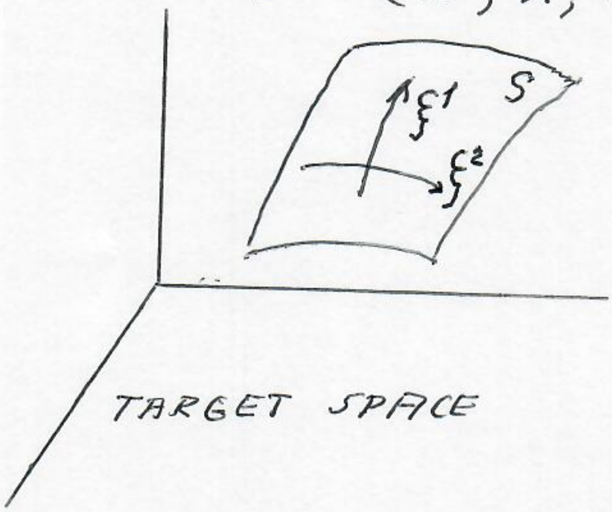
Strings – first „impressions"

Introduction

2-surface in D-dimensional space („target")

$\vec{x} = (x^1, x^2, \dots, x^D)$ $\quad S: \ \vec{x} = \vec{x}(\xi^1, \xi^2)$



$x^\mu, \mu = 1, \dots, D \qquad \xi^i \ \ i = 1, 2$

the metric on S:

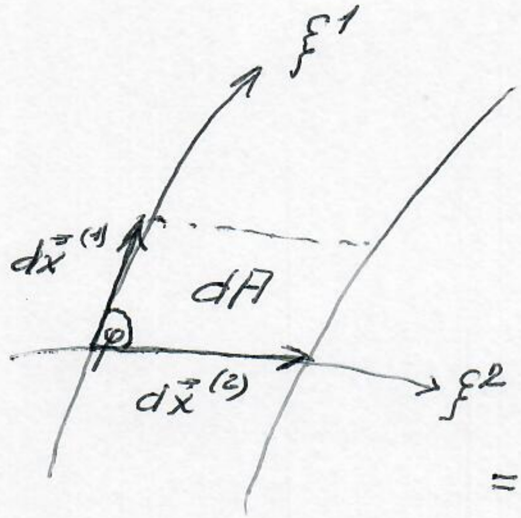
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \Big|_S =$$

$$= \underbrace{g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j}}_{= g_{ij} \ \dots \text{induced metric on } S} d\xi^i d\xi^j =$$

$$= g_{ij} \, d\xi^i d\xi^j$$

In the variational principle for the motion of a (point) particle (the role of) the Lagrangian density is the line element ds along its worldline.

In the variational principle for the motions of a 1-dimensional string along its worldsheet will the Lagrangian d. be $\sim dA$ – element of its worldsheet



surface spanned by vectors $d\vec{x}^{(1)}$ and $d\vec{x}^{(2)}$ is dA, where

$$dA^2 = |d\vec{x}^{(1)}|^2 |d\vec{x}^{(2)}|^2 \sin^2\varphi$$

$$= |d\vec{x}^{(1)}|^2 |d\vec{x}^{(2)}|^2 (1 - \cos^2\varphi) =$$

$$= |d\vec{x}^{(1)}|^2 |d\vec{x}^{(2)}|^2 - \underbrace{(d\vec{x}^{(1)} \cdot d\vec{x}^{(2)})^2}_{\text{scalar product}} =$$

$$= \left[\left(\frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^1} \right) \cdot \left(\frac{\partial X^\nu}{\partial \xi^2} \frac{\partial X_\nu}{\partial \xi^2} \right) - \left(\frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^2} \right)^2 \right] (d\xi^1)^2 (d\xi^2)^2$$

$$\Rightarrow \boxed{A = \int d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^1} \right) \left(\frac{\partial X^\nu}{\partial \xi^2} \frac{\partial X_\nu}{\partial \xi^2} \right) - \left(\frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^2} \right)^2} \quad (*)}$$

Simplifying $(*)$ in terms of the induced metric:

$$g_{ij}(\xi) = \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j}$$

$$g_{ij} = \begin{pmatrix} \frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^1}, & \frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^2} \\ \frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^2}, & \frac{\partial X^\mu}{\partial \xi^2} \frac{\partial X^\mu}{\partial \xi^2} \end{pmatrix}$$

