Hokkai-Gakuen University

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Energy conditions

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based on the works in recent 5 years

1. HM & Martinez, PTEP. 4 (2020) 043E02 e-Print:1810.02487 [gr-qc]
2. HM, Gen. Rel. Grav. 53 (2021) 10, 90 e-Print:2001.11335 [gr-qc]
3. HM, Phys. Rev. D 104 (2021) 8, 084088 e-Print:2107.01455 [gr-qc]
4. HM, JHEP 11 (2022) 108 e-Print:2107.04791 [gr-qc]
5. HM, Class. Quant. Grav. 39 (2022) 7, 075027 e-Print:2107.07052 [gr-qc]
6. HM & Harada, Class. Quant. Grav. 39 (2022) 19, 195002 e-Print:2205.12993 [gr-qc]
7. HM, Class. Quant. Grav. 40 (2023) 19, 195009 e-Print:2306.07326 [gr-qc]

Energy conditions and their consequences

We consider n($\geq$3) dimensions in the units $8\pi G=c=1$
the Einstein equations $G_{\mu\nu}=T_{\mu\nu}$

Standard Energy Conditions (ECs)

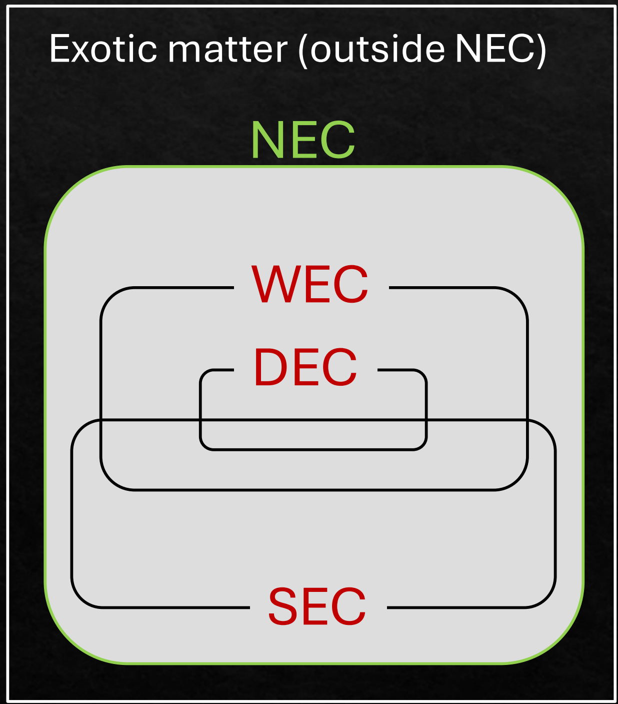
- Imposed on $T_{\mu\nu}$ to single out physically reasonable matter fields
- Null (NEC): $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any null k^μ
- Weak (WEC): $T_{\mu\nu}v^\mu v^\nu \geq 0$ for any timelike v^μ

} Non-negative energy density observed

- Dominant (DEC): WEC + $[J_\mu J^\mu \leq 0]$ for any timelike v^μ, where $J^\mu := -T^\mu{}_\nu v^\nu$
  - Energy flux (J^μ) does not propagate faster than the speed of light
- Strong (SEC): $(T_{\mu\nu} - \frac{1}{n-2} g_{\mu\nu} T) v^\mu v^\nu \geq 0$ for any timelike v^μ
  - Gravity is attractive: Violation of the SEC is not so pathological (ex. Inflationary Universe, or around regular center)

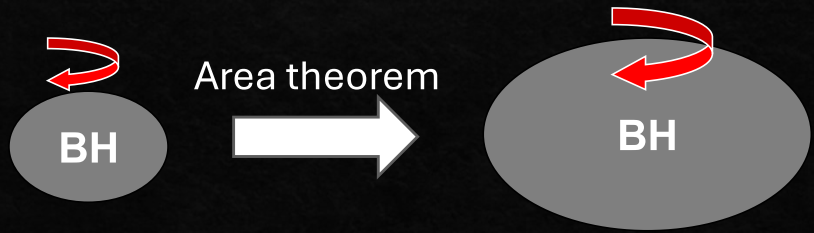
In GR, the SEC is equivalent to the timelike convergence condition (TCC)

TCC: $R_{\mu\nu}v^\mu v^\nu \geq 0$ for any timelike v^μ



Strong results in GR under the ECs

- NEC
  - Trapped surface is inside event horizon of asymptotically flat BH (Wald '84)
  - Black-hole area theorem (Hawking '72)
  - Penrose's singularity theorem ('64)
- WEC
  - Third law of BH thermodynamics (Israel ` 86)
- SEC
  - Various singularity theorems ('66-'70 by Geroch, Hawking, Penrose)
- DEC
  - BH topology theorem (Hawking '72)
  - Positive mass theorem (Schoen-Yau '79 '81, Nester '81, Witten '81)
  - Black-hole positive mass theorem (Gibbons-Hawking-Horowitz-Perry '83)
  - Zeroth law of BH thermodynamics (Bardeen-Carter-Hawking '73)
- See "A Primer on Energy Conditions" by Curiel in 2014 for more

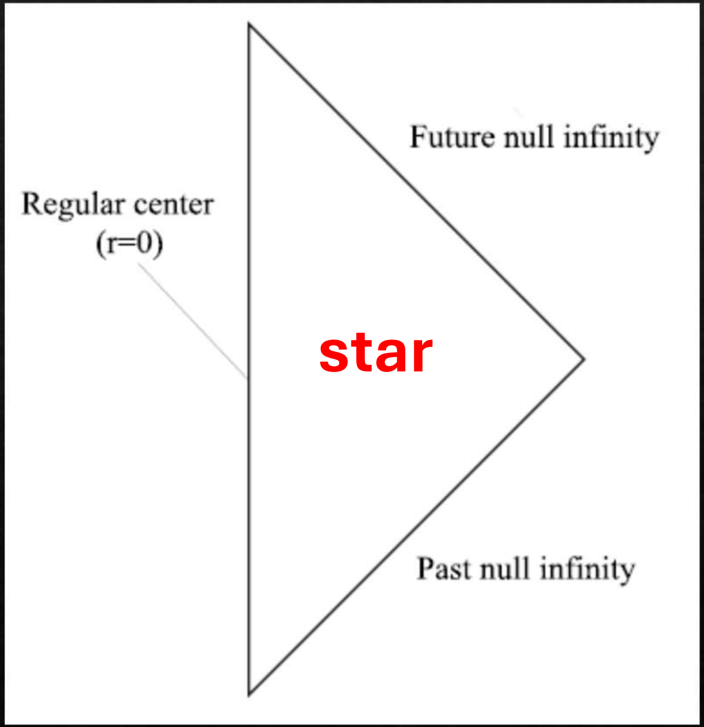


SEC violation is common in GR

- Consider a regular center in a spherical static spacetime
- Expansion around r=0 (a regular center for p≥0)

$$ds^2 \simeq -(1-\lambda r^{2+p})dt^2 + \frac{dr^2}{1-\lambda r^{2+p}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

