Conservation laws in the field theories and in General Relativity

Motivations – „integrals of motion"

(eqs. of motion, e.g. $m\ddot{x} = -\frac{dV(x)}{dx} \Big/ \dot{x}$

$$\Rightarrow \frac{d}{dt}\Big[\underbrace{\tfrac{1}{2}m\dot{x}^2 + V(x)}\Big] = 0$$

$$\Rightarrow \qquad = \text{const}$$

usually associated with some symmetry

Start from variational principle for general field

$y_A(x)$, $A = 1, \dots N$, $x \equiv \{x^\mu\}$, $\mu = 0,1,2,3$

Lagrangian $L = L(x; y_A, y_{A,\alpha}, y_{A,\alpha\beta})$ ... scalar

Lagrangian density $\mathcal{L} = \sqrt{-g}\, L$

Action $S = \int_\Omega \mathcal{L}\, d^4x$

Variation. princ. $\delta S = \int_\Omega L^A \delta y_A\, d^4x = 0 \qquad \delta y_A\big|_{\partial\Omega} = \delta y_{A,\alpha}\big|_{\partial\Omega} = 0$

$$L^A = \frac{\delta S}{\delta y_A} = \frac{\partial \mathcal{L}}{\partial y_A} - \partial_\rho \frac{\partial \mathcal{L}}{\partial y_{A,\rho}} + \partial_\sigma \partial_\rho \frac{\partial \mathcal{L}}{\partial y_{A,\rho\sigma}} \qquad (1)$$

$\delta S = 0$ for arbitrary $\delta y_A \Rightarrow$

equations of motion (EOM): $\boxed{L^A = 0}$ $\qquad (1^*)$

$\equiv$ field eqs.

The same physical system can be described in various $\{x^\alpha\}$, but we can change also y_A (indep.)

We call these <u>gauge transformations</u> ("general")

(i) $x^\alpha \to x'^\alpha = x'^\alpha(x^\beta)$, $x^\beta = x^\beta(x'^\alpha)$

interconnected with

$$y'_A(x') = Y_A(y(x); x)$$

for example for a vector $A_\mu(x)$ (like elmg. potential)

$$A'_\mu(x') = \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}}_{\text{functions of } x} A_\alpha(x)$$

(ii) $x'^\alpha = x^\alpha$ but change of y_A,

e.g. $$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$$

"usual" gauge transformation in electromagnetism

general function

In the new gauge the action is

$$S' = \int_{\Omega'} \mathcal{L}'(x', y'_A(x'))\, d^4x'$$

in general form of $\mathcal{L}'$ different from $\mathcal{L}$, EOM may change

Basic Proposition:

Two actions S and S' describe the same physical situation if $\mathcal{L}$ and $\mathcal{L}'$ differ by a divergence, i.e. when there exist functions $Q^\alpha(x; y_A)$ such that

$$S' = \underbrace{\int_\Omega \Big[\mathcal{L}(x; y_A(x))}_{=S} - \underbrace{\partial_\alpha Q^\alpha(x; y_A(x))\Big] d^4x}_{\text{after } \delta:\ \int_{\partial\Omega} \delta Q^\alpha \to 0} \qquad (*)$$

$$\Rightarrow \quad \delta S = \delta S' \quad \Rightarrow \quad L^A = 0,\ L'^A = 0$$

(„Divergence" is sufficient, not necessary)

From possible gauge transformations we select

„Symmetry transformations"

These are such gauge transf. which do not change the form of EOM

$$L^A(x; y_B(x)) = L'^A(x; y(x)) \qquad \text{e.g. } F^{\mu\nu}{}_{,\nu} = 0$$

$$\to F'^{\mu\nu}{}_{,\nu} = 0$$

Sufficient condition for this is that the form of the Lagrangian does not change:

$$\boxed{\mathcal{L}'(x; y(x)) = \mathcal{L}(x; y(x))}$$

we shall assume this

e.g. elmag: $\mathcal{L}'(F'_{\mu\nu}) = F'_{\mu\nu} F'^{\mu\nu}$, write $F'_{\mu\nu} \to F_{\mu\nu}$, so $\mathcal{L}'(F_{..}) = \mathcal{L}(F_{..})$

Example of Lagrangians differing by a divergence

simple case of scalar field in Minkowski: $\varphi(x^\mu)$

$$\mathcal{L} = \varphi_{,\alpha}\varphi^{,\alpha} = \eta^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}$$

"Wave Equation"

$$\Rightarrow \quad \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi}}_{=0} - \frac{\partial}{\partial x^\alpha}\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}} = -\varphi^{,\alpha}{}_{\alpha} = 0, \text{ i.e. WE: } \Box\varphi = 0$$

field eq.