$g_{\mu\nu} = \eta_{\mu\nu} = (-, +, +, + \dots)$ in the simplest case

So $(*)$ can be written in terms of g_{ij} simply as

$$\boxed{A = \int d\xi^1 d\xi^2 \sqrt{\det g_{ij}}}$$

In this form A is clearly invariant under reparametrization $\xi^i \to \tilde{\xi}^i(\xi^j)$

$$\sqrt{g}\, d\xi^1 d\xi^2 = \sqrt{\tilde{g}}\, d\tilde{\xi}^1 d\tilde{\xi}^2$$

where $g \equiv \det g_{ij}$

Details: denoting $M = (M_{ij}) = \frac{\partial \xi^i}{\partial \tilde{\xi}^j}$, $\tilde{M} = (\tilde{M}_{ij}) = \frac{\partial \tilde{\xi}^i}{\partial \xi^j}$,

transf. eq. $g_{ij}(\xi) = \tilde{g}_{pq}(\tilde{\xi}) \frac{\partial \tilde{\xi}^p}{\partial \xi^i} \frac{\partial \tilde{\xi}^q}{\partial \xi^j}$ can be written

as $g_{ij} = \tilde{g}_{pq} \tilde{M}_{pi} \tilde{M}_{qj} = (\tilde{M}^T)_{ip} \tilde{g}_{pq} \tilde{M}_{qj}$

writing in terms of 3 matrices

$$\Rightarrow \quad g = (\det \tilde{M}^T)\, \tilde{g}\, (\det \tilde{M}) = \tilde{g}\,(\det \tilde{M})^2$$

(the transformation of a density

$$\sqrt{g} = \sqrt{\tilde{g}}\, |\det \tilde{M}|$$ transformation of a density

and $\int d\xi^1 d\xi^2 \sqrt{g} = \int d\tilde{\xi}^1 d\tilde{\xi}^2 \underbrace{|\det M|\, \sqrt{\tilde{g}}\, |\det \tilde{M}|}_{=1}$

$$= \int d\tilde{\xi}^1 d\tilde{\xi}^2 \sqrt{\tilde{g}}$$

Towards physics: surfaces in spacetime (area)

Assume S to be a 2-dim surface (worldsheet) in $D = (d+1)$ - spacetime $\quad x^\mu = (x^0, x^1, \ldots, x^d)$

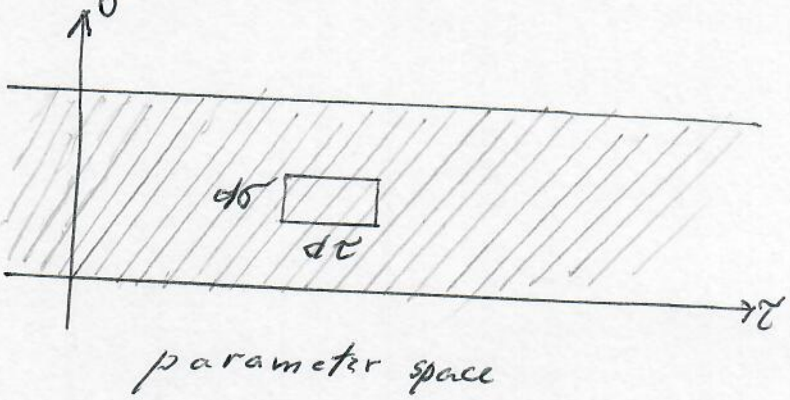
it is parameterized by 2 parameters τ, σ (previous ξ^1, ξ^2),