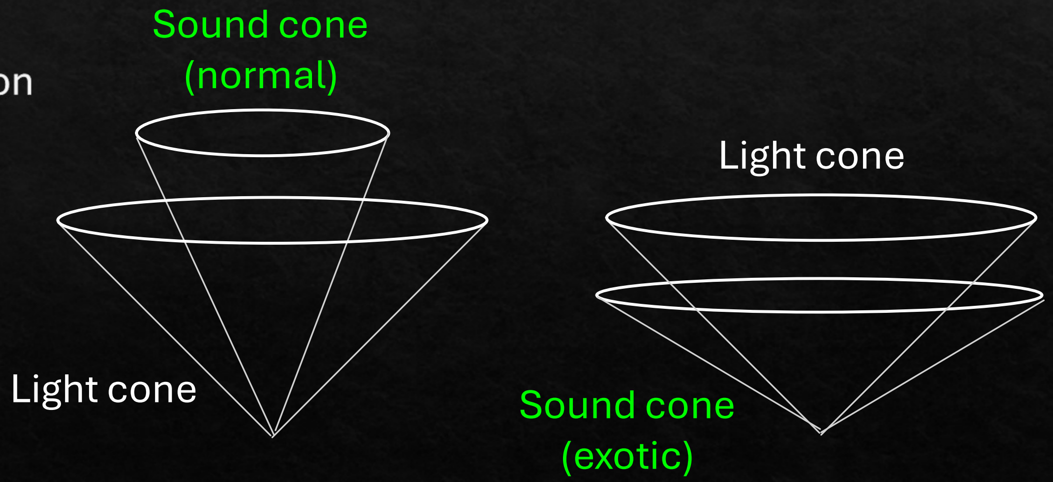
- For p=0 with $\lambda>0$, only the SEC is violated
  - It occurs in many stellar models
  - $\lambda<0$ satisfies the SEC but violates the WEC (so DEC as well)



Example of the DEC violation

- A relativistic perfect fluid
- Energy-momentum tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu},$
- Sound speed $v_\mathrm{s} := \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}}$
  - Propagation speed of the density perturbation
- A possible DEC violation is $v_\mathrm{s} > 1 (= c)$
  - A sound cone is wider than a light cone
  - Information comes out from the BH horizon

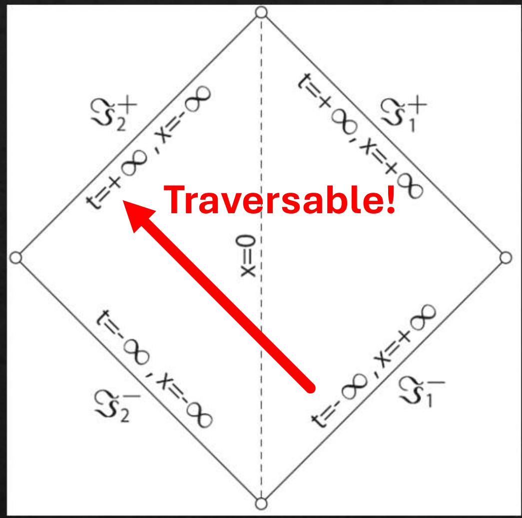
p: pressure
ρ: energy density
u^μ: four-velocity of the fluid element



NEC violation

- NEC violation = Negative energy density
  - Wormholes require violation of even weaker condition (averaged NEC) by the topological censorship theorem (Friedman-Schleich-Witt '93)
- Apparent horizon can shrink by capturing a matter field violating NEC
  - Apparent horizon = Marginally trapped surface (outer boundary of the union of trapped regions)
  - Consequence: One can escape from trapped regions (namely, from a "BH")
- This is explicitly shown by the Schwarzschild-Vaidya solution

Wormhole spacetime



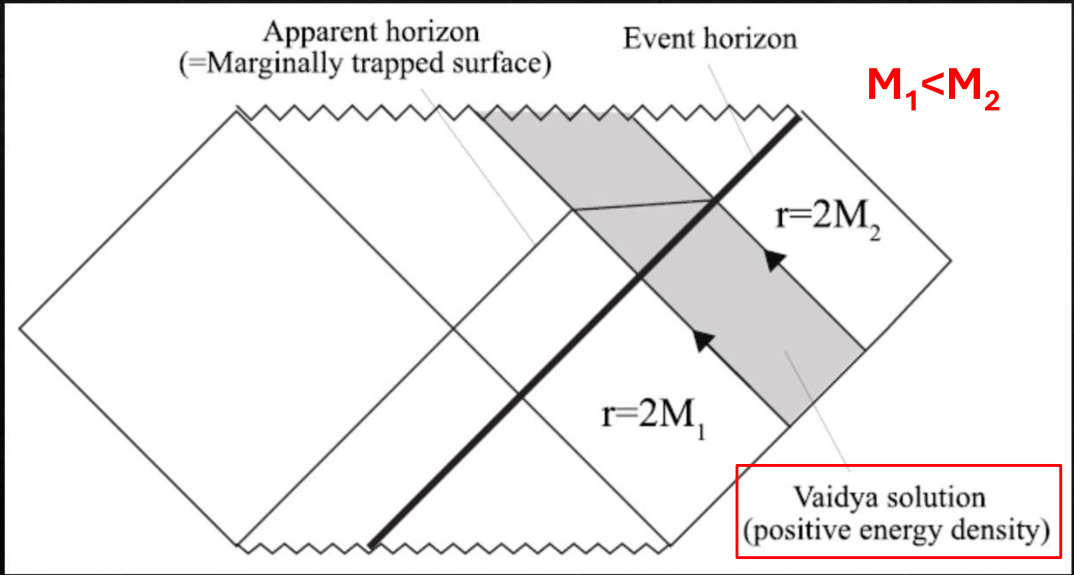
BH dynamics in Schwarzschild-Vaidya solution