Let there is a simple gauge transformation

$$\varphi \to \tilde{\varphi} = \varphi + k_\alpha x^\alpha,$$ where k_α is a constant null vector, $k_\alpha k^\alpha = 0$

$$\Rightarrow \tilde{\varphi}_{,\alpha} = \varphi_{,\alpha} + k_\alpha$$

Lagrangian after gauge transformation ($\mathcal{L}(\varphi) = \tilde{\mathcal{L}}(\varphi)$ so symmetry)

$$\tilde{\mathcal{L}} = \tilde{\varphi}_{,\alpha}\tilde{\varphi}^{,\alpha} = (\varphi_{,\alpha} + k_\alpha)(\varphi^{,\alpha} + k^\alpha) = \varphi_{,\alpha}\varphi^{,\alpha} + 2\varphi_{,\alpha}k^\alpha + \underbrace{k_\alpha k^\alpha}_{=0}$$

$$= \underbrace{\varphi_{,\alpha}\varphi^{,\alpha}}_{\mathcal{L}(\varphi)} + \underbrace{(2k^\alpha\varphi)_{,\alpha}}_{\text{divergence}}$$

$$\boxed{\tilde{\mathcal{L}} = \mathcal{L} + \text{divergence}}$$

Field eq. after gauge transf.

$$\underbrace{\frac{\partial\tilde{\mathcal{L}}}{\partial\tilde{\varphi}}}_{=0} - \frac{\partial}{\partial x^\alpha}\frac{\partial\tilde{\mathcal{L}}}{\partial\tilde{\varphi}_{,\alpha}} = -\frac{\partial}{\partial x^\alpha}\left[\frac{\partial}{\partial\tilde{\varphi}_{,\alpha}}\left(\eta^{\rho\sigma}\tilde{\varphi}_{,\rho}\tilde{\varphi}_{,\sigma}\right)\right] = \frac{\partial}{\partial x^\alpha}2\tilde{\varphi}^{,\alpha}$$

$$= 2\tilde{\varphi}^{,\alpha}{}_{\alpha} = 2(\varphi^{,\alpha}{}_{\alpha} + \underbrace{k^\alpha{}_{,\alpha}}_{=0,\ k^\alpha = \text{const.}}) = 2\Box\varphi = 0$$

So, field eq. does not change

Assume the symmetry transformations form a <u>continuous group</u> – so any finite transformation can be achieved by <u>composing infinitesimal (coordinate) transformations</u>:

$$x'^{\alpha} = x^{\alpha} - \varepsilon\xi^{\alpha} = x^{\alpha} + \overset{*}{\delta} x^{\alpha} \qquad \overset{*}{\delta} x^{\alpha} \equiv -\varepsilon\xi^{\alpha}$$

point has x^{α}
after transf. the same point has x'^{α}

introduce $\overset{*}{\delta} y_A$ and $\bar{\delta} y_A$:

$$y_A'(x') = y_A(x) + \overset{*}{\delta} y_A$$

$$\bar{\delta} y_A = y_A'(x) - y_A(x)$$ the change of the form of y_A

Expressing relations:

$$\bar{\delta} y_A = y_A'(x) - y_A(x) = y_A'(x') - y_A(x) + y_A'(x) - y_A'(\underbrace{x'}_{x+\overset{*}{\delta}x})$$

$$= \underbrace{\overset{*}{\delta} y_A}_{\text{tensor transf.}} \underbrace{- y'_{A,\alpha}(x)\, \overset{*}{\delta} x^{\alpha}}_{\cong -y_{A,\alpha}(x)\, \overset{*}{\delta} x^{\alpha} = +y_{A,\alpha}\varepsilon\xi^{\alpha}}$$

hence $$\boxed{\bar{\delta} y_A = \overset{*}{\delta} y_A + y_{A,\alpha}\varepsilon\xi^{\alpha} = \underset{\varepsilon\xi}{\pounds}\, y_A}$$

Lie derivative

<u>Recall:</u> consider infinit. mapping of point P (with x^{μ}) to point P' (with x'^{μ}), $x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}(x)$ "point transf."

