Conservation laws in given spacetimes, i.e. on the „backgrounds" (e.g. fields in Kerr spacetime)

Reminders:

- For a Lie derivative of any quantity, it is true that all partial derivatives can be replaced by covariant ones (scriptum 2.129, 2.132, etc):

For example,

$$\underset{\xi}{\pounds} T_{\mu\nu} = T_{\mu\nu,\rho}\,\xi^\rho + T_{\rho\nu}\,\xi^\rho{}_{,\mu} + T_{\mu\rho}\,\xi^\rho{}_{,\nu}$$

$$= T_{\mu\nu;\rho}\,\xi^\rho + T_{\rho\nu}\,\xi^\rho{}_{;\mu} + T_{\mu\rho}\,\xi^\rho{}_{;\nu}$$

we'll often write $T_{\mu\nu;\rho} \equiv \nabla_\rho T_{\mu\nu}$

For the metric:

$$\underset{\xi}{\pounds} g_{\mu\nu} = \underbrace{g_{\mu\nu;\rho}}_{=0!}\,\xi^\rho + g_{\rho\nu}\,\xi^\rho{}_{;\mu} + g_{\mu\rho}\,\xi^\rho{}_{;\nu}$$

$$= \xi_{\nu;\mu} + \xi_{\mu;\nu} \qquad (= 0 \text{ for a Killing vector})$$

For an arbitrary physical field ψ_K, we write in general

$$\boxed{\underset{\xi}{\pounds} \psi_K = \psi_{K,\alpha}\,\xi^\alpha - \underbrace{F_{K\alpha}{}^{\ell\beta}}_{\text{coefficients depending on properties of } \psi}\,\psi_\ell\,\xi^\alpha{}_{,\beta}}$$

Note: in coordinates in which $\xi^\alpha = (1, 0, 0, 0)$

$\pounds \psi_K = \psi_{K,0}$ just partial derivative

But also can write „covariantly"

(1) $\underset{\xi}{\pounds} \psi_K = \nabla_\alpha \psi_K \xi^\alpha - F_{K\alpha}{}^{\ell\beta} \nabla_\beta \xi^\alpha$

For a scalar density (of weight 1), $\mathcal{L} = \sqrt{-g}\, L$,

sc. density of weight 1 — scalar

(2) $\boxed{\underset{\xi}{\pounds} \mathcal{L} = \partial_\alpha (\mathcal{L} \xi^\alpha)}$ see (2.130) for $\pounds$ of density of weight w

but $\mathcal{L}\xi^\alpha$ is a vector density, so $\partial_\alpha \mathfrak{A}^\alpha = \nabla_\alpha \mathfrak{A}^\alpha$, hence

$$\underset{\xi}{\pounds} \mathcal{L} = \partial_\alpha (\mathcal{L}\xi^\alpha) = \underline{\nabla_\alpha (\mathcal{L}\xi^\alpha)}$$

<u>Conservation laws for a field theory described</u>

by $\mathcal{L} = \mathcal{L}(g_{\alpha\beta};\, \psi_K,\, \partial_\alpha \psi_K)$ (action $S = \int_\Omega \mathcal{L}\, d^4x$)

Starting from (2) and writing out $\underset{\xi}{\pounds}\mathcal{L}$, we get

$$0 = \underbrace{\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}}}_{\frac{1}{2}\mathfrak{T}^{\alpha\beta}} \underset{\xi}{\pounds} g_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial \psi_K} \underset{\xi}{\pounds} \psi_K + \frac{\partial \mathcal{L}}{\partial \psi_{K,\alpha}} \pounds \psi_{K,\alpha} - \partial_\alpha (\mathcal{L}\xi^\alpha)$$

$\frac{1}{2}\mathfrak{T}^{\alpha\beta}$ tensor density (sym.)