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

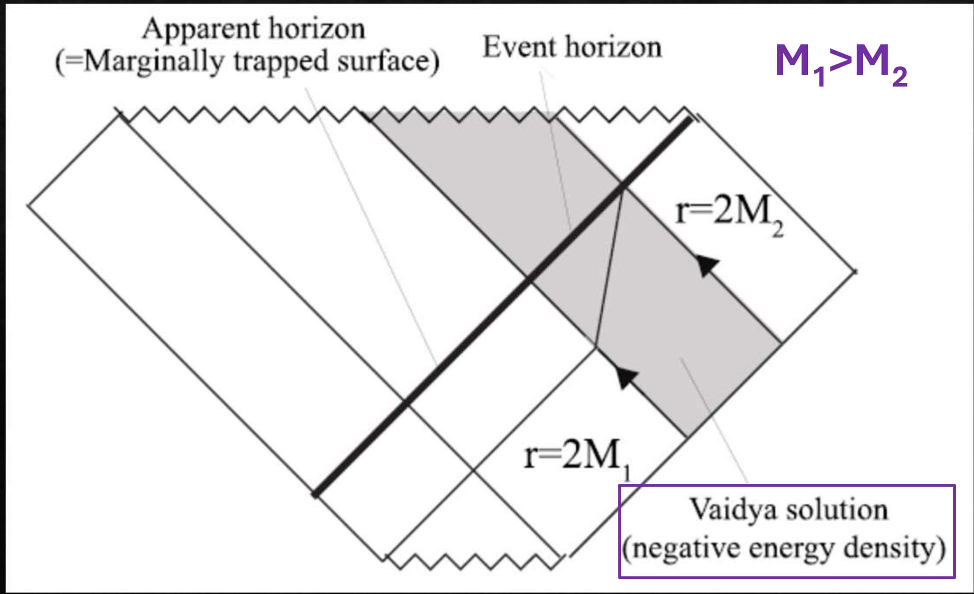
$$T_{\mu\nu} = \rho k_\mu k_\nu, \qquad \rho = \frac{M_{,v}}{r^2}, \qquad k_\mu k^\mu = 0$$

Energy density of a null dust

Growth of a Schwarzschild BH (Normal)



Shrink of a Schwarzschild BH (Exotic)



Hawking-Ellis type of $T_{\mu\nu}$ and energy conditions

Hawking-Ellis classification

- For symmetric real two-tensors $T_{\mu\nu}$
- Consider their components in an orthonormal frame $T^{(a)(b)} = T^{\mu\nu} E^{(a)}_{\mu} E^{(b)}_{\nu}$

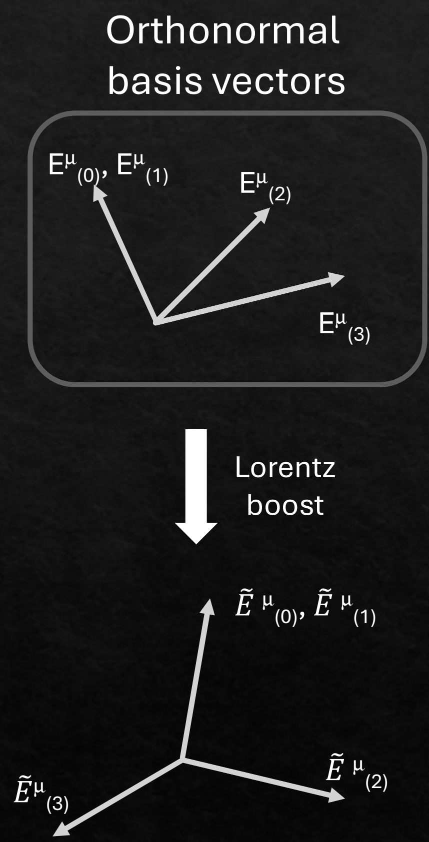
  Basis vectors

  $E^{\mu}_{(a)} = (E^{\mu}_{(0)}, E^{\mu}_{(1)}, \cdots, E^{\mu}_{(n-1)})$ where $E^{\mu}_{(a)} E_{(b)\mu} = \eta_{(a)(b)} = \mathrm{diag}(-1, 1, \cdots, 1)$

  - There is a freedom to choose basis vectors by local Lorentz transformations

  $\tilde{E}^{\mu}_{(a)} := L_{(a)}{}^{(b)} E^{\mu}_{(b)}$ where $L_{(a)}{}^{(c)} L_{(b)}{}^{(d)} \eta_{(c)(d)} = \eta_{(a)(b)}$

- Idea: To how much extent $T^{(a)(b)}$ can be simplified by choosing basis vectors?
  - Answer with the Euclidean signature: $T^{(a)(b)}$ can always be diagonal
- Answer with the Lorentzian signature: Diagonalization is not always possible
  - $T^{(a)(b)}$ are can be classified into 4 types



Hawking-Ellis type of $T_{\mu\nu}$

- Hawking-Ellis classification : Depending on the signature of eigenvectors n^μ of $T_{\mu\nu}$
  - Eigenvalue equations $$T^{(a)(b)}n_{(b)} = \lambda\eta^{(a)(b)}n_{(b)} \quad \Leftrightarrow \quad T^{\mu\nu}n_\nu = \lambda g^{\mu\nu}n_\nu,$$
  - There are 4 types in arbitrary n(≥3) dimensions (Hawking & Ellis for n=4)

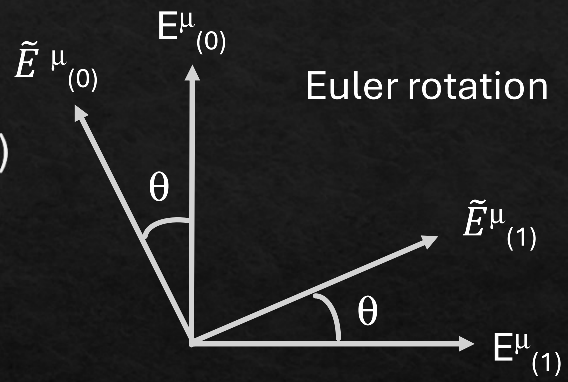
Type	Eigenvectors
I	1 timelike, $n-1$ spacelike
II	1 null (doubly degenerated), $n-2$ spacelike
III	1 null (triply degenerated), $n-3$ spacelike
IV	2 complex, $n-2$ spacelike

(Santos, Reboucas, Teixeira ’95, ‘04, Hall, Reboucas, Santos, Teixeira ’96)

Two dimensions (n=2) in the Euclidean case

- Consider a symmetric real matrix $T^{(a)(b)} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$
- Basis transformation with the Euclidean signature (Euler rotation)

$$\tilde{E}^{(0)}_{\mu} := \cos\theta E^{(0)}_{\mu} - \sin\theta E^{(1)}_{\mu},$$
$$\tilde{E}^{(1)}_{\mu} := \sin\theta E^{(0)}_{\mu} + \cos\theta E^{(1)}_{\mu}.$$



- Components with new basis vectors

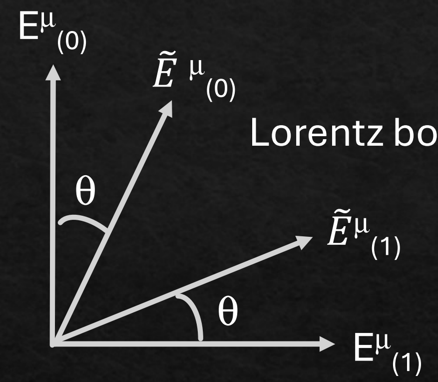
$$\tilde{T}^{(0)(0)} := T^{\mu\nu}\tilde{E}^{(0)}_{\mu}\tilde{E}^{(0)}_{\nu} = \cos^2\theta T^{(0)(0)} - 2\cos\theta\sin\theta T^{(0)(1)} + \sin^2\theta T^{(1)(1)},$$
$$\tilde{T}^{(0)(1)} := T^{\mu\nu}\tilde{E}^{(0)}_{\mu}\tilde{E}^{(1)}_{\nu} = \frac{1}{2}\sin 2\theta T^{(0)(0)} + \cos 2\theta T^{(0)(1)} - \frac{1}{2}\sin 2\theta T^{(1)(1)},$$
$$\tilde{T}^{(1)(1)} := T^{\mu\nu}\tilde{E}^{(1)}_{\mu}\tilde{E}^{(1)}_{\nu} = \sin^2\theta T^{(0)(0)} + 2\cos\theta\sin\theta T^{(0)(1)} + \cos^2\theta T^{(1)(1)}.$$