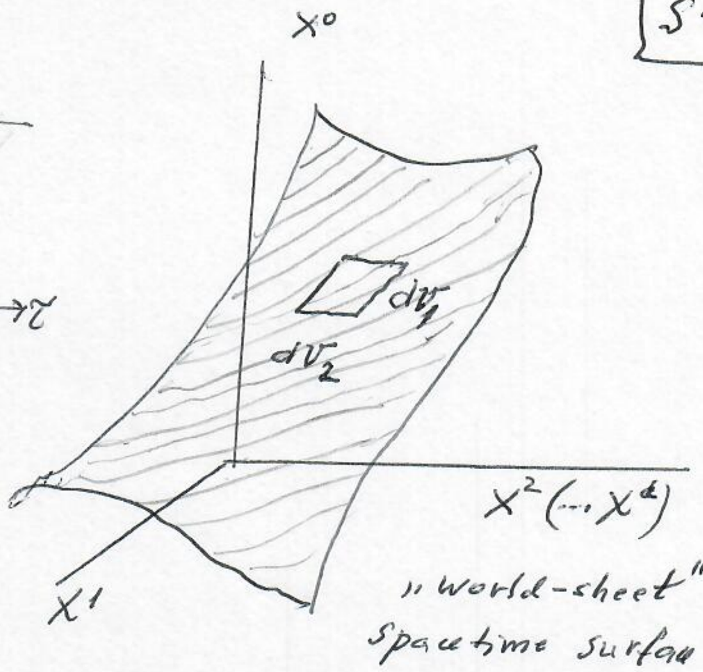
so $\quad x^\mu = X^\mu(\tau, \sigma)$

given a point in the parameter space τ, σ

it is mapped in $(d+1)$ dim. spacetime to a point with coordinates

$$\{ X^0(\tau,\sigma), X^1(\tau,\sigma), \ldots X^d(\tau,\sigma) \}$$ string coordinates





σ ... within a finite interval

τ -- from $-\infty$ to $+\infty$

„time on the string"

(details later)

endpoints of the string have $\sigma = \text{const}$, parameterized by τ.

One assumes: $\left.\dfrac{\partial X^0}{\partial \tau}\right|_{\text{endpoint}} \neq 0$

A rectangle $(d\tau, d\sigma)$ in parameter space becomes (under mapping $X^\mu(\tau,\sigma)$) an element spanned by the vectors dv_1^μ, dv_2^μ, where

$$dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau, \qquad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma$$

In analogy with (*), p S2 for the 2-surface in E^d we may try to define

$$dA = \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2}$$

$$dv_1 \cdot dv_1 = \frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \tau} d\tau^2, \text{ etc}$$

We find that the proper is given in the relativistic case by

$$\boxed{A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau}\right)\left(\frac{\partial X^\nu}{\partial \sigma}\frac{\partial X_\nu}{\partial \sigma}\right)}}$$

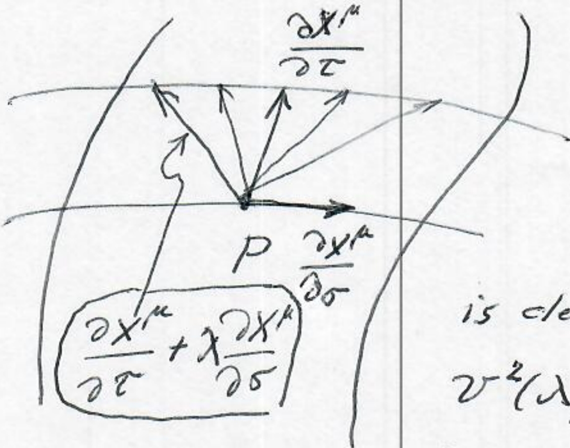
notice the interchange $\leftrightarrow$ ( )

This is often written just as:

$$A = \int d\tau\, d\sigma \sqrt{\underbrace{\left(\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2\left(\frac{\partial X}{\partial \sigma}\right)^2}_{\equiv \mathcal{A}}} \qquad (\ \ )$$

Proof that $\mathcal{A} > 0$ so that the square root is real:

We require that at any point P of the worldsheet there is a timelike and spacelike vector (See more below):



The set of tangent vectors $v^\mu(\lambda)$ ($\lambda \in \mathbb{R}$) at P is:

$$v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}$$

The character (timelike or spacelike) of v^μ is determined by the sign of

$$v^2(\lambda) = v^\mu v_\mu = \lambda^2\left(\frac{\partial X}{\partial \sigma}\right)^2 + 2\lambda\left(\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial \sigma}\right) + \left(\frac{\partial X}{\partial \tau}\right)^2$$

to have both $v^2 > 0$ and $v^2 < 0$, the equation $v^2(\lambda) = 0$ must have two real roots. Hence the discriminant of the quadratic equation $v^2(\lambda) = 0$ must be > 0. But the discriminant is just $\mathcal{A}$. So

$$\mathcal{A} > 0 \qquad \text{q.e.d.}$$

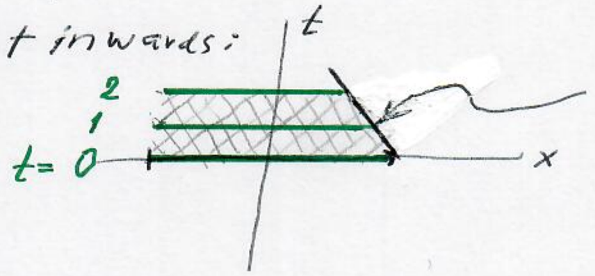
("worldsheet")

