Problem of energy and energy-momentum complex for gravitational field in General Relativity and generalized theories of gravity

- „perpetual" question - why?
- from variational considerations àla preceding parts
- directly from field equations - Landau-Lifshitz complex → seminar, Monday 15th afternoon
- Komar approach

(i) Why a problem?

Principle of equivalence! conserved energy should represent an integral of motion

In electromagnetism $\varepsilon = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$ is a part of energy-momentum tensor $T^{\alpha\beta}$ which is conserved, $T^{\alpha\beta}{}_{,\beta} = 0$. If it is $\neq 0$ at given spacetime point, it cannot be made 0 by going to some other frame. In addition

$$\int_\Sigma T_{00}\, d^3x = \text{const} \quad \text{is "integral of motion"}$$

because T_{00} involves $\vec{E}$ and $\vec{B}$ but not their derivatives whereas Maxwell's equations do contain $\partial\vec{E}, \partial\vec{B}$

In GR Einstein's eqs. involve $\partial\partial g_{..}$ so energy-momentum should contain only $g_{..}$ and $\partial g_{..}$

But there is no quantity which is a tensor and contains only $g_{..}$ and $\partial g_{..}$, or equivalently, $\Gamma^{\alpha}_{\beta\gamma}$.

As a consequence of the principle of equivalence in a local inertial frame $\Gamma^{\cdot}_{..} = 0$ ("force vanishes") and $g = \eta$ (special relativity).

thanks, or a consequence of, universal character of gravity

So in order to get „integrals of motion"
we must accept also non-tensorial quantities
so-called "complexes".

There are many other approaches – "superenergy Bell-Robinson tensor" from $R_{--} \times R_{---}$ quadratic in Riemann t

– energy with respect to a background $g_{..} = \bar{g} + \delta g$ (may be large)
KBL superpotential

– quasilocal mass-energy of Penrose

– ...

Conservation of energy of sources $T_{\mu\nu}$ for stationary systems

In general from rules of covariant diff.:

$$T_\alpha{}^\beta{}_{;\beta} = T_\alpha{}^\beta{}_{,\beta} + \Gamma^\beta_{\rho\beta} T_\alpha{}^\rho - \Gamma^\rho_{\alpha\beta} T_\rho{}^\beta$$

Note:

Recall that $\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\beta} = \Gamma^\sigma_{\beta\sigma}$ $\quad (*)$

how to remember this?

for $\forall A^\mu$: $A^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,A^\mu\right)_{,\mu}$

$$\Rightarrow \cancel{A^\mu{}_{,\mu}} + \Gamma^\mu_{\mu\nu} A^\nu = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\mu} A^\mu + \cancel{A^\mu{}_{,\mu}}$$

$$\Rightarrow \Gamma^\mu_{\mu\nu} A^\nu = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\mu} A^\mu = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\nu} A^\nu$$

this is true for any A^ν $\Rightarrow \Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\nu}$

which is $(*)$

using $(*)$

$$\text{So, } T_\alpha{}^\beta{}_{;\beta} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T_\alpha{}^\beta\right)_{,\beta} - \cancel{\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\beta} T_\alpha{}^\beta}$$

$$+ \cancel{\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\right)_{,\rho} T_\alpha{}^\rho} - \Gamma^\rho_{\alpha\beta} T_\rho{}^\beta$$

$$\rho \to \beta \qquad \rho \to \beta$$

$$= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T_\alpha{}^\beta\right)_{,\beta} - \frac{1}{2} g^{\rho\sigma}\left(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}\right) T_\rho{}^\beta$$

$$= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T_\alpha{}^\beta\right)_{,\beta} - \frac{1}{2} T^{\sigma\beta}\left(\cancel{g_{\alpha\sigma,\beta}} + g_{\beta\sigma,\alpha} - \cancel{g_{\alpha\beta,\sigma}}\right)$$

$$= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T_\alpha{}^\beta\right)_{,\beta} - \frac{1}{2} g_{\beta\sigma,\alpha} T^{\beta\sigma}$$

take $\alpha = 0$ ("time" comp.):

$$0 = T_0{}^{\beta}{}_{;\beta} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, T_0{}^{\beta}\right)_{,\beta} - \frac{1}{2} g_{\rho\sigma,0}\, T^{\rho\sigma}$$