Can always be diagonal by choosing θ as $\tan 2\theta = \dfrac{2T^{(0)(1)}}{T^{(1)(1)} - T^{(0)(0)}}$

Two dimensions (n=2) in the Lorentzian case

- Consider a symmetric real matrix $T^{(a)(b)} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$
- Basis transformation with the Lorentzian signature (Lorenz boost)

$$\tilde{E}^{(0)}_\mu := \cosh\theta E^{(0)}_\mu - \sinh\theta E^{(1)}_\mu,$$
$$\tilde{E}^{(1)}_\mu := -\sinh\theta E^{(0)}_\mu + \cosh\theta E^{(1)}_\mu.$$



- Components with new basis vectors

$$\tilde{T}^{(0)(0)} := T^{\mu\nu}\tilde{E}^{(0)}_\mu\tilde{E}^{(0)}_\nu = \cosh^2\theta T^{(0)(0)} - 2\cosh\theta\sinh\theta T^{(0)(1)} + \sinh^2\theta T^{(1)(1)},$$
$$\tilde{T}^{(0)(1)} := T^{\mu\nu}\tilde{E}^{(0)}_\mu\tilde{E}^{(1)}_\nu = -\frac{1}{2}\sinh 2\theta T^{(0)(0)} + \cosh 2\theta T^{(0)(1)} - \frac{1}{2}\sinh 2\theta T^{(1)(1)},$$
$$\tilde{T}^{(1)(1)} := T^{\mu\nu}\tilde{E}^{(1)}_\mu\tilde{E}^{(1)}_\nu = \sinh^2\theta T^{(0)(0)} - 2\cosh\theta\sinh\theta T^{(0)(1)} + \cosh^2\theta T^{(1)(1)}.$$

can be diagonal by choosing θ as $\tanh 2\theta = \frac{2T^{(0)(1)}}{T^{(0)(0)} + T^{(1)(1)}}$ but $-1 < \tanh 2\theta < 1$

so not always possible

Two, three, and many

- ⟐ Two dimensions (n=2)

$$T^{(a)(b)} = \begin{pmatrix} T^{(0)(0)} & T^{(0)(1)} \\ T^{(0)(1)} & T^{(1)(1)} \end{pmatrix}$$

  - ⟐ If $T^{(0)(1)}=0$, $T^{(a)(b)}$ is diagonal => type I
  - ⟐ Hawking-Ellis types for $T^{(0)(1)} \neq 0$

$$\begin{aligned}
(T^{(0)(0)} + T^{(1)(1)})^2 > 4(T^{(0)(1)})^2 &\Rightarrow \text{Type I}, \\
(T^{(0)(0)} + T^{(1)(1)})^2 = 4(T^{(0)(1)})^2 &\Rightarrow \text{Type II}, \\
(T^{(0)(0)} + T^{(1)(1)})^2 < 4(T^{(0)(1)})^2 &\Rightarrow \text{Type IV}.
\end{aligned}$$

Type	Eigenvectors
I	1 timelike, $n-1$ spacelike
II	1 null (doubly degenerated), $n-2$ spacelike
III	1 null (triply degenerated), $n-3$ spacelike
IV	2 complex, $n-2$ spacelike

- ⟐ Three dimensions (n=3): Another type (type III) appears
- ⟐ Four and higher dimensions (n≥4): Same as three dimensions (4 Hawking-Ellis types)
  - ⟐ Reason: Additional eigenvectors are all spacelike

- For each type, the canonical form of $T_{(a)(b)}$ is obtained by local Lorentz transformations (Martin-Moruno & Visser '17)
- Equivalent representations of the energy conditions for each type are available (HM & Martinez '20)

Hawking-Ellis type I and energy conditions

- Type I: Ex. Perfect fluid, Cosmological constant (Maxwell & Scalar field also can be)
- Canonical form (diagonal)

$$T^{(a)(b)} = \begin{pmatrix} \rho & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Characteristic equation to obtain eigenvalues

$$(\lambda + \rho)(\lambda - p_1)\cdots(\lambda - p_{n-1}) = 0,$$

Equivalent representation of the energy conditions

$$\begin{aligned} \text{NEC}: &\quad \rho + p_i \geq 0 \quad \text{for} \quad i = 1, 2, \cdots, n-1, \\ \text{WEC}: &\quad \rho \geq 0 \text{ in addition to NEC}, \\ \text{DEC}: &\quad \rho - p_i \geq 0 \quad \text{for} \quad i = 1, 2, \cdots, n-1 \text{ in addition to WEC}, \\ \text{SEC}: &\quad (n-3)\rho + \textstyle\sum_{j=1}^{n-1} p_j \geq 0 \quad \text{in addition to NEC} \end{aligned}$$

Hawking-Ellis type II and energy conditions

- Type II : Ex. Null dust (Maxwell & Scalar field also can be)
- Canonical form

$$T^{(a)(b)} = \begin{pmatrix} \rho+\nu & \nu & 0 & 0 & \cdots & 0 \\ \nu & -\rho+\nu & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Characteristic equation to obtain eigenvalues

$$(\lambda+\rho)^2(\lambda-p_2)\cdots(\lambda-p_{n-1})=0,$$

Equivalent representations of the energy conditions

$$\begin{aligned} &\text{NEC}: \ \nu \geq 0 \text{ and } \rho+p_i \geq 0 \ \text{ for } \ i=2,3,\cdots,n-1, \\ &\text{WEC}: \ \rho \geq 0 \text{ in addition to NEC}, \\ &\text{DEC}: \ \rho-p_i \geq 0 \ \text{ for } \ i=2,3,\cdots,n-1 \text{ in addition to WEC}, \\ &\text{SEC}: \ (n-4)\rho+\textstyle\sum_{j=2}^{n-1}p_j \geq 0 \quad \text{in addition to NEC} \end{aligned}$$