Denote $\mathcal{L}^K = \dfrac{\partial \mathcal{L}}{\partial \psi_K} - \dfrac{\partial}{\partial x^\alpha}\left(\dfrac{\partial \mathcal{L}}{\partial \psi_{K,\alpha}}\right)$ eqs. of motion for field ψ_K are $\mathcal{L}^K = 0$

$\Longrightarrow$

(3) $$0 = \frac{1}{2}\mathfrak{T}^{\alpha\beta} \underset{\xi}{\pounds} g_{\alpha\beta} + \mathcal{L}^K \pounds \psi_K + \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{K,\alpha}} \pounds \psi_K - \mathcal{L}\xi^\alpha \right)$$

In the last term $\partial_\alpha \to \nabla_\alpha$ since derivatives applied to vector density; similarly use ∇_α in $\pounds \psi_k$ (see (1) on CL2)
The last term (the divergence) in (3) on previous page can thus be written as follows:

$$(4)\quad \nabla_\alpha\Big[\underbrace{\Big(\frac{\partial\mathcal{L}}{\partial\psi_{k,\alpha}}\nabla_\rho\psi_k - \mathcal{L}\,\delta_\rho^\alpha\Big)}_{\equiv -t_\rho^{\alpha}}\xi^\rho \underbrace{- \frac{\partial\mathcal{L}}{\partial\psi_{k,\alpha}} F_{k\rho}{}^{l\sigma}\psi_l}_{\equiv +S_\rho^{\sigma\alpha}}\nabla_\sigma\xi^\rho\Big]$$

The whole equation (3) at the bottom of p. CL2 becomes

$$(5)\quad \boxed{\begin{aligned} 0 = {}& \tfrac{1}{2}\mathfrak{T}^{\alpha\beta}(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha) + \mathcal{L}^k\big(\nabla_\alpha\psi_k\,\xi^\alpha - F_{k\alpha}{}^{l\beta}\psi_l\nabla_\beta\xi^\alpha\big) \\ & - \nabla_\alpha t_\rho{}^\alpha\,\xi^\rho - t_\rho{}^\alpha\nabla_\alpha\xi^\rho + \nabla_\alpha S_\rho{}^{\sigma\alpha}\nabla_\sigma\xi^\rho \\ & + S_\rho{}^{\sigma\alpha}\nabla_\alpha\nabla_\sigma\xi^\rho \end{aligned}}$$

This condition must be satisfied for arbitrary functions ξ^ρ, arbitrary $\xi^\rho{}_{,\alpha}$ and $\xi^\rho{}_{,\alpha\beta}$; in all these quantities, thus, it must be satisfied identically.
The 2nd derivatives only in the last term – after writing explicitly $\nabla\nabla$ in terms of $\partial\partial$ we just find that

$$S_\rho{}^{\sigma\alpha}\underbrace{\partial_\alpha\partial_\sigma}_{\text{is symmetric, hence}}\xi^\rho = 0 \quad\Rightarrow\quad \boxed{S_\rho{}^{\sigma\alpha} + S_\rho{}^{\alpha\sigma} = 0}\quad(6)$$

or $S_\rho{}^{(\alpha\sigma)},\ S_\alpha{}^{(\beta\gamma)} = 0$

Now we have to satisfy (4) in the 1st deriv. $\xi^\rho{}_{,\alpha}$

we know that $\pounds_\xi \, g_{\mu\nu} = \xi_{\nu;\mu} + \xi_{\mu;\nu}$ but also

$$= g_{\mu\nu,\rho}\,\xi^\rho + g_{\rho\nu}\,\xi^\rho{}_{,\mu} + g_{\mu\rho}\,\xi^\rho{}_{,\nu}$$

everywhere in (4) we now put $\nabla_\alpha \xi^\rho = \partial_\alpha \xi^\rho \cdots$

notice that the 1st derivative will not be in the last term

because $S_\rho{}^{\sigma\alpha}\,\nabla_\alpha \nabla_\sigma \xi^\rho = \frac{1}{2} S_\rho{}^{\sigma\alpha} \underbrace{(\nabla_\alpha \nabla_\sigma - \nabla_\sigma \nabla_\alpha)\,\xi^\rho}_{\sim R^{\cdot}{}_{\cdots}\,\xi, \ \text{no } \xi^\rho{}_{,\alpha}}$