Local characterization of surface in spacetime describing a moving string:

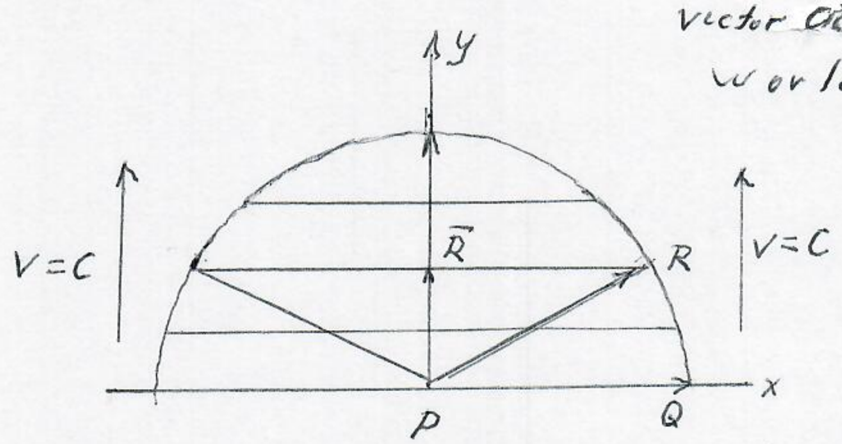
At each point of the surface there is 2-dimensional tangent vector space. We require that in this space 2 vectors form the basis: one spacelike and one timelike vector

Exceptional points - there may be finite set of points where the tangents do not include a timelike vector. Here the strings moves with $v = c$.

For example: „the right endpoint moves with the speed of light inwards:



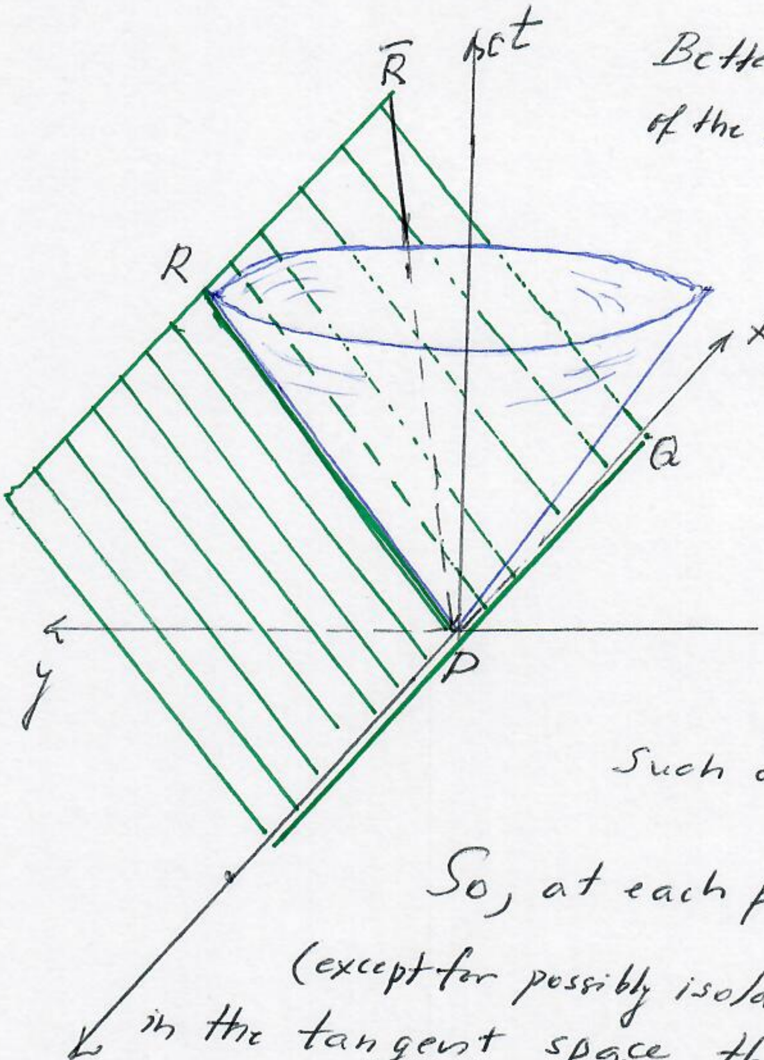
But the string cannot have finite (not just point-like) pieces moving with the speed of light because then timelike vectors at given points would not exist and there could not be introduced local instantaneous Lorentz observers with respect to whom the point would be at rest.