Hawking-Ellis type III and energy conditions

- Type III : Ex. Gyraton (Null dust with angular momentum), Rotating pressureless null shell
- Canonical form

$$T^{(a)(b)} = \begin{pmatrix} \rho+\nu & \nu & \zeta & 0 & 0 & \cdots & 0 \\ \nu & -\rho+\nu & \zeta & 0 & 0 & \cdots & 0 \\ \zeta & \zeta & -\rho & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & p_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Characteristic equation to obtain eigenvalues

$$(\lambda+\rho)^3(\lambda-p_3)\cdots(\lambda-p_{n-1})=0,$$

Type III matter field violates all the energy conditions

Hawking-Ellis type IV and energy conditions

- Type IV : Ex. Quantum vacuum expectation value $\langle T_{\mu\nu} \rangle$ can be of type IV (Roman '86)
- Canonical form

$$T^{(a)(b)} = \begin{pmatrix} \rho & \nu & 0 & 0 & \cdots & 0 \\ \nu & -\rho & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Characteristic equation to obtain eigenvalues

$$[(\lambda + \rho)^2 + \nu^2](\lambda - p_2) \cdots (\lambda - p_{n-1}) = 0,$$

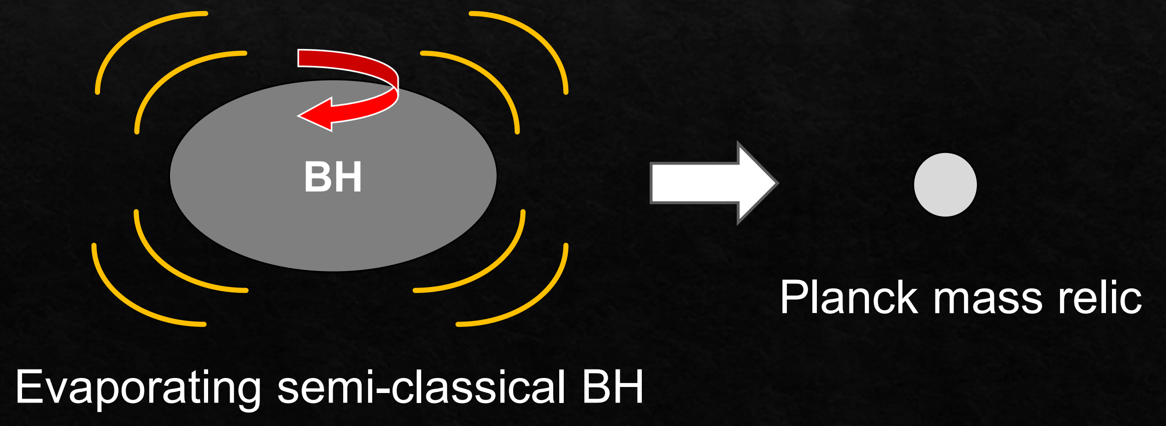
Type IV matter field violates all the energy conditions

A note: Matter field in static spacetimes

- Static solutions are consistent only with type I matter field in a large class of gravity (Hall ‘93, HM ‘21)

$$S=\frac{1}{2}\int \mathrm{d}^n x\sqrt{-g}f(R_{\mu\nu\rho\sigma},g^{\mu\nu})+S_{\mathrm{m}},$$

  - Quantum vacuum expectation value $\langle T_{\mu\nu}\rangle$ in the static background can be of type IV (Roman ‘86, Martin-Moruno & Visser ‘13)
  - However, a type IV matter is not possible if back-reaction is taken into account
- A consequence for evaporating BHs
  - A static Planck mass relic is possible as the final state only if the semi-classical matter field is of type I



A note: Hawking-Ellis type on the Killing horizon

- Spherically symmetric spacetime in the diagonal coordinates

$$ds^2 = -H(x)dt^2 + \frac{dx^2}{H(x)} + r(x)^2\gamma_{ij}(z)dz^i dz^j,$$

- $T^{(a)(b)}$ is diagonal, where $T_{\mu\nu}(:= G_{\mu\nu})$
- However, $T^{(a)(b)}$ is NOT of type I everywhere
  - Because the coordinates do not cover Killing horizons
    - Killing horizon: Regular null hypersurface given by H(x)=0
  - A typical pitfall in the GR research (many have fallen)
  - One has to use coordinates covering horizons to obtain a correct result

$$E^{(0)}_\mu dx^\mu = \begin{cases} -\sqrt{H}dt & (\text{if } H(x) > 0) \\ -\sqrt{-H^{-1}}dx & (\text{if } H(x) < 0) \end{cases}$$

$$E^{(1)}_\mu dx^\mu = \begin{cases} -\sqrt{H^{-1}}dx & (\text{if } H(x) > 0) \\ -\sqrt{-H}dt & (\text{if } H(x) < 0) \end{cases},$$

$$E^{(k)}_\mu dx^\mu = re^{(k)}_i dz^i,$$

Basis one-forms

$$ds^2 = -H(x)dv^2 + 2dvdx + r(x)^2\gamma_{ij}(z)dz^i dz^j,$$

$$v := t + \int H(x)^{-1}dx,$$

Single null coordiantes

Result (HM '21) : The matter field on the horizon is of type I if $r''(x_h)=0$ and of type II if $r''(x_h)\neq 0$

Energy conditions for various matter fields

Fluid and Λ

- Perfect fluid $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu},$

- NEC: $\rho + p \geq 0.$
- WEC: $\rho \geq 0$ in addition to NEC.
- DEC: $\rho - p \geq 0$ in addition to WEC.
- SEC: $(n-3)\rho + (n-1)p \geq 0$ in addition to NEC.

- Cosmological constant

  $\rho = \Lambda$ and $p = -\Lambda.$

  - Positive Λ : Only the SEC is violated
  - Negative Λ : Only the NEC & SEC are satisfied

- Null dust fluid $T_{\mu\nu} = \mu k_\mu k_\nu,$ $k_\mu k^\mu = 0$
  - All the energy conditions are equivalent to $\mu \geq 0$

Minimally coupled scalar field

- Lagrangian density: $\mathcal{L}_{\rm m} = -\left(\frac{1}{2}\varepsilon(\nabla\phi)^2 + V(\phi)\right),$ ⇒ $T_{\mu\nu} = \varepsilon(\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left(\frac{1}{2}\varepsilon(\nabla\phi)^2 + V(\phi)\right).$
  - $\varepsilon = 1$: Real scalar field, $\varepsilon = -1$: Ghost scalar field
- Equivalent representations to the energy conditions (HM-Harada '22):
  - If $\nabla_\mu\phi = 0$: NEC holds, WEC is $V \geq 0$, DEC is $V \geq 0$, SEC is $V \leq 0$
  - If $\nabla_\mu\phi \neq 0$: with $\varepsilon = -1$: All ECs are violated
  - If $\nabla_\mu\phi \neq 0$: with $\varepsilon = 1$: NEC holds, and others depend on signature of $\nabla_\mu\phi$ as

$\nabla_\mu\phi(\neq 0)$ *with* $\varepsilon = 1$	*WEC*	*DEC*	*SEC*
Timelike	$V \geq (\nabla\phi)^2/2$	$V \geq 0$	$V \leq -(n-2)(\nabla\phi)^2/2$
Spacelike	$V \geq -(\nabla\phi)^2/2$	$V \geq 0$	$V \leq 0$
Null	$V \geq 0$	$V \geq 0$	$V \leq 0$

All the ECs are satisfied if and only if the scalar field is real ($\varepsilon = 1$) and massless ($V=0$)