Then (4) implies (keeping terms $\simeq \xi^\rho{}_{,\alpha}$):

$$0 = \frac{1}{2}\mathcal{T}^{\alpha\beta}\left(g_{\rho\alpha}\,\xi^\rho{}_{,\beta} + g_{\rho\beta}\,\xi^\rho{}_{,\alpha}\right) + \mathcal{L}^\kappa\left(-F_{\kappa\alpha}{}^{l\beta}\,\psi_l\,\xi^\alpha{}_{,\beta}\right)$$

$$- t_\rho{}^\alpha\,\xi^\rho{}_{,\alpha} + \nabla_\alpha S_\rho{}^{\sigma\alpha}\,\xi^\rho{}_{,\sigma}$$

$\Rightarrow$

$$0 = \underbrace{\left[\mathcal{T}_\rho{}^\alpha - t_\rho{}^\alpha + \nabla_\gamma S_\rho{}^{\alpha\gamma} - \mathcal{L}^\kappa F_{\kappa\rho}{}^{l\alpha}\,\psi_l\right]}_{=0}\underbrace{\xi^\rho{}_{,\alpha}}_{\text{arbitrary}}$$

$\Rightarrow$

change $\alpha \to \beta$, $\rho \to \alpha$ and find

$$\boxed{\mathcal{T}_\alpha{}^\beta = t_\alpha{}^\beta - \nabla_\gamma S_\alpha{}^{\beta\gamma} + \mathcal{L}^\kappa F_{\kappa\alpha}{}^{l\beta}\,\psi_l} \qquad (7)$$

canonical energy-momentum tensor

symmetric en-mom tensor

Let us now assume the field eqs. to be satisfied, $\mathcal{L}^K = 0$, so the last term in (7) disappears and, after multiplication by ξ^α, Eq. (7) becomes

$$\underbrace{\mathcal{T}_\alpha{}^\beta \xi^\alpha}_{\stackrel{\text{dfn.}}{=} \mathcal{T}^\beta} = \underbrace{t_\alpha{}^\beta \xi^\alpha}_{\text{dfn.}} - (\nabla_\gamma S_\alpha{}^{\beta\gamma})\xi^\alpha$$

$$= t^\beta - \nabla_\gamma(\xi^\alpha S_\alpha{}^{\beta\gamma}) + (\nabla_\gamma \xi^\alpha) S_\alpha{}^{\beta\gamma}$$

Now let us go back to page (CL3) and substitute for $t_\alpha{}^\beta$ and for $S_\alpha{}^{\beta\gamma}$ (in the last term) according to Eq. (4):

$$\mathcal{T}^\beta = -\frac{\partial\mathcal{L}}{\partial\psi_{K,\beta}}\nabla_\alpha\psi_K\xi^\alpha + \mathcal{L}\xi^\beta - \nabla_\gamma\xi^\alpha\frac{\partial\mathcal{L}}{\partial\psi_{K,\gamma}}F_{K\alpha}{}^{\ell\beta}\psi_\ell$$
$$- \nabla_\gamma(\xi^\alpha S_\alpha{}^{\beta\gamma})$$

$$\Rightarrow \mathcal{T}^\beta = -\frac{\partial\mathcal{L}}{\partial\psi_{K,\beta}}\nabla_\alpha\psi_K\xi^\alpha - \underbrace{\frac{\partial\mathcal{L}}{\partial\psi_{K,\gamma}}F_{K\alpha}{}^{\ell\beta}\psi_\ell}_{= -\frac{\partial\mathcal{L}}{\partial\psi_{K,\beta}}F_{K\alpha}{}^{\ell\gamma}\psi_\ell}\nabla_\gamma\xi^\alpha$$
$$+ \mathcal{L}\xi^\beta - \nabla_\gamma(\xi^\alpha S_\alpha{}^{\beta\gamma})$$

because of (6), antisym of $S^{\cdots}$

$$\Rightarrow \mathcal{T}^\beta = -\frac{\partial\mathcal{L}}{\partial\psi_{K,\beta}}\underbrace{\left(\nabla_\alpha\psi_K\xi^\alpha - F_{K\alpha}{}^{\ell\gamma}\psi_\ell\nabla_\gamma\xi^\alpha\right)}_{= \pounds_\xi\psi_K} + \mathcal{L}\xi^\beta -$$
$$- \nabla_\gamma(\xi^\alpha S_\alpha{}^{\beta\gamma})$$

$$\Rightarrow \mathcal{T}^\alpha = t^\alpha - \nabla_\gamma(\xi^\beta S_\beta{}^{\hat{\alpha}\gamma}) = t^\alpha + \nabla_\gamma(\xi^\beta S_\beta{}^{\gamma\alpha}) \quad (7)$$