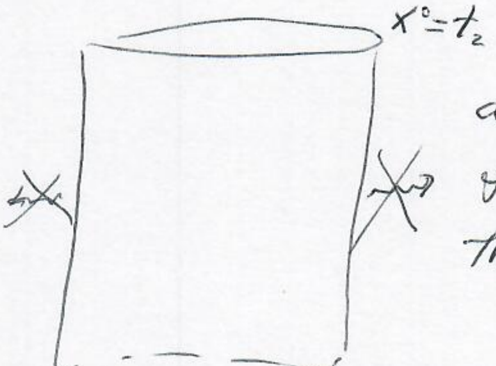
For stationary metrics in adapted coordinates

$$g_{\rho\sigma,0} = 0$$

so $\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, T_0{}^{\beta}\right)_{,\beta} = 0$ implies global conservation law for "sources"

$$\int_{\Omega} \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, T_0{}^{\beta}\right)_{,\beta} \underbrace{\sqrt{-g}\, d^4x}_{\text{invariant 4-volume}} = 0$$

$$\overset{\text{Gauss}}{\Longrightarrow} \int_{\partial\Omega} \left(\sqrt{-g}\, T_0{}^{\beta}\right) \underset{-1,+1}{\varepsilon}\, d\Sigma_{\beta} = 0$$



assume no fluxes through | |

$$\Rightarrow \boxed{\int \sqrt{-g}\, T_0{}^{0}\, d^3x \Big|_{t_1} = \int \sqrt{-g}\, T_0{}^{0}\, d^3x \Big|_{t_2}}$$

Conservation of energy

Conservation laws in GR and generalized theories

(– „variational" approach „like" for field theories previously)

Start out from action

$$S_g = \int_\Omega \hat{G}(g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\delta})\, d^4x$$

„ $\wedge$ " denotes scalar density

in GR" $\hat{G} = \frac{1}{16\pi}\sqrt{-g}\,R$ „ $\frac{1}{16\pi}$ " $= \frac{c^3}{16\pi G}$ but $c = 1 = G$

but we may assume more general $\hat{G}$

Variational derivative of S_g reads

$$\frac{\delta S_g}{\delta g_{\alpha\beta}} = \frac{\partial \hat{G}}{\partial g_{\alpha\beta}} - \partial_\gamma \frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma}} + \partial_\gamma \partial_\delta \frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma\delta}} \qquad (*)$$

Vacuum field equations are

$$\frac{\delta S_g}{\delta g_{\alpha\beta}} \overset{\text{def.}}{\equiv} \frac{1}{16\pi} \hat{G}^{\alpha\beta} = 0$$

in GR $\hat{G}^{\alpha\beta}$ is the density of Einstein's tensor

GE6

Since $\hat{G}$ is a scalar density, its Lie derivative w.r.t. general infinit. vector field ξ satisfies (cp. for $\mathcal{L}$ earlier) $\underset{\xi}{\mathcal{L}} \hat{G} = (\hat{G}\xi^\alpha)_{,\alpha}$, so the following identity is valid:

$$0 = \underset{\xi}{\mathcal{L}} \hat{G} - (\hat{G}\xi^\alpha)_{,\alpha} \qquad (**)$$

but $$\underset{\xi}{\mathcal{L}} \hat{G} = \underbrace{\frac{\partial \hat{G}}{\partial g_{\alpha\beta}} \underset{\xi}{\mathcal{L}} g_{\alpha\beta}}_{1} + \underbrace{\frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma}} \underset{\xi}{\mathcal{L}} g_{\alpha\beta,\gamma}}_{2} + \underbrace{\frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma\delta}} \underset{\xi}{\mathcal{L}} g_{\alpha\beta,\gamma\delta}}_{3}$$

The term 2 is (cp. "before")

$$\frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma}} \underset{\xi}{\mathcal{L}} g_{\alpha\beta,\gamma} = \underbrace{\frac{\partial}{\partial x^\gamma}\left(\frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma}} \underset{\xi}{\mathcal{L}} g_{\alpha\beta}\right)}_{\text{divergence } \frac{\partial}{\partial x^\gamma} \hat{\tau}^\gamma} - \left[\frac{\partial}{\partial x^\gamma}\left(\frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma}}\right)\right] \underset{\xi}{\mathcal{L}} g_{\alpha\beta}$$

Similarly, with term 3 as the divergence of a vector density and the 3rd term in (*), p. GE5, in $\delta S_g / \delta g_{\alpha\beta}$

One can thus demonstrate that the identity (**) above implies the relation

$$\boxed{0 = \frac{1}{16\pi} \hat{G}^{\alpha\beta} \underset{\xi}{\mathcal{L}} g_{\alpha\beta} + \partial_\alpha \hat{\tau}^\alpha} \qquad (\mathrm{I})$$

where $\hat{\tau}^\alpha$ contains terms like $\frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma}} \underset{\xi}{\mathcal{L}} g_{\alpha\beta}$

or $-\partial_\beta \frac{\partial \hat{G}}{\partial g_{\alpha\beta,\gamma\delta}} \underset{\xi}{\mathcal{L}} g_{\alpha\beta}$