vector at all points tangent to the world sheet:

Vector associated with PQ spacelike but also that wit PR since $P\bar{R}$ is null (P moves "up" with $v = c$ ... gets to $\bar{R}$)

snapshots at different times of a string along x-axis moving with $v = c$ along the y-axis - not allowed motion



Better is the spacetime picture of the string along x moving with c in dir. of y

PQ is spacelike

$P\bar{R}$ is spacelike – it is outside the null-cone (with vertex) at P

PR is null – the generator of the null cone with vertex at P

Such a world-sheet is forbidden

So, at each point of the worldsheet (except for possibly isolated points moving with $v=c$) in the tangent space there exists a timelike and spacelike vector –

however, in contrast to a particle with timelike tangent vector to its worldline where we can go to a Lorentz frame at which the particle is at rest, in case of strings we cannot say that it constitutes of particles which move in some definite way

One cannot say how individual particles on the strings move (except of endpoints of open strings)

closed string

open string

at given t

Nambu-Goto string action

The action for the relativistic string
– proportional to the proper area

We take $[\tau] = T$, $[\sigma] = L$ $\quad$ (T sec, L cm)
$\qquad$ time $\qquad$ length

$[X^\mu] = L$

$[A] = L^2$ indeed area

viz $A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau}\frac{\partial X}{\partial \sigma}\right)^2 - \dots}$

$d\tau d\sigma$ cancels with $\sqrt{\left(\frac{}{d\tau d\sigma}\right)^2}$, so

$A \sim \cancel{\Delta\tau\Delta\sigma} \frac{X^2}{\cancel{\Delta\tau\Delta\sigma}}$, $[A] = L^2$

The action dimension is

erg. sec = F.L.sec =
joule

$= F \cdot L \cdot T = M\frac{L}{T^2} \cdot LT =$

(= "ma") $\quad$ "a" acceler. $\quad$ "mass"

$= M\frac{L^2}{T}$ $\quad \to = \frac{ML^2}{T}$ $\quad (*)$

So we must multiply area by a quantity with $[\;] = \frac{M}{T}$

tension in the string T_0 $\quad [T_0] = [F] = M\frac{L}{T^2}$
$\qquad$ force

dividing this by a velocity $\frac{L}{T}$ (of course c)

gives with $[A] = L^2$ correct dimension $(*)$

Hence, the action for the relativistic string is

$$(NG) \quad \boxed{S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X'^2)}}$$

Here T_0 is the string tension, c velocity of light

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \quad X^{\mu\prime} \equiv \frac{\partial X^\mu}{\partial \sigma}$$

" – " similarly to the action for a particle

$$S = -mc\int_a^b ds \qquad \left(\int ds \text{ for straight line is maximum} \right.$$
$$\left. \Rightarrow -\int ds \text{ -- minimum} \right)$$

(NG) is the Nambu - Goto action for relativistic string

Writing (NG) in manifestly reparameterization invariant form:

$$-ds^2 = \eta_{\mu\nu}\, dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta$$

Minkowski metric in the „target space"

$$\xi^1 = \tau, \; \xi^2 = \sigma$$

Induced metric on the string's worldsheet is

$$\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} = \frac{\partial X}{\partial \xi^\alpha}\cdot\frac{\partial X}{\partial \xi^\beta}$$

$$\gamma_{\alpha\beta} = \begin{bmatrix} \dot{X}^2, & \dot{X}\cdot X' \\ \dot{X}\cdot X', & X'^2 \end{bmatrix}$$

(NG) in the manifestly reparameterization inv. form:

$$S = -\frac{T_0}{c}\int d\tau\, d\sigma \sqrt{-\gamma}$$

$$\gamma = \det(\gamma_{\alpha\beta})$$

Equations of motion from (NG), D-branes

EOM following from the variation of the string action, need of boundary conditions at the ends of open string.