Skripta p. 71–78
Eq. (2.120)

$$\underset{\varepsilon\xi}{\pounds}\phi = \lim \frac{1}{\varepsilon}\left(\phi(x') - \phi'(x')\right) \quad \text{... definition}$$

compare with "above":

$$= \lim \frac{1}{\varepsilon}\left(\phi(x) + \phi_{,\alpha}\varepsilon\xi^{\alpha} - \phi(x)\right)$$

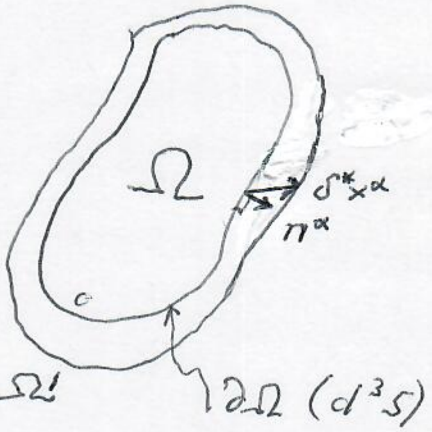
$$\bar{\delta}\phi = \phi'(x) - \phi(x) =$$

$$= \underbrace{\phi'(x') - \phi(x)}_{=0\ \text{(scalar)}} + \underbrace{\phi'(x) - \phi'(x')}_{} \qquad = \phi_{,\alpha}\xi^{\alpha}$$

$= 0$ (scalar) $\phi'(x) - \phi'(x) \dots \phi \dots \xi^{\alpha}$

Now, in order 2 descriptions of a system differ by symmetry transformation, the actions must differ by a divergence (see (*) on p. 3) - we do infinitesimal transf., so actions differ by a "small divergence" - for arbitrary region Ω in V^4

$$(+) \quad \underbrace{\int_{\Omega'} \mathcal{L}(x';y'(x'))\, d^4x'}_{I} = \int_{\Omega} \left[\mathcal{L}(x;y(x)) - \partial_\alpha \bar{\delta} Q^\alpha(x;y(x))\right] d^4x$$



I: $\int_{\Omega'} \mathcal{L}(x';y'(x'))\, d^4x' =$ | x' is just variable back: $x' \to x$

$$= \int_{\Omega} \mathcal{L}(x;y'(x))\, d^4x + \underbrace{\int_{\partial\Omega} \mathcal{L}(\delta^* x^\alpha n_\alpha)\, dS}_{= \int_{\Omega} \partial_\alpha (\mathcal{L}\, \delta^* x^\alpha)\, d^4x}$$

Using this result for integral I and substituting into (+) above we obtain

$$\underbrace{\int_{\Omega} \mathcal{L}(x,y'(x))\, d^4x - \int_{\Omega} \mathcal{L}(x,y)\, d^4x}_{= \bar{\delta}\mathcal{L}} + \int_{\Omega} \partial_\alpha (\mathcal{L}\, \delta^* x^\alpha)\, d^4x + \int_{\Omega} \partial_\alpha \bar{\delta} Q^\alpha\, d^4x = 0$$

$\Rightarrow$ basic equation:

$$(A) \quad \boxed{\int_{\Omega} \left[\bar{\delta}\mathcal{L} + \partial_\alpha \left(\bar{\delta} Q^\alpha + \mathcal{L}\, \delta^* x^\alpha\right)\right] d^4x = 0}$$

If (A) should be true for any Ω - which we require -

$\Rightarrow$

$$\bar{\delta}\mathcal{L} + \partial_\alpha\left(\bar{\delta}Q^\alpha + \mathcal{L}\,\delta^* x^\alpha\right) = 0 \qquad (A^*)$$