Maxwell & Yang-Mills field

- Maxwell field $\mathcal{L}_{\rm m} = -\frac{\alpha}{4} F_{\mu\nu} F^{\mu\nu},$ ⇨ $T_{\mu\nu} = \alpha\left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right).$

- Yang-Mills field with symmetry group SU(N)

$$\mathcal{L}_{\rm m} = -\frac{\alpha}{2} {\rm Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{\alpha}{4} F^{a}_{\mu\nu} F^{a\mu\nu}, \quad \Rightarrow \quad T_{\mu\nu} = \alpha\left(F^{a}_{\mu\rho} F^{a\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F^{a}_{\rho\sigma} F^{a\rho\sigma}\right).$$

where $\mathbf{A} = A_{\mu} {\rm d}x^{\mu} = A^{a}_{\mu} \tau^{a} {\rm d}x^{\mu},$ $\quad {\rm Tr}(\tau^{a}\tau^{b}) = \frac{1}{2}\delta^{ab}, \quad [\tau^{a}, \tau^{b}] = \tau^{a}\tau^{b} - \tau^{b}\tau^{a} = i f^{abc} \tau^{c}.$

$$F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + i\zeta f^{bca} A^{b}_{\mu} A^{c}_{\nu}.$$

- All the energy conditions are equivalent to α≥0 (HM-Martinez '20)

The result for Maxwell field has been recently generalized for a p-form field (Bernardo-Brahma-Faruk '22)

A lemma for more complicated matter fields

- Consider the sum of several matter fields

$$T_{\mu\nu} = \sum_{A=1}^{p} \Pi^A T^A_{\mu\nu}$$

(A=1,2,…,p)

Π^A Non-negative functions

$T^A_{\mu\nu}$ Energy-momentum tensors

$J^A_\mu = j^A_{(a)} E^{(a)}_\mu$ Associated current vector

- If $T^A_{\mu\nu}$ satisfy the NEC, WEC, or SEC for all p, then $T_{\mu\nu}$ satisfies the same energy condition
- If $T^A_{\mu\nu}$ satisfy the DEC for all p and $j^A_{(0)} j^B_{(0)} \geq 0$ is satisfied for any set of A and B,

  then $T_{\mu\nu}$ also satisfies the DEC

Maxwell field + Something

- Proca field

$$\mathcal{L}_{\rm m} = -\alpha\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu\right),$$

$$T_{\mu\nu} = \alpha\left\{F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + m^2\left(A_\mu A_\nu - \frac{1}{2}g_{\mu\nu}A^\rho A_\rho\right)\right\}.$$

  - All the energy conditions are equivalent to $\alpha \geq 0$

- Proca-dilaton field

$$\mathcal{L}_{\rm m} = -\left(\frac{1}{2}\varepsilon(\nabla\phi)^2 + V(\phi)\right) - e^{-\gamma\phi}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu\right),$$

$$T_{\mu\nu} = \varepsilon(\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left(\frac{1}{2}\varepsilon(\nabla\phi)^2 + V(\phi)\right) + e^{-\gamma\phi}\left\{F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + m^2\left(A_\mu A_\nu - \frac{1}{2}g_{\mu\nu}A^\rho A_\rho\right)\right\}.$$

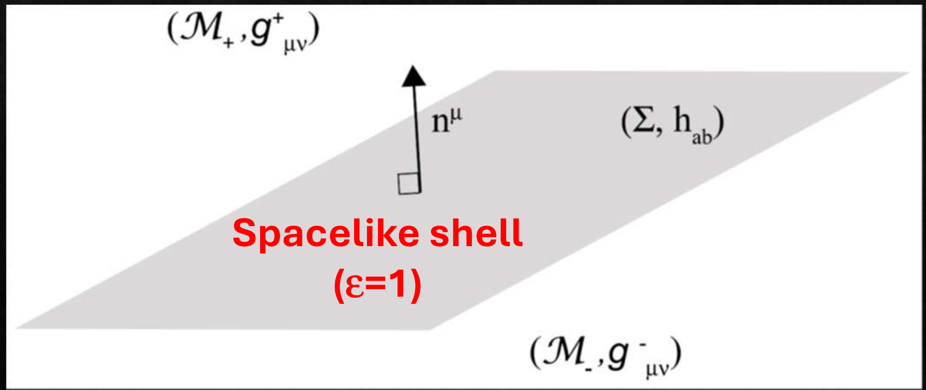
If the scalar field satisfies an energy condition, the Proca-dilaton field satisfies as well

Energy conditions for thin shell

- Attach two spacetimes at a hypersurface Σ
  - Regular matching: No matter field induced on Σ (Metric is $C^{1,1}$ at Σ)
  - Thin shell: An induced matter field $t_{\mu\nu}$ on Σ (Metric is $C^{0,1}$ at Σ)
  - Useful for model building of physical phenomena (Gravitational collapse, cosmic bubble, braneworld, etc)
- $t_{\mu\nu}$ on Σ is determined by junction conditions
  - Localized version of the gravitational equations
- Junction condition for non-null Σ in GR (Israel `66)

$$-\varepsilon\left([K_{\mu\nu}] - h_{\mu\nu}[K]\right) = \kappa_n t_{\mu\nu}$$

$\kappa_n = 8\pi G$, $h_{\mu\nu}$: Induced metric on Σ
$K_{\mu\nu}$: Extrinsic curvature of Σ, ε: Signature of Σ
$t_{\mu\nu}$: Induced energy-momentum tensor on Σ



Matter field on a lightlike thin shell

μ: surface density
j_A: surface current (A=2,3,...,n-1)
p: isotropic surface pressure