Assume now the field eqs. $\mathcal{L}^K = 0$ are valid and also covariant conserv. law for symmetric $\mathcal{T}_\alpha{}^\beta$, i.e $\nabla_\beta \mathcal{T}_\alpha{}^\beta = 0$ ("earlier" this directly from the variational principle)

Now for the 'canonical' tensor:

$$\nabla_\beta t_\alpha{}^\beta = \nabla_\beta \nabla_\gamma S_\alpha{}^{\beta\gamma} = \frac{1}{2}\left(\nabla_\beta \nabla_\gamma - \nabla_\gamma \nabla_\beta\right) S_\alpha{}^{\beta\gamma}$$

in general, for commutation of covariant derivatives of $S_\cdot{}^{\cdot\cdot}$ we get 3 terms:

$$S_\alpha{}^{\beta\gamma}{}_{;\mu\nu} - S_\alpha{}^{\beta\gamma}{}_{;\nu\mu} = \dots =$$

$$= R^\beta{}_{\sigma\nu\mu} S_\alpha{}^{\sigma\gamma} + R^\gamma{}_{\sigma\nu\mu} S_\alpha{}^{\beta\sigma} - R^\sigma{}_{\alpha\nu\mu} S_\sigma{}^{\beta\gamma}$$

put $\mu = \gamma$, $\nu = \beta$

$$\Rightarrow S_\alpha{}^{\beta\gamma}{}_{;\gamma;\beta} - S_\alpha{}^{\beta\gamma}{}_{;\beta;\gamma} = \overbrace{R^\beta{}_{\sigma\beta\gamma}}^{-R^\beta{}_{\sigma\gamma\beta}} S_\alpha{}^{\sigma\gamma} + R^\sigma{}_{\sigma\beta\gamma} \underbrace{S_\alpha{}^{\beta\sigma}}_{=-S_\alpha{}^{\sigma\gamma}}$$

$$- R^\sigma{}_{\alpha\beta\gamma} S_\sigma{}^{\beta\gamma}$$

$$= -R^\sigma{}_{\alpha\beta\gamma} S_\sigma{}^{\beta\gamma}$$

$$\Rightarrow \boxed{\nabla_\beta t_\alpha{}^\beta = -R^\sigma{}_{\alpha\beta\gamma} S_\sigma{}^{\beta\gamma}} \qquad (8)$$

so, the canonical en.-mom. tensor is not covariantly conserved in a generally curved spacetime ($R^\sigma{}_{\alpha\beta\gamma} \neq 0$)

Until now we consindered a physical (non-gravitational) field characterized by variables ψ_K, $k=1,\dots N$, on a curved background spacetime described by metric tensor $g_{\alpha\beta}$. Since gravity is not dynamical ("background") the action (see p. CL2) is given by

$$\underset{\text{"field"}}{S_f} = \int_\Omega \mathcal{L}\left(g_{\alpha\beta};\ \psi_K, \partial_\alpha \psi_K\right) d^4x \qquad \mathcal{L} = \sqrt{-g}\,L \qquad (9)$$

Now imagine <u>a (point) particle moving along the worldline $z^\alpha(\tau)$</u>, τ its proper time which moves in $g_{\alpha\beta}$ (not in general along geodesics), and interacts in general with field. The action for the particle, S_p, has the form

$$S_p = \int_\Omega \left[\int_{-\infty}^{\infty} \Lambda\left(\psi_K, \dot{z}^\alpha\right) \underbrace{\delta(x - z(\tau))}_{\text{4 dim } \delta\text{-function}} d\tau \right] d^4x \qquad (10)$$

(example: el. current interacting with elmag potential: $\underset{\psi_K}{A^\mu} \underset{e\dot{z}_\mu}{j_\mu}$)