After expressing $\bar{\delta}\mathcal{L} = \mathcal{L}(x, y'(x)) - \mathcal{L}(x, y(x))$

this (A^*) becomes the basic identity

since $y'(x) = y(x) + \bar{\delta}y$, we obtain

$$\bar{\delta}\mathcal{L} = \mathcal{L}(x, y + \bar{\delta}y) - \mathcal{L}(x, y) =$$

$$= \mathcal{L}(x, y) + \frac{\partial\mathcal{L}}{\partial y_A}\bar{\delta}y_A + \frac{\partial\mathcal{L}}{\partial y_{A,\alpha}}\bar{\delta}y_{A,\alpha} + \frac{\partial\mathcal{L}}{\partial y_{A,\alpha\beta}}\bar{\delta}y_{A,\alpha\beta} - \mathcal{L}(x,y)$$

$$= L^A\bar{\delta}y_A + \partial_\alpha\left[\left(\frac{\partial\mathcal{L}}{\partial y_{A,\alpha}} - \partial_\beta\frac{\partial\mathcal{L}}{\partial y_{A,\alpha,\beta}}\right)\bar{\delta}y_A + \frac{\partial\mathcal{L}}{\partial y_{,\alpha\beta}}\bar{\delta}y_{A,\beta}\right]$$

EOM, see (1) p. 1

$$\frac{\partial\mathcal{L}}{\partial y_A} - \partial_\beta\frac{\partial\mathcal{L}}{\partial y_{A,\beta}} + \partial_\sigma\partial_\beta\frac{\partial\mathcal{L}}{\partial y_{A,\beta\sigma}}$$

divergence

So basic identity (A^*) becomes

$$\boxed{L^A\bar{\delta}y_A + \partial_\alpha\bar{\delta}t^\alpha = 0,} \qquad (B)$$

where

$$\bar{\delta}t^\alpha = \left[\left(\frac{\partial\mathcal{L}}{\partial y_{A,\alpha}} - \partial_\beta\frac{\partial\mathcal{L}}{\partial y_{A,\alpha\beta}}\right)\bar{\delta}y_A + \frac{\partial\mathcal{L}}{\partial y_{A,\alpha\beta}}\bar{\delta}y_{A,\beta}\right. + \bar{\delta}Q^\alpha + \mathcal{L}\,\delta^* x^\alpha$$

If all y_A are dynamical quantities, $L^A = 0$ satisfied,

(B) implies the conservation law $\boxed{\partial_\alpha\bar{\delta}t^\alpha = 0}$ (C)

I. Noether theorem - Transfs. form Lie group

Emmy Noether * March 23, 1882 in Erlangen, Bavaria
1933 to US, died 1935

Einstein: "the most significant creative mathematical genius thus far produced since the higher education of women began."

Assume symmetry transformation form r-dimensional Lie group with generators ξ^α_i, $i = 1, 2, \dots r$

e.g. Lorentz transform - characterized by finite numbers of independent parameters ε^i, $i = 1, \dots r$

In LT $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ $\quad a^\mu \dots$ 4 indep. parameters

$\Lambda^\mu{}_\nu \dots$ 6 indep. parameters since constrained by

$\eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$ (10 eqs.)

in general $\delta^* x^\alpha = -\varepsilon^i \xi^\alpha_i$ and assume

(~) $\bar{\delta} y_A = \varepsilon^i \eta_{Ai}$, $\bar{\delta} t^\alpha = \varepsilon^i t^\alpha_i$ $\quad$ this defines η_{Ai}, t^α_i

For example: take $y_A \equiv \phi$ $\quad \bar{\delta}\phi = \phi'(x) - \phi(x) =$

$= \phi'(x) - \phi'(x') = \phi'(x) - \phi'(x + \delta^* x) = -\phi_{,\rho}\, \delta^* x^\rho$

$= +\phi_{,\rho} \xi^\rho_i \varepsilon^i$ comparing with (~): $\eta_{Ai} = \phi_{,\rho} \xi^\rho_i$

Therefore - provided the EOM are satisfied, $L^A = 0$ we obtain r conservation laws (see (C), p. 7)

$$\boxed{\partial_\alpha t^\alpha_i = 0, \; i = 1, 2, \dots r}$$ for each independent symmetry

WEAK CONSERV. LAWS ... TRUE IF $L^A = 0$

Example

Scalar field in special relativity

$$\mathcal{L}_{KG} = -\frac{1}{2}\sqrt{-g}\left(g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + m^2\phi^2\right)$$

take $g_{\alpha\beta} = \eta_{\alpha\beta}$ (Minkowski)

we already found $\bar{\delta} y_A = \bar{\delta}\phi = \phi_{,\rho}\, \varepsilon^i \xi^\rho_i$