Now integrate the identity (I) over 4-region Ω, substitute standard relation $\underset{\xi}{\pounds} g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$, assume $\xi, \partial\xi = 0$ on $\partial\Omega$, use Gauss on $\partial_\alpha \hat{\iota}^\alpha$ so that because of boundary conditions this term after integration does not contribute, so integrated (I) yields

$$0 = \int_\Omega \hat{G}^{\alpha\beta} (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) d^4x = 2 \int_\Omega \hat{G}^{\alpha\beta} \nabla_\alpha \xi_\beta d^4x$$

again, integrating by parts,

$$0 = \int_\Omega \hat{G}^{\alpha\beta} \nabla_\alpha \xi_\beta d^4x = \int_\Omega \underbrace{(\hat{G}^{\alpha\beta} \xi_\beta)_{;\alpha}} d^4x - \int_\Omega \nabla_\alpha \hat{G}^{\alpha\beta} \xi_\beta d^4x$$

covar. div of vector density using $\hat{v}^\alpha{}_{;\alpha} = \hat{v}^\alpha{}_{,\alpha}$ and Gauss — for $\xi \to 0$ on $\partial\Omega$ this gives 0

$$\Rightarrow \quad 0 = \int_\Omega \nabla_\alpha \hat{G}^{\alpha\beta} \xi_\beta d^4x$$

but ξ is arbitrary $\Rightarrow$

$$\boxed{\nabla_\alpha \hat{G}^{\alpha\beta} = 0} \quad (\text{or } \nabla_\beta \hat{G}^{\alpha\beta} = 0)$$

<u>"generalized" contracted Bianchi identities</u>

In GR $\quad \hat{G}^{\alpha\beta} = \sqrt{-g}\,(R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R)$

$$\hat{G}^{\alpha\beta}{}_{;\beta} = 0$$

Now let us go back to identity (I), p. GE6, where substitute $\underset{\xi}{\mathcal{L}} g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$, use the possibility $\partial_\alpha \leftrightarrow \nabla_\alpha$ for a vector density and that $\nabla_\alpha \hat{G}^{\alpha\beta} = 0$, and write (I) in the form

$$\nabla_\alpha \left(\frac{1}{8\pi} \hat{G}^{\alpha\beta} \xi_\beta + \hat{\tau}^\alpha \right) = 0 \qquad (I^*)$$

note here is not 16π because 2 drops out from $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$

Since in (I^*) is vector density, we may also write $\partial_\alpha (\dots) = 0$

This is <u>strong</u> conservation law - valid identically in g and ∂g, $\partial^2 g$

<u>However</u>, when we substitute from field equations, e.g., $\hat{G}^{\alpha\beta} = 8\pi \hat{T}^{\alpha\beta}$ (i.e. Einstein) and $\nabla_\alpha \to \partial_\alpha$ we obtain conservation law

$$\boxed{\partial_\alpha \left(\hat{T}^\alpha + \hat{\tau}^\alpha \right) = 0,} \qquad (II)$$

where $\hat{T}^\alpha \equiv \hat{T}^{\alpha\beta} \xi_\beta$

It is evident from (I*) on previous page that this identity can be in GR easily satisfied if we choose $\hat{\tau}^\alpha$ simply so that it automatically cancels the first term, i.e., we take

$$\hat{\tau}^\alpha = -\frac{1}{8\pi}\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\xi_\beta$$

$$\overset{\text{using field eqs}}{=} -\frac{1}{8\pi}\sqrt{-g}\,8\pi\,T^{\alpha\beta}\xi_\beta = -\hat{T}^{\alpha\beta}\xi_\beta = -\hat{T}^\alpha$$

and (II) on previous page is automatically satisfied since

$$\boxed{\underbrace{\hat{T}^\alpha}_{\text{matter}} + \underbrace{\hat{\tau}^\alpha}_{\text{gravity}} = 0}$$

Possible interpretation:

gravitational energy-momentum annihilates energy-momentum of matter so that <u>total</u> current of energy-momentum $= 0$.

View of H. A. Lorentz, W. Pauli

<u>Note</u>: in fact this just a "reinterpration" of Einstein's field eqs. multiplied by $\sqrt{-g}\,\xi^\alpha$

Present-day viewpoint(s) different!