Then, varying the worldline z^α of the particle (here not ψ_K) we get

$$(10)\quad \delta S_p = \int_\Omega d^4x \left\{ \int_{-\infty}^{\infty} \left[\frac{\partial \Lambda}{\partial z^\alpha}\,\delta z^\alpha + \frac{\partial \Lambda}{\partial \dot{z}^\alpha}\,\delta \dot{z}^\alpha\right] d\tau + \Lambda\,\delta\, d\tau \right\} \delta(x - z)$$

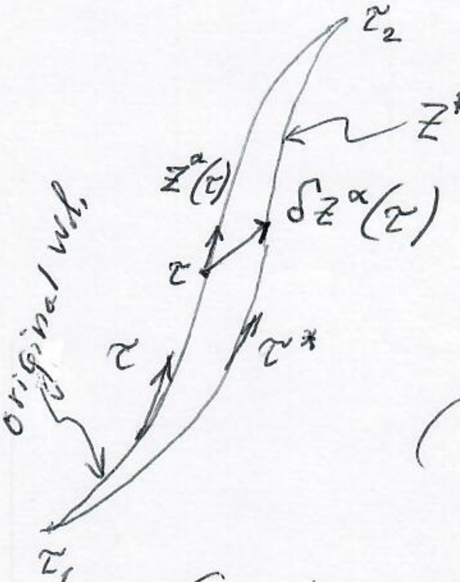
We now calculate "separately" $\delta \dot{z}^\alpha$ and $\delta d\tau$ and substitute the results into (10)

a little "t.b." but let's do it!

(reminder: "t.b." = "tedious and boring")

Within Special Relativity, see V. Votruba, p. 287–289; see also O. Semerák, Ch. 7, p. 77 –

Variation of a world line:



$$z^{*\alpha}(\tau) = z^\alpha(\tau) + \delta z^\alpha(\tau)$$

$$\delta z^\alpha(\tau_1) = \delta z^\alpha(\tau_2) = 0$$

(but with general $x^\alpha \to x'^\alpha = x^\alpha - \xi^\alpha$ we can consider $\tau_1 \to -\infty$, $\tau_2 \to +\infty$)

On the <u>varied</u> worldline the <u>real</u> shift corresponds to

$$dz^{*\alpha}(\tau) = z^{*\alpha}(\tau + d\tau) - z^{*\alpha}(\tau) \stackrel{\text{following (definition of } z^*}{=} =$$

not τ^*

$$= z^\alpha(\tau + d\tau) + \delta z^\alpha(\tau + d\tau) - (z^\alpha(\tau) + \delta z^\alpha(\tau))$$

$$= \underbrace{z^\alpha(\tau + d\tau) - z^\alpha(\tau)} + \underbrace{\delta[z^\alpha(\tau + d\tau) - z^\alpha(\tau)]} \Rightarrow$$

$$dz^{*\alpha}(\tau) = dz^\alpha(\tau) \quad + \quad \delta dz^\alpha(\tau) \qquad (\sim)$$

on the other hand $\underbrace{\delta z^\alpha(\tau + d\tau) - \delta z^\alpha(\tau)}_{= \delta dz^\alpha(\tau)} = d\delta z^\alpha(\tau)$

but from $\rightsquigarrow$ in $(\sim)$

$\Rightarrow$ „expected" relation: $\boxed{\delta dz^\alpha(\tau) = d\,\delta z^\alpha(\tau)}$

What does correspond, on the varied worldline, to the increase of proper time $d\tau = \sqrt{-g_{\alpha\beta}\, dz^\alpha dz^\beta}$ on the „original" worldline?

it is

$$d\tau^* = \sqrt{-g_{\alpha\beta}\, dz^{*\alpha} dz^{*\beta}}\,,$$

but $dz^{*\alpha} = dz^\alpha + \delta dz^\alpha = dz^\alpha + d\,\delta z^\alpha$