From p. 7, below (B):

$$\bar{\delta} t^\alpha = \frac{\partial \mathcal{L}}{\partial y_{A,\alpha}} \bar{\delta} y_A - \theta + \theta + \underbrace{\bar{\delta} Q^\alpha}_{\text{can take } = 0}$$

$$+ \mathcal{L}\underbrace{\left(-\varepsilon^i \xi^\alpha_i\right)}_{= \delta^* x^\alpha}$$

$$= -\frac{1}{2}\eta^{\alpha\beta}\phi_{,\beta}\, 2\cdot\left(\phi_{,\rho}\xi^\rho_i\right)\varepsilon^i - \varepsilon^i \xi^\alpha_i \underbrace{\left(-\frac{1}{2}\eta^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} - \frac{1}{2}m^2\phi^2\right)}_{\mathcal{L}_{KG}}$$

Consider now generators along 4 translations (t, x, y, z),

so take $\xi^\rho_i = \delta^\rho_i$ (Killing vectors in t, x, y, z directions)

then

$$\bar{\delta} t^\alpha = -\varepsilon^i\left[\eta^{\alpha\beta}\phi_{,\beta}\phi_{,\rho}\delta^\rho_i - \delta^\alpha_i \frac{1}{2}\left(\phi_{,\rho}\phi^{,\rho} + m^2\phi^2\right)\right]$$

ε^i are 4 indep. parameters can choose $-\varepsilon^i > 0$

change $i \to \varkappa$ and get

$$\underset{KG}{t}{}^\alpha{}_\varkappa = \phi^{,\alpha}\phi_{,\varkappa} - \frac{1}{2}\delta^\alpha_\varkappa\left(\phi_{,\rho}\phi^{,\rho} + m^2\phi^2\right)$$

energy-momentum tensor of scalar ϕ

$\Rightarrow$ conserv. laws of [illegible], under

II Symmetry transformation form continuous ∞ - dimensional (pseudo)group

examples – coordinate transformations in GR $x'^{\alpha} = x'^{\alpha}(x^{\beta})$

gauge transformations in electromagn. $A_\mu \to A'_\mu(x) = A_\mu + \partial_\mu \Lambda(x)$

Assume that they are given by s functions of x^α $(s = 4 \text{ in GR})$

Assume that infinitesimal transformations of y_A have the form

$$(*) \qquad \bar{\delta} y_A = \varepsilon^j(x)\, \underbrace{\gamma_{Aj} - \varepsilon^j{}_{,\alpha} \gamma^{\alpha}_{Aj}}_{\text{constant parameters}}$$

$\varepsilon^j(x)$ general functs of x

$j = 1, 2, \dots s$

similarly for $\bar{\delta} t^\alpha \dots$

Example – gauge transf. in electromagnetism

$$\bar{\delta} A_\alpha = -\partial_\alpha \underbrace{\varepsilon(x)}_{\Lambda(x)} \qquad \Rightarrow j = 1, \ \gamma_{Aj} = 0, \ \gamma^{\alpha}_{Aj} = \delta^{\alpha}_{A}$$

Then from (A), p. 6, i.e. $\int_\Omega [\bar{\delta}\mathcal{L} + \partial_\alpha(\bar{\delta} Q^\alpha + \mathcal{L}\, \delta^* x^\alpha)]\, d^4x = 0$ (•)

divergences

see p. 7, above (B): $= L^A \bar{\delta} y_A + \partial_\alpha(\dots)$

Substitute $\varepsilon^j(x)$, are arbitrary, assume $\varepsilon, \partial\varepsilon, \dots \big|_{\partial\Omega} = 0$

so

$$(\bullet) \Rightarrow 0 = \int_\Omega L^A \bar{\delta} y_A\, d^4x = \int_\Omega L^A (\varepsilon^j \gamma_{Aj} - \varepsilon^j{}_{,\alpha} \gamma^{\alpha}_{Aj})\, d^4x \overset{\text{b.p.}}{=}$$

$$\overset{\text{by parts}}{=} \int_\Omega \varepsilon^j [L^A \gamma_{Aj} + (L^A \gamma^{\alpha}_{Aj})_{,\alpha}]\, d^4x$$

viz

$(L^A \varepsilon^j \gamma^{\alpha}_{Aj})_{,\alpha} =$

$= \dots$

ε^j is arbitrary $\Rightarrow$

$$(B1) \qquad \boxed{L^A \gamma_{Aj} + (L^A \gamma^{\alpha}_{Aj})_{,\alpha} = 0}$$