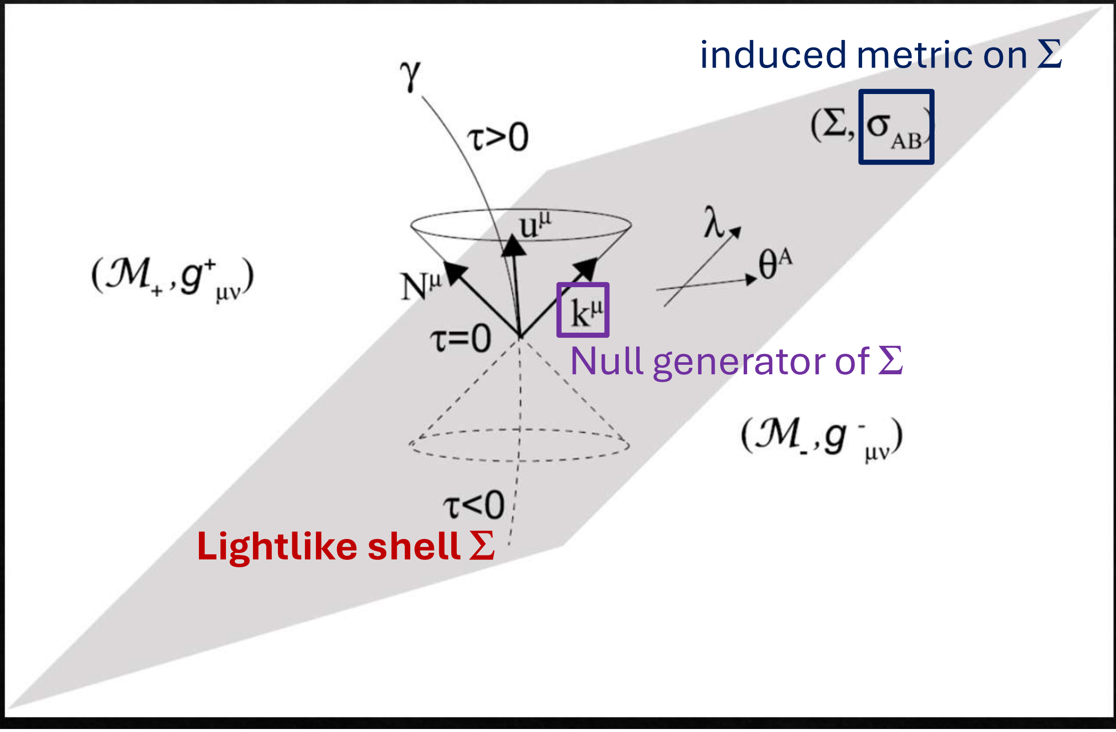
- Junction conditions for lightlike (null) Σ
- General form of $t_{\mu\nu}$: $t_{\mu\nu} = (-k_\eta u^\eta)^{-1} S_{\mu\nu}$, where $S_{\mu\nu} := \mu k_\mu k_\nu + j_A(k_\mu e^A_\nu + e^A_\mu k_\nu) + p\sigma_{AB} e^A_\mu e^B_\nu$. (Poisson '02)
  - In any theory of gravity

e_ν^A is a basis pointing transverse direction to the generator

- Junction conditions for null Σ in GR (Barrabes-Israel `91, Poisson `02)

$$\kappa_n \mu = -\sigma^{AB}[C_{AB}], \qquad \kappa_n j^A = \sigma^{AB}[C_{\lambda B}], \qquad \kappa_n p = -[C_{\lambda\lambda}].$$

  - κ_n=8πG
  - C_{AB}: Transverse curvature of Σ



Energy conditions for a lightlike thin shell

- General form of $t_{\mu\nu}$ in general gravitation theory

$$t_{\mu\nu} = (-k_\eta u^\eta)^{-1} S_{\mu\nu}, \qquad S_{\mu\nu} := \mu k_\mu k_\nu + j_A(k_\mu e^A_\nu + e^A_\mu k_\nu) + p\sigma_{AB} e^A_\mu e^B_\nu.$$

- Equivalent representations of the energy conditions (HM '23)

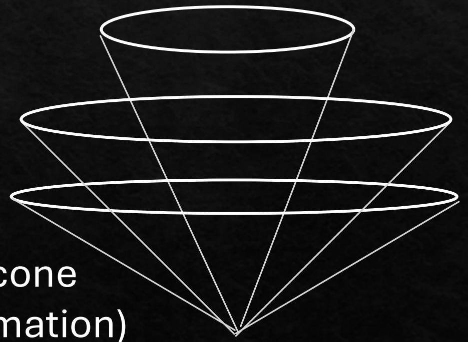
$$J^2 := j_A j_B \sigma^{AB}.$$

	Hawking-Ellis type	NEC, WEC, SEC	DEC
$J = 0$	II	$\mu \geq 0,\ p \geq 0$	$\mu \geq 0,\ p = 0$
$J \neq 0,\ p = 0$	III	violated	violated
$Jp \neq 0,\ J^2 \neq \mu p$	II	$\mu p > J^2,\ p > 0$	violated
$Jp \neq 0,\ J^2 = \mu p$	I	$p > 0$	violated

- Rotating pressureless null shell is of type III
- This result can be used in any gravitation theory in arbitrary dimensions

Caution: Slow-rotation approximation

- Slow-rotation approximation is sometimes used to analyze rotating BHs
  - Only up to linear order of a/r is taken into account
  - Location of the event horizon remains the same
- Approximation may mislead to different types
  - Ex. Type III (Full-order) -> Type II (linear approximation)
  - because light cones in the full order and under the approximation may be different



Applications in modified gravity

Effective Energy Conditions

- Modified gravity: Field equations are not $G_{\mu\nu}=T_{\mu\nu}$
  - Scalar-tensor theories, Higher-curvature theories (Einstein-Gauss-Bonnet, Lovelock gravity, etc)
- Definition [Effective energy-momentum tensor]: $\bar{T}_{\mu\nu}=G_{\mu\nu}$
  - If $\bar{T}_{\mu\nu}$ violates energy conditions, something (interesting) happens
  - Wormhole, non-singular BH, etc
- Recently, a model-building study has been very active in astrophysics
  - Consider not a solution but just a metric to find something new beyond GR
  - $\bar{T}_{\mu\nu}$ of such a model metric <u>should</u> satisfy the Energy Conditions in Asymptotically Flat regions

Asymptotic Effective Energy Conditions

- Definition [Asymptotic effective energy conditions]:
  ECs for an effective energy-momentum tensor $\bar{T}_{\mu\nu}$=$G_{\mu\nu}$ in asymptotically flat regions
- Proposal:
  Physically reasonable solutions must satisfy all the asymptotic effective energy conditions
  - Originally proposed in HM '22 in the context of non-singular BHs
  - AEEC can single out physically reasonable metrics without specifying the theory
  - Example: A variety of metrics describing non-singular BHs

4 non-singular BHs with a regular center

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2),$$

$$f(r) := 1 - \frac{2M(r)}{r},$$

- Parameters: m & l
- f(r)=0 : Two Killing horizons
- Asymptotically flat as r→ ∞
- Regular center (r=0): de Sitter core

- Bardeen ('68): $M(r) = \frac{mr^3}{(r^2+l^2)^{3/2}}$
- Hayward ('06): $M(r) = \frac{mr^3}{r^3+2ml^2}$
- Dymnikova ('04): $M(r) = \frac{2m}{\pi}\left\{\arctan\left(\frac{r}{l}\right) - \frac{lr}{r^2+l^2}\right\}$
- Fan & Wang ('16): $M(r) = \frac{mr^3}{(r+l)^3}$

A model building study

Where are Energy Conditions respected?