First write NG-action as the integral of a Lagrangian density:

$$S = \int_{\tau_i}^{\tau_f} L\, d\tau = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \mathcal{L}(\dot{X}^\mu, X'^\mu)$$

where

$$\mathcal{L} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}$$

(v) $$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\frac{\partial(\delta X^\mu)}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X'^\mu}\frac{\partial(\delta X^\mu)}{\partial \sigma}\right],$$

where it was used that one can write

$$\delta \dot{X}^\mu = \delta\left(\frac{\partial X^\mu}{\partial \tau}\right) = \frac{\partial(\delta X^\mu)}{\partial \tau}$$

$$\delta X'^\mu = \delta\left(\frac{\partial X^\mu}{\partial \sigma}\right) = \frac{\partial(\delta X^\mu)}{\partial \sigma}$$

Natural notation

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \quad \mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu}$$

Employing this notation and integrating in δS in (V) by parts, we get

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \Big[\big(P^\tau_\mu \delta x^\mu\big)^{\cdot} + \big(P^\sigma_\mu \delta x^\mu\big)' - \big(\dot{P}^\tau_\mu + P'^\sigma_\mu\big)\delta x^\mu \Big] = 0$$

$$\Rightarrow \int_0^{\sigma_1} d\sigma \Big[P^\tau_\mu \delta x^\mu\Big]_{\tau_1}^{\tau_f} + \int_{\tau_i}^{\tau_f} d\tau \Big[P^\sigma_\mu \delta x^\mu\Big]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \delta X^\mu \big(\dot{P}^\tau_\mu + P'^\sigma_\mu\big) = 0$$

We restrict ourselves to variations for which δX^μ vanishes at τ_i and τ_f so that the first term $\sim \Big[P^\tau_\mu \delta x^\mu\Big]_{\tau_1}^{\tau_f} = 0$

Then

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \Big[P^\sigma_\mu \delta x^\mu\Big]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \underbrace{\delta X^\mu}_{\text{arbitrary}} \big(\dot{P}^\tau_\mu + P'^\sigma_\mu\big)$$

Hence, from $\delta S = 0 \Longrightarrow$

$$\dot{P}^\tau_\mu + P'^\sigma_\mu = 0 \quad \text{or, explicitly} \quad \boxed{\frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} = 0} \qquad (\text{I})$$

This is the EOM of the relativistic string – both closed and open.

For open strings $\delta S = 0$ also implies that on boundary

$$\boxed{\int_{\tau_i}^{\tau_f} d\tau \Big[P^\sigma_\mu \delta x^\mu\Big]_0^{\sigma_1} = 0} \qquad (\text{II})$$

Explicit, straightforward calculations give

$$P^{\tau}_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = -\frac{T_0}{c} \frac{(\dot{X}\cdot X')X'_{\mu} - (X')^2 \dot{X}_{\mu}}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}} \quad (\mathrm{III})$$

$$P^{\sigma}_{\mu} = \frac{\partial \mathcal{L}}{\partial X'^{\mu}} = -\frac{T_0}{c} \frac{(\dot{X}\cdot X')\dot{X}_{\mu} - (\dot{X})^2 X'_{\mu}}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}} \quad (\mathrm{IV})$$

notice that P^{σ}_{μ} is P^{τ}_{μ} with the change $' \longleftrightarrow \cdot$

Because of (III), (IV) EOM (I) are very complicated.

A Simplification will come from a suitable parametrization.

For open strings we have still to satisfy the boundary conditions (II) which explicitly in detail read:

$$\int_0^{\tau_f} d\tau \Big[\delta X^0(\tau,\sigma_1) P^{\sigma}_0(\tau,\sigma_1) - \delta X^0(\tau,0) P^{\sigma}_0(\tau,0)$$
$$+ \delta X^1(\tau,\sigma_1) P^{\sigma}_1(\tau,\sigma_1) - \delta X^1(\tau,0) P^{\sigma}_1(\tau,0)$$
$$+ \dots$$
$$+ \delta X^d(\tau,\sigma_1) P^{\sigma}_d(\tau,\sigma_1) - \delta X^d(\tau,0) P^{\sigma}_d(\tau,0) \Big]$$

This is <u>$2D = 2(d+1)$ boundary conditions</u>
– for <u>each term individually</u>.

