2M/\kappa$. The expansion near $u = 0_-$ is

$$f = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{N_i} f_{i,j} \, (-M^{-1} \ln\left[\sinh(-u)\right])^j \, \sinh^{2i/M}(-u) . \tag{29}$$

If $2/M$ is an integer then the function f belongs to the class $C^{(30/M)-1}$, otherwise it is of the class $C^{\{2/M\}}$.

5 Possible modifications and applications

The Robinson–Trautman pure radiation solutions in the region $u_0 \leq u \leq 0$ approaching the Vaidya metric near $u = 0$, which can be extended (albeit non-smoothly) to flat Minkowski space in the region $u \geq 0$ as in Fig. 1, may be used for construction of various models of radiative spacetimes. For example, it is natural to further extend the solution "backwards" into the region $u_1 < u \leq u_0$ by the Robinson–Trautman *vacuum* solution with a constant mass $m_0 = m(u_0)$, such that the function f is continuous on u_0. This is shown in Fig. 2. In such a case the spacetime may describe the process of "evaporation" of a white hole (with a different character of the singularity at $r = 0$ when $\mu \leq 1/16$) with its mass decreasing from the value m_0 to zero. Let us emphasize that the region $u < u_0$ does not represent the Schwarzschild solution because the spacetime is *not spherically symmetric* there ($f \neq 1$). In fact, this is the region where the original Chruściel theorems on the behaviour of the Robinson–Trautman vacuum spacetimes with constant mass apply (cf. (11), (12)). However, the spacetime in this region can not be extended up to the past conformal infinity $\mathcal{I}^-$ because the metric function f diverges as $u \to -\infty$.

In the presence of the cosmological constant Λ one obtains a family of exact spacetimes that describe evaporation of a white hole in the (anti-)de Sitter universe. In this case the

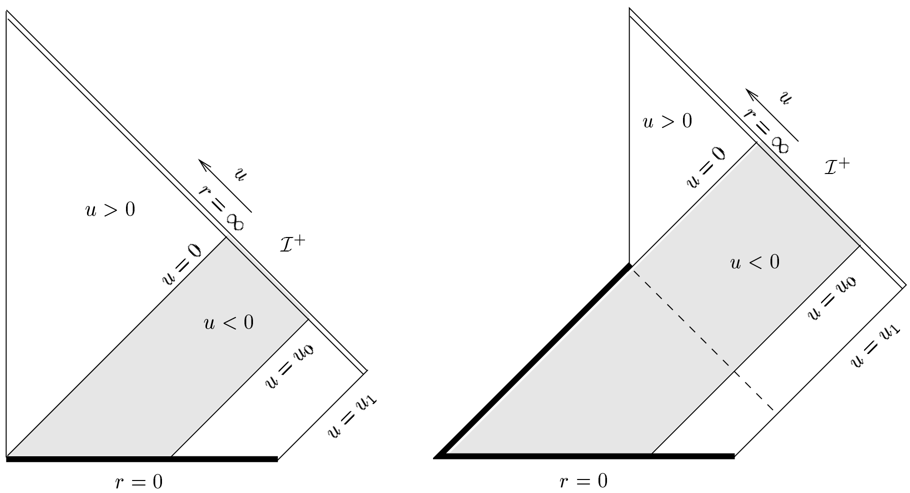

Figure 2: Possible extensions of the Robinson–Trautman radiative spacetimes into the region $u < u_0$. Pure radiation is present only in the shaded region, everywhere else it is a vacuum solution. For $u \in (u_1, u_0)$ the mass function is constant, $m(u_0) = -\mu\, u_0$, but the spacetime is not spherically symmetric — it is *not* the Schwarzschild solution ($\mu > 1/16$ on the left, $\mu \leq 1/16$ on the right).

schematic conformal diagram on Fig. 2 has to be modified in such a way that for all values of u the conformal infinity $\mathcal{I}^+$ becomes timelike (for $\Lambda > 0$) or spacelike (for $\Lambda < 0$).