	NEC	WEC	DEC	SEC
Bardeen ($m > 0$)	everywhere	everywhere	$0 \leq r \leq 2l$	$r \geq \sqrt{2/3}l$
Bardeen ($m < 0$)	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Hayward ($m > 0$)	everywhere	everywhere	$0 \leq r \leq (4ml^2)^{1/3}$	$r \geq (ml^2)^{1/3}$
Hayward ($m_s < m < 0$)	everywhere	everywhere	$\emptyset$	everywhere
Dymnikova ($m > 0$)	everywhere	everywhere	everywhere	$r \geq l$
Dymnikova ($m < 0$)	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Fan-Wang ($m > 0$)	everywhere	everywhere	everywhere	$r \geq l$
Fan-Wang ($m < 0$)	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

- Bardeen & Hayward BHs don’t respect the DEC at infinity: Discarded
- Dymnikova & Fan-Wang BHs respect the DEC everywhere
    - We will focus on the rotating counterparts of these two BHs

Rotating counterparts: Metric ansatz

- Gurses-Gursey (GG) metric (`74): M(r) is a function and Δ(r)=0 is a Killing horizon

$$
\begin{aligned}
\mathrm{d}s^2 = &-\left(1-\frac{2M(r)r}{\Sigma(r,\theta)}\right)\mathrm{d}t^2 - \frac{4aM(r)r\sin^2\theta}{\Sigma(r,\theta)}\mathrm{d}t\mathrm{d}\phi \\
&+ \frac{\Sigma(r,\theta)}{\Delta(r)}\mathrm{d}r^2 + \Sigma(r,\theta)\mathrm{d}\theta^2 + \left(r^2+a^2+\frac{2a^2M(r)r\sin^2\theta}{\Sigma(r,\theta)}\right)\sin^2\theta\mathrm{d}\phi^2, \\
&\Sigma(r,\theta) := r^2 + a^2\cos^2\theta, \qquad \Delta(r) := r^2 + a^2 - 2rM(r).
\end{aligned}
$$

- Effective energy-momentum tensor: Hawking-Ellis type I (Gurses & Gursey `74)

$$\tilde{T}^{(a)(b)} := \tilde{T}^{\mu\nu}E^{(a)}_{\mu}E^{(b)}_{\nu} = \mathrm{diag}(\rho, p_1, p_2, p_3),$$

$$\boxed{\rho = -p_1} = \frac{2r^2M'}{\Sigma^2},$$

$$p_2 = p_3 = -\frac{rM''\Sigma + 2M'a^2\cos^2\theta}{\Sigma^2}$$

Rotating counterparts with GG metric

- Rotating Dymnikova BH:

$$M(r) = \frac{2m}{\pi}\left\{\arctan\left(\frac{r}{l}\right) - \frac{lr}{r^2+l^2}\right\}$$

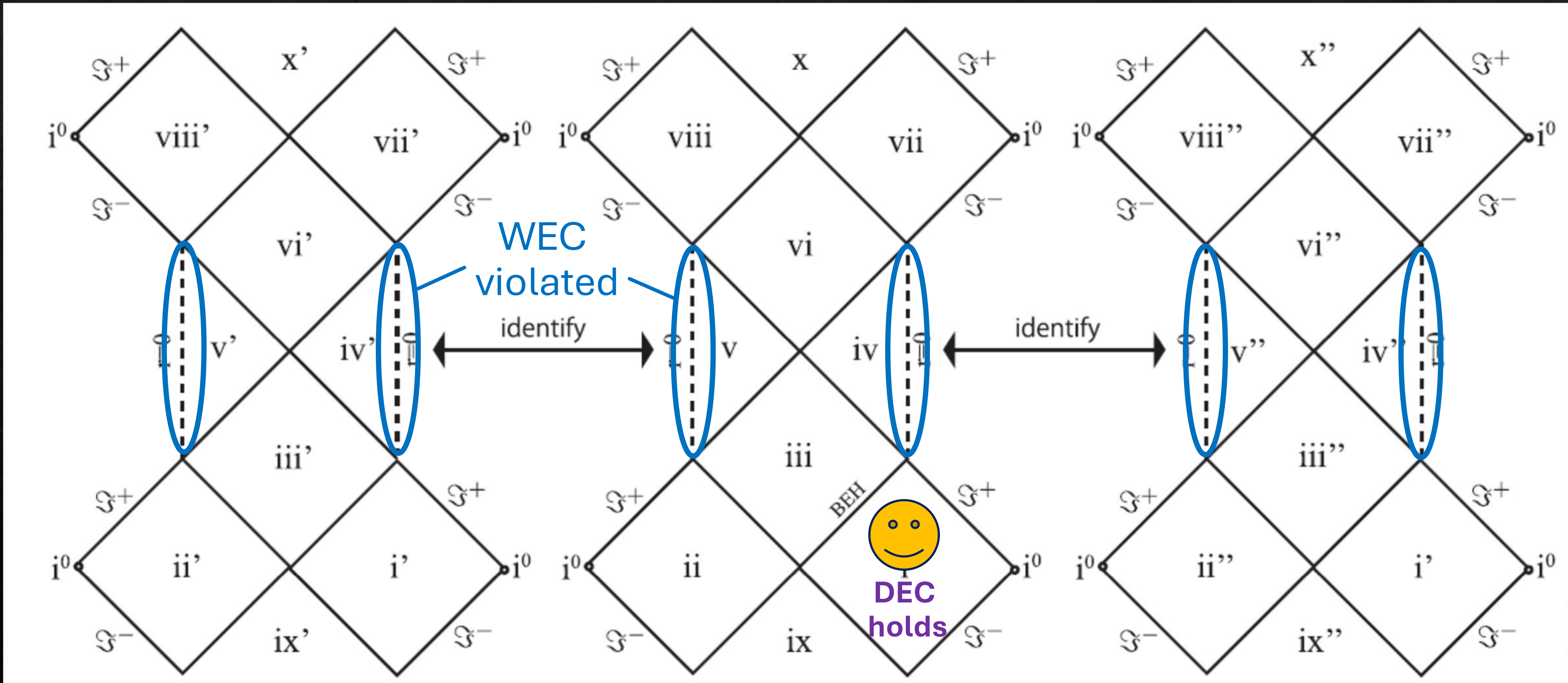
  - Singularity-free in $-\infty<r<\infty$ except for the ring $(r,\theta)=(0,\pi/2)$
  - All ECs are respected at spatial infinity
  - DEC is respected on & outside the event horizon

- Rotating Fan-Wang BH:

$$M(r) = \frac{mr^3}{(r+l)^3}$$

  - Curvature singularity at $r=-l(<0)$
  - Domain of r is $r_s<r<\infty$
  - All ECs are respected at spatial infinity
  - Discarded by its singular nature

Rotating Dymnikova BH for $m > m_{ex}$



4 non-degenerate Killing horizons in $-\infty<r<\infty$

The ring $(r,\theta)=(0,\pi/2)$ could be a p.p. curvature singularity

Summary

1. $T_{\mu\nu}$ is classified into 4 Hawking-Ellis types
   - Type III and IV violate all the standard energy conditions
   - Equivalent representations of the energy conditions are available for type I and II
   - Only type I is compatible with static spacetime
2. Some criteria are available to check energy conditions
   - Minimally coupled scalar field, fluid, lightlike shell, etc
   - Be careful for approximation that changes light cones
3. Asymptotic effective energy conditions (AEEC) have been proposed
   - They should be checked in the model-building study in astrophysics
   - The meaning of the violation of effective DEC should be clarified

FIN