Let $\sigma^* = \begin{cases} \sigma = 0 \\ \sigma = \sigma_1 \end{cases}$ σ^* denotes the endpoint

Two – types of boundary conditions at σ_*

α) Dirichlet boundary condition:

(α) $$\boxed{\frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0}$$

$\mu \neq 0$, $\mu = 1, \ldots d$ we had the condition $\frac{\partial X^0}{\partial \tau}(\sigma_1^*, \tau) \neq 0$ before

⇒ The endpoint of the string remains fixed during the motion

so $\delta X^\mu(\tau, \sigma_*) = 0$

β) a free endpoint condition (von Neumann (-type))

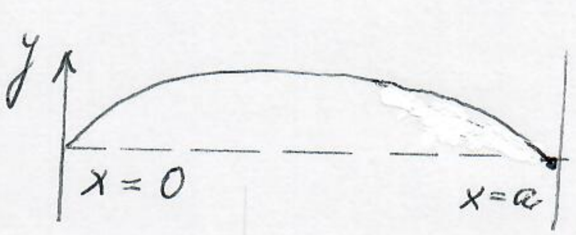
(β) $$\boxed{P^\sigma_\mu(\tau, \sigma_*) = 0}$$

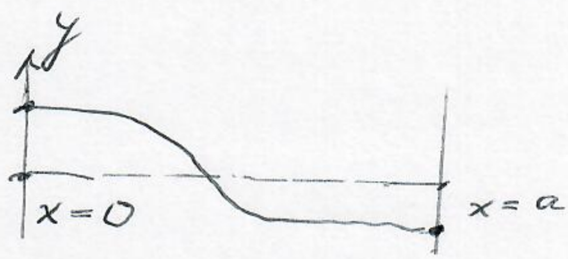
no constraint on $\delta X^\mu(\tau, \sigma_*)$

(β) is true also $\mu = 0$, $P^\sigma_0(\tau, \sigma_1) = P^\sigma_0(\tau, 0) = 0$

For <u>„classical" non-relativistic string:</u>

Dirichlet boundary condition   von Neumann





(α)   (β)

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0$$

μ_0 - mass per unit length T_0 tension T_0 T_0 (force)

$[T_0] = [\text{Energy}]$

D- branes

Dirichlet boundary conditions arise if string endpoints attached to some object

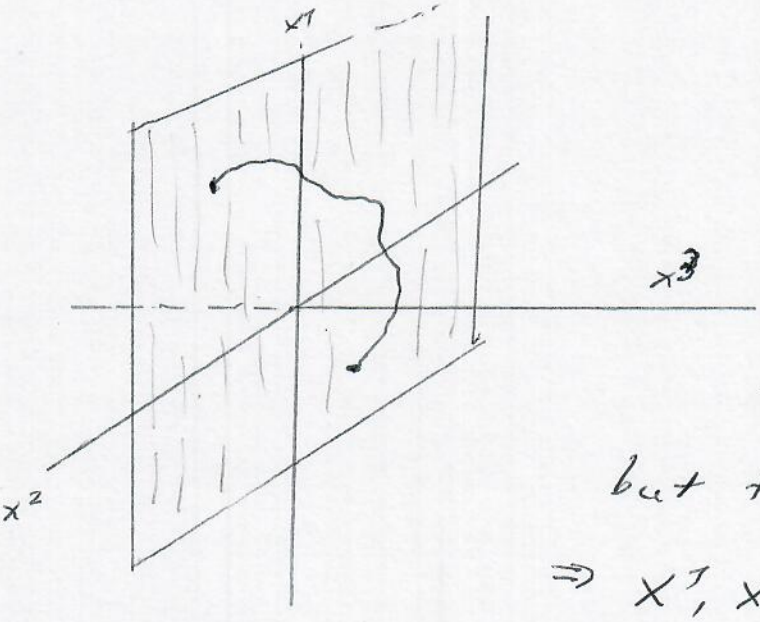
In Fig.-(α) on previous page it is attached to 2 points