Generalized Bianchi identities

Clearly, these identities imply constraints on field equations, i.e. on L^A – L^A are 'bound' by these identities – this makes initial value problem (Cauchy problem) for such systems more complicated

Example: $\mathcal{L} = \mathcal{L}(x, y_A, y_{A,\alpha})$ no $y_{A,\alpha\beta}$

(LI) $$L^A = \frac{\partial \mathcal{L}}{\partial y_A} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial y_{A,\alpha}} \right) = - \frac{\partial^2 \mathcal{L}}{\partial y_{A,\alpha} \partial y_{B,\beta}} \underbrace{y_{B,\alpha\beta}}_{\text{2nd derivatives}} + \text{terms without 2nd deriv. of } y_A$$

We wish to choose initial values for y_A and $y_{A,\alpha}$ on $x^0 = \text{const}$ knowing these we can calculate (on $x^0 = \text{const}$) also $y_{A,\alpha i}$ $(i = 1,2,3)$ but $y_{A,00}$ should be expressed from field eqs. (EOM)

Can write $L^A = 0$ and calculate $y_{A,00}$?

(LII) $$L^A = - \underbrace{\frac{\partial^2 \mathcal{L}}{\partial y_{A,0} \partial y_{B,0}}}_{} y_{B,00} + \underline{\text{terms without } y_{B,00}} \text{ (only } y_{B,0i}, y_{B,\alpha}, \dots)$$

if this matrix is singular, we cannot from $L^A = 0$ determine $y_{B,00}$ in terms of $y_{B,\alpha i}$, $y_{B,\alpha}$, y_B i.e. in terms of „initial values",

$y_B, y_{B,\alpha}$ – also called „Cauchy data"

note „Cauchy in Prague"

Substitute expression for L^A given in (L1) p. 11
into generalized Bianchi identities (BI), bottom p. 10
3rd derivative will appear; change $\alpha \to \rho$ in (BI)
and execute the derivative ∂_ρ. One gets

$$L^A \gamma^{\,}_{Aj} + L^A \gamma^{\,\rho}_{Aj;\rho} + \underbrace{L^A_{\,;\rho}}_{\text{express using (L1), p. 11}} \gamma^{\,\rho}_{Aj} = 0$$

$\Rightarrow$

$$L^A \gamma_{Aj} + L^A \gamma^{\,\rho}_{Aj;\rho} \underbrace{- \frac{\partial^2 \mathcal{L}}{\partial y_{A,\alpha}\, \partial y_{B,\beta}}\, y_{B,\alpha\beta\rho}\, \gamma^{\,\rho}_{Aj}}_{\substack{\text{the only term which contains} \\ \text{the 3rd derivatives of } y_A}} + \frac{\partial^3 \mathcal{L}}{\partial \partial \partial} \cdots = 0$$

The last relation must be valid also when $L^A = 0$,
in addition, identically in y_A so that the coefficient
at the 3rd derivative (which at given point can be chosen
arbitrarily) must vanish:

$$\frac{\partial^2 \mathcal{L}}{\partial y_{A,(\alpha}\, \partial y_{B,\beta}}\, \gamma^{\,\rho)}_{Aj} = 0$$

here symmetrization ( )
since in $\underbrace{y_{B,\alpha\beta\rho}}_{\text{symmetrical}}$

but we use this only for $\alpha = \beta = \rho = 0$

$$\Rightarrow \quad \frac{\partial^2 \mathcal{L}}{\partial y_{A,0}\, \partial y_{B,0}}\, \gamma^{\,0}_{Aj} = 0 \qquad \text{(C1)}$$

Hence, if there exist $\gamma^{\,0}_{Aj} \neq 0$ (which is always generally true)
$\Rightarrow$ $\partial^2 \mathcal{L} / \partial y_{A,0}\, \partial y_{B,0}$ is singular $\Rightarrow$ problem for Cauchy