Hence, up to the first order we get

$$d\tau^* = \sqrt{-g_{\alpha\beta}\left(dz^\alpha + d\,\delta z^\alpha\right)\left(dz^\beta + d\,\delta z^\beta\right)} =$$

$$\cong \sqrt{-g_{\alpha\beta}\, dz^\alpha dz^\beta - 2 g_{\alpha\beta}\, dz^\alpha d\,\delta z^\beta} = \sqrt{d\tau^2\left(1 - 2 g_{\alpha\beta}\, \dot{z}^\alpha \frac{d\,\delta z^\beta}{d\tau}\right)}$$

$$\cong d\tau\left(1 - g_{\alpha\beta}\, \dot{z}^\alpha \frac{d\,\delta z^\beta}{d\tau}\right)$$

$$\Rightarrow \delta d\tau \overset{\text{def}}{=} d\tau - d\tau^* = -g_{\alpha\beta}\, \dot{z}^\alpha\, d\,\delta z^\beta$$

Variation of $\dot{z}^\alpha$:

$$\delta \dot{z}^\alpha = \delta\left(\frac{dz^\alpha}{d\tau}\right) \underset{\uparrow}{=} \frac{\delta dz^\alpha}{d\tau} - \frac{1}{d\tau^2}\, dz^\alpha\, \delta d\tau$$

„derivative of a quotient"

$$\Rightarrow \delta \dot{z}^\alpha = \frac{\delta dz^\alpha}{d\tau} + g_{\rho\beta}\, \dot{z}^\rho \frac{d\,\delta z^\beta}{d\tau} \frac{dz^\alpha}{d\tau} =$$

$$= \frac{d\,\delta z^\alpha}{d\tau} + g_{\rho\beta}\, \dot{z}^\rho \frac{d\,\delta z^\beta}{d\tau}\, \dot{z}^\alpha$$

$$= \frac{d\,\delta z^\alpha}{d\tau} + g_{\rho\beta}\, \dot{z}^\rho \dot{z}^\alpha \frac{d\,\delta z^\beta}{d\tau}$$ (cp. Smerall (7.7))

Writing down again δS_p given in (10), p. CL7 and substituting there for $\delta \dot{z}^\alpha$ just derived, we obtain

$$\delta S_p = \int_\Omega d^4x\, \delta(x-z) \left\{ \int_{-\infty}^{\infty} \left[\frac{\partial \Lambda}{\partial z^\alpha} \delta z^\alpha + \frac{\partial \Lambda}{\partial \dot{z}^\alpha} \left(\frac{d\delta z^\alpha}{d\tau} + g_{\rho\beta} \dot{z}^\rho \dot{z}^\alpha \frac{d\delta z^\beta}{d\tau} \right) \right] d\tau \right.$$

$$\left. + \underbrace{\Lambda \left(- g_{\alpha\beta} \dot{z}^\alpha \frac{d\delta z^\beta}{d\tau} \right) d\tau}_{(\delta\, d\tau)/d\tau} \right\} \qquad (\tilde{10})$$

The "— —" term can be rewritten "by parts" as

$$\frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} g_{\rho\beta} \dot{z}^\rho \dot{z}^\alpha \delta z^\beta \right) d\tau - \frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} g_{\rho\beta} \dot{z}^\rho \dot{z}^\alpha \right) \delta z^\beta d\tau$$

$$= \underbrace{\frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} g_{\rho\beta} \dot{z}^\rho \dot{z}^\alpha \right) \delta z^\beta d\tau}_{\sim\sim} + \frac{\partial \Lambda}{\partial \dot{z}^\alpha} g_{\rho\beta} \dot{z}^\rho \dot{z}^\alpha \frac{d\delta z^\beta}{d\tau} d\tau$$

$$\underbrace{- \frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} g_{\rho\beta} \dot{z}^\rho \dot{z}^\alpha \right) \delta z^\beta d\tau}_{\sim\sim}$$

$\sim\sim$ terms drop out even if we generally consider $\frac{d g_{\rho\beta}}{d\tau} \neq 0$

(so not only $\eta_{\rho\beta}$)