Fig. (β) it can slide up and down

Dp - brane is object with p spatial dimensions

In case (α) points are zero-dimensional - D0-branes

(β) endpoints must lie on y = const D1-branes



D2- brane

$x^3 = 0$ for the endpoints (Dirichlet)

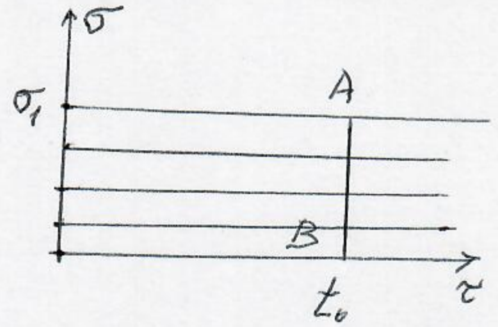
but their x^1, x^2 can vary freely

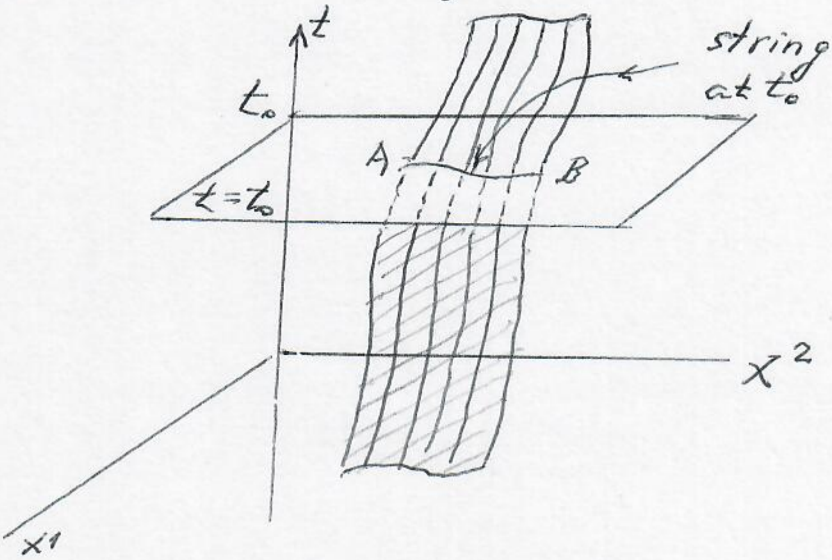
$\Rightarrow$ x^1, x^2 satisfy free boundary conditions

The static gauge

as in electrodynamics, a suitable choice of coordinates (gauge) on a string can simplify „physics" (understanding).

We shall now fix the lines of constant string parameter τ by relating it to the time coordinat. X^0 in some Lorentz frame in the target spacetime.





Hyperplane $t=t_0$ on the target space intersects the string worldsheet along a curve: the string at time $t=t_0$ of our chosen Lorentz frame $(t, x^1, x^2, \dots)$

We take it to be the curve $\tau = t_0$ (see "left" figure)

Extending this definition to all times t, take

$$\tau(Q) = t(Q)$$

for any Q on the worldsheet

<u>This choice of τ parameterization called static gauge</u>

lines of $\tau = $ const are "static strings" in the given Lorentz frame

For open strings one endpoint will have $\sigma = 0$, the other $\sigma = \sigma_1$ (our choice otherwise σ lines arbitr.)

For <u>closed</u> strings we must do identification

$$(\tau, \sigma) \sim (\tau, \sigma + \sigma_c)$$ σ direction is made into a circle

σ_c denotes the circumference of the σ circle

$\sigma \in [0, \sigma_c]$

Since $\tau = t$, the string coordinates in the static gauge can be written as

$$X^\mu(\tau, \sigma) = X^\mu(t, \sigma) = \{ct, \vec{X}(t, \sigma)\}$$

the derivatives are then

$$\frac{\partial X^\mu}{\partial \sigma} = \left(\frac{\partial X^0}{\partial \sigma}, \frac{\partial \vec{X}}{\partial \sigma}\right) = \left(0, \frac{\partial \vec{X}}{\partial \sigma}\right)$$

$$\frac{\partial X^\mu}{\partial \tau} = \left(\frac{\partial X^0}{\partial t}, \frac{\partial \vec{X}}{\partial t}\right) = \left(c, \frac{\partial \vec{X}}{\partial t}\right)$$