We cannot express $y_{B,00}$ in terms of $y_{B,\alpha i}$ and $y_{B,\alpha}$

Moreover, multiplying L^A given in (LII), p. 11 by $\gamma^{1\,0}_{Aj}$, we get

$$L^A \gamma^{1\,0}_{Aj} = \underbrace{-\frac{\partial^2 \mathcal{L}}{\partial y_{A,0}\, \partial y_{B,0}} \gamma^{1\,0}_{Aj}}_{=0 \text{ due to (C1), p.12}} y_{B,00} + \text{terms} \sim y_A, y_{A,0} \text{ but not } y_{B,00}$$

$\Rightarrow$ $\boxed{L^A \gamma^{1\,0}_{Aj} = 0}$ represents the constraint on y_A, $y_{A,0}$, i.e. on the Cauchy data

since $L^A \gamma^{1\,0}_{Aj}$ contains only y_A, $y_{A,0}$ but <u>not</u> $y_{B,00}$

Example: Elmag. field in terms of $A^\mu(x)$ – in M_4

$$\bar{\delta} A_\alpha = - \partial_\alpha \Lambda(x^\mu) \quad , \quad \gamma^{\ \alpha}_{Aj} = \delta^{\ \alpha}_{A}$$

see p. 10

$j = 1$

$$\mathcal{L} = \underset{\text{put}}{\underset{c=1}{\text{const}}} \; F_{\mu\nu} F^{\mu\nu} = \left(A_{\nu,\mu} - A_{\mu,\nu}\right)\left(A^{\nu,\mu} - A^{\mu,\nu}\right)$$

$$= A_{\nu,\mu} A^{\nu,\mu} - A_{\nu,\mu} A^{\mu,\nu} - A_{\mu,\nu} A^{\nu,\mu} + A_{\mu,\nu} A^{\mu,\nu}$$

$$= 2\left(A_{\mu,\nu} A^{\mu,\nu} - A_{\mu,\nu} A^{\nu,\mu}\right) =$$

$$= 2 A_{\mu,\nu} \left(A_{\rho,\sigma} - A_{\sigma,\rho}\right) \eta^{\rho\mu} \eta^{\sigma\nu}$$

$$\frac{\partial \mathcal{L}}{\partial y_A} = 0 \; , \quad \frac{\partial \mathcal{L}}{\partial y_{A,\alpha}} \widetilde{=} \frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = 2\left(A_{\rho,\sigma} - A_{\sigma,\rho}\right) \eta^{\rho\mu} \eta^{\sigma\nu}$$

$$+ \underbrace{2 A_{\mu,\nu} \eta^{\rho\mu} \eta^{\sigma\nu}}_{A^{\rho,\sigma}} \left(\delta^{\ \mu}_{\rho} \delta^{\ \nu}_{\sigma} - \delta^{\ \mu}_{\sigma} \delta^{\ \nu}_{\rho}\right)$$

$$= 2 F^{\nu\mu} + 2\left(A^{\mu,\nu} - A^{\nu,\mu}\right) = \boxed{4 F^{\nu\mu}}$$

$$\frac{\partial \mathcal{L}}{\partial y_A} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial y_{A,\alpha}}\right) = 0 - \frac{\partial}{\partial x^\nu} F^{\nu\mu} = 0 \qquad A \equiv \mu$$

$\alpha \to \nu$

$$\Rightarrow L^{(A)} \gamma^{\ 0}_{Aj} = 0 \xRightarrow{A \equiv \mu} L^{\mu} \gamma^{\ 0}_{\mu,1} = 0$$

$$\Rightarrow \frac{\partial}{\partial x^\nu} F^{\nu\mu} \cdot \delta^{\ 0}_{\mu} = 0 \quad \underline{\text{are constraints!}}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad \text{no } \frac{\partial E}{\partial t}$$

$$F^{0\nu}_{\ \ ,\nu} = 0 \Rightarrow \operatorname{div} \vec{E} = 0$$

$F_{0\nu} = A_{\nu,0} - A_{0,\nu}$

$i: \; \Rightarrow$ constraints $A_{i,0}$, $A_{0,i}$

III More generally, assume gauge transformations forming ∞-dim group G_∞ which cotains r-dim Lie group G_r

consider, e.g. GR stationary spacetime (1 Killing vect)

then
$$\varepsilon^j(x) = \varepsilon^i \xi^j_i(x) \qquad i = 1, \dots r \quad j = 1, \dots s$$