Another possible modification is to consider the "advanced" form of the spacetimes (which describes an ingoing flow) rather than the "retarded" form (corresponding to outgoing flow) employed above (see, e.g., [13] for more details). This time-reversed form is obtained formally by a simple substitution $u \to -v$ in the metrics and corresponding functions. The Robinson–Trautman metric thus reads

$$\mathrm{d}s^2 = -\Big(K + 2r(\ln P)_{,v} - 2\frac{m}{r} - \frac{\Lambda}{3}r^2\Big)\,\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + 2\frac{r^2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}\;, \tag{30}$$

where $m(v)$ is an increasing mass function in $v \in [0, v_0]$. This is joined with flat Minkowskian region $v < 0$, and extended to the region $v \geq v_0$ by the corresponding Robinson–Trautman–(anti-)de Sitter black hole vacuum solution, see Fig. 3. It is a non-spherical generalization of the gravitational collapse of a shell of null dust forming a naked singularity [9,37,38] — in these works the mass function was taken to be $m(v) = \mu\, v$ (with $m(v) = 0$ for $v \leq 0$, and $m(v) = M = \mu\, v_0$ for $v \geq v_0$). The metric function P is now given by $P = f P_0$ where f is analogous to (13),

$$f = \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{i,j}\,\Big(-\mu^{-1}\ln v\Big)^j\, v^{2i/\mu}\;, \tag{31}$$

so that the smoothness of the metric on the boundary $v = 0$ depends on the parameter μ. For $v \in (v_0, v_1)$ the spacetime is vacuum but not spherically symmetric. The metric diverges as $v \to \infty$. Our results can thus be interpreted in such a way that — at least within the Robinson–Trautman family of solutions — the model [9] of collapse to a naked shell-focusing singularity which is based on the spherically symmetric Vaidya metric *is not stable* against perturbations.

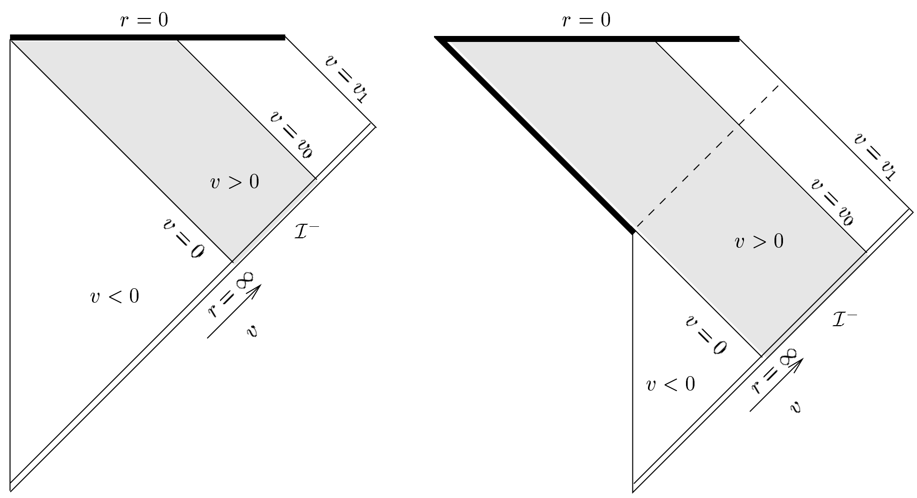

Figure 3: Time-reversed version of Fig. 2 represents the "advanced" form of the Robinson–Trautman spacetimes (30) which describes an ingoing flow of radiation.

6 Concluding remarks

In our contribution we have analyzed exact solutions of the Robinson–Trautman class which contain homogeneous pure radiation and a cosmological constant. This is a natural extension of previous works [19–34] on properties of vacuum spacetimes of this family. We have demonstrated that these solutions exist for any smooth initial data, and that they approach the spherically symmetric Vaidya–(anti-)de Sitter metric. It generalizes previous results according to which vacuum Robinson–Trautman spacetimes approach asymptotically the spherically symmetric Schwarzschild–(anti-)de Sitter metric. We have investigated extensions of these solutions into Minkowski region, and we have shown that its order of smoothness is in general only finite. Finally, we suggested some applications of the results. For example, it follows that the model of gravitational collapse of a shell of null dust diverges as $v \to \infty$ which indicates that investigations of such process based on the spherically symmetric Vaidya metric are, in fact, not stable against "non-linear perturbations", at least within the Robinson–Trautman family of exact solutions.

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