Similarly the term "green" above in $(\tilde{10})$ can be rewritten by parts:

$$\int \frac{\partial \Lambda}{\partial \dot{z}^\alpha} \frac{d\delta z^\alpha}{d\tau} d\tau \overset{\text{b.p.}}{=} \int \left[\frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} \delta z^\alpha \right) - \frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} \right) \delta z^\alpha \right] d\tau$$

and

$$\frac{\partial \Lambda}{\partial \dot{z}^\alpha} \frac{d\delta z^\alpha}{d\tau} - \Lambda g_{\alpha\beta} \dot{z}^\alpha \frac{d\delta z^\beta}{d\tau}$$

$$= \left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} - \Lambda g_{\beta\alpha} \dot{z}^\beta \right) \frac{d\delta z^\alpha}{d\tau} = \frac{d}{d\tau}(\cdots) - \delta z^\beta \frac{d}{d\tau}(\cdots)$$

Putting the terms together, few times interchanging $\alpha \leftrightarrow \beta$, we finally arrive at the following result:

$$\delta S_p =$$

$$= \int_\Omega d^4x\, \delta(x-z) \underbrace{\left\{ \int_{-\infty}^{+\infty} \left[\frac{\partial \Lambda}{\partial z^\alpha} - \frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} + \frac{\partial \Lambda}{\partial \dot{z}^\beta} g_{\rho\alpha} \dot{z}^\rho \dot{z}^\beta - \Lambda g_{\alpha\beta} \dot{z}^\beta \right) \right]}_{\stackrel{\text{def.}}{\equiv} \Lambda_\alpha} \underbrace{\delta z^\alpha}_{=-\xi^\alpha} d\tau$$

$= -\xi^\alpha$ infinit. coord. transf. induces variation

$$+ \int_\Omega d^4x \int_{-\infty}^{+\infty} \delta(x-z) \frac{d}{d\tau} \underbrace{\left[\left(\frac{\partial \Lambda}{\partial \dot{z}^\alpha} + \frac{\partial \Lambda}{\partial \dot{z}^\beta} \dot{z}^\beta \dot{z}_\alpha - \Lambda \dot{z}_\alpha \right) \delta z^\alpha \right]}_{\stackrel{\text{def.}}{=} \bar{\delta} P} d\tau \qquad \delta z^\alpha = -\xi^\alpha$$

So $$\bar{\delta} P = \left[\frac{\partial \Lambda}{\partial \dot{z}^\alpha} + \left(\frac{\partial \Lambda}{\partial \dot{z}^\beta} \dot{z}^\beta - \Lambda \right) \dot{z}_\alpha \right] \underbrace{(-\xi^\alpha)}_{=\delta z^\alpha}$$

(See O. Semerák above (7.9) – with $\eta_{\alpha\beta}$)

Equations of motion $\Lambda_\alpha = 0$, ...

The basic identity (B) on p. 7 becomes
$(L^A \bar\delta y_A + \partial_\alpha \bar\delta t^\alpha = 0)$ becomes

$$-\frac{1}{2} T^{\alpha\beta} \bar\delta g_{\alpha\beta} + \mathcal{L}^k \delta\psi_k + \int_{-\infty}^{+\infty} \Lambda_\alpha \delta x^\alpha \delta(x-z)\, d\tau$$

$$+ \bar\delta t^\alpha{}_{,\alpha} + \int_{-\infty}^{\infty} \delta(x-z) \frac{d}{d\tau} \bar\delta P \, d\tau = 0 \qquad (11)$$

recall $-T^{\alpha\beta} = \dfrac{\delta \mathcal{L}^k}{\delta g_{\alpha\beta}}$

Let $\mathcal{L}^k = 0 = \Lambda_\alpha$ and $\bar\delta g_{\alpha\beta} = 0$

$\Rightarrow$ there are Killing vectors satisfying

$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

Let ξ_i^α, $i = 1, \dots p$ are p independent Killing vectors
we may write (cp. p. 8)

$$\xi^\alpha = \varepsilon^i \xi_i^\alpha, \quad \bar\delta t^\alpha = \varepsilon^i t_i^\alpha, \quad \bar\delta P = \varepsilon^i P_i$$

then (11) leads to p conservation laws in the form

$$\boxed{t_{i\,,\alpha}^{\alpha} + \int_{-\infty}^{\infty} \dot P_i \, \delta(x-z)\, d\tau = 0}$$