ε^i are independent parameters (can take $\varepsilon^1 = (1, 0, 0 \dots)$, $\varepsilon^2 = (0, 1 \dots)$ etc)

Then Eq. (*) p. 10 reads
$$\bar\delta y_A = \varepsilon^i \left(\xi^j_i \gamma_{Aj} - \partial_\alpha \xi^j_i \gamma^{\alpha}_{Aj} \right)$$

but we also have (compare (~) on p. 8)
$$\bar\delta t^\alpha = \varepsilon^i t^\alpha_i \qquad \text{(for transforms forming Lie g.)}$$

After substituting into basic identity (B), p. 7:

(remind $L^A \bar\delta y_A + \partial_\alpha \bar\delta t^\alpha = 0$)
$$\varepsilon^i L^A \left(\xi^j_i \gamma_{Aj} - \partial_\alpha \xi^j_i \gamma^{\alpha}_{Aj} \right) + \varepsilon^i \partial_\alpha t^\alpha_i = 0$$

since ε^i are now independent parameters $\Rightarrow$
$$(\circ) \qquad \partial_\alpha t^\alpha_i + L^A \left(\xi^j_i \gamma_{Aj} - \partial_\alpha \xi^j_i \gamma^{\alpha}_{Aj} \right) \equiv 0$$

But $L^A \gamma_{Aj}$ enters also Bianchi identities (BI) p. 10; multiplying (BI) by ξ^j_i, we have
$$(\circ\circ) \qquad L^A \xi^j_i \gamma_{Aj} + \xi^j_i \left(L^A \gamma^{\alpha}_{Aj} \right)_{,\alpha} = 0$$

Subtracting (oo) from (o):

$$\Rightarrow \quad \boxed{\partial_\alpha \left(t_i^{\alpha} - L^A \gamma^{\alpha}_{Aj} \xi^j_i \right) = 0} \qquad (S)$$

STRONG CONSERVATION LAW(S) $i = 1, \dots r$

is true regardless of validity of EOM - for arbitr. L^A

Rewrite (S) into the form

$$(S') \quad \boxed{\partial_\alpha \, \Theta_i^{\alpha} = 0}, \quad i = 1, \dots r \qquad \Theta_i^{\alpha} = t_i^{\alpha} - L^A \gamma^{\alpha}_{Aj} \xi^j_i$$

since (S') is identity there must exist "superpotential" such that $\Theta_i^{\alpha} = U_{i\,,\beta}^{\alpha\beta}$, $U_i^{\alpha\beta} = -U_i^{\beta\alpha}$ ... then automatically

$$\Theta^{\alpha}_{i,\alpha} = U^{\alpha\beta}_{i\,,\underline{\beta\alpha}} = 0$$

So energy-momentum of a physical theory which admits G_∞ containing Lie group G_r can be written in the form

$$\underline{t_i^{\alpha} = U^{\alpha\beta}_{i\,,\beta} + L^A \gamma^{\alpha}_{Aj} \xi^j_i}$$

If EOM satisfied, $L^A = 0$, (S) becomes a weak conservation law

$$t^{\alpha}_{i,\alpha} = 0$$

"improper" since t_i^{α} is divergence

Summary

- The invariance of a theory with respect to an infinitely dimensional (pseudogroup) $G_{\infty s}$ (s general functions, e.g., in GR $s=4$) leads to s differential identities, generalized Bianchi identities, which are linear in L^A.
- Finite dimensional Lie group G_r of r dimensions implies r weak conservation laws (i.e. valid only if $L^A=0$, eqs. of motion satisfied).
- These weak cons. laws can be extended to the strong cons. laws if the theory is also invariant under $G_{\infty s}$ which contains G_r.
- Proper weak conservation laws are valid in the theories for which the finite-dimensional G_r cannot be extended to general $G_{\infty s}$ without introduction of auxiliary <u>non</u>-dynamical fields (e.g. STR with $g_{\alpha\beta}$ which is <u>not</u> dynamical - given backgrounds).