Our goal for Einstein's equations: bring them into the form of a quasilinear, hyperbolic system of the 2nd order:

$$(E\,\overline{\underline{III}})\quad \boxed{g^{\alpha\beta}(x;\phi_j,\nabla_\gamma\phi_j)\,\nabla_\alpha\nabla_\beta\phi_i = F_i(x;\phi_j,\nabla_\gamma\phi_j) \qquad i=1,\dots n}$$

smooth Lorentzian metric

(i.e. can be explicitly $-+++$) implies hyperbolic character

more general than previously:

- $g^{\alpha\beta}$ may depend on $\phi_j, \nabla_\gamma\phi_j$

$-(\ast)\nabla_0\nabla_0\phi_i + (\ast)\nabla_i\nabla_j\phi$

the highest derivatives linearly

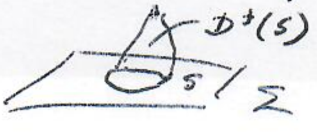
Theorem Leray (1952)

Let $(\phi_0)_1 \dots (\phi_0)_n$ be solution of $(E\overline{\underline{III}})$ on M which is globally hyperbolic (i.e. contains a global Cauchy hypersurface), Σ smooth hypersurf. in $(M,(g_0)_{\alpha\beta})$, $(g_0)^{\alpha\beta} = g^{\alpha\beta}(x;(\phi_0)_j,\nabla_\gamma(\phi_0)_j)$

Then I.V.P for $(E\overline{\underline{III}})$ is well-posed in the following sense:

If we give initial data sufficiently close to data for $(\phi_0)_i$, then there exists a neighborhood $\mathcal{O}$ of hypersurface Σ (so "leave Σ upwards in time") such that $(E\overline{\underline{III}})$ has solution $\phi_1, \dots \phi_n$ in neigh. $\mathcal{O}$ and $(\mathcal{O}, g_{\alpha\beta}(x;\phi_j,\nabla_\gamma\phi_j))$ is globally hyperbolic ("new piece of spacetime").

In addition, the solution is causal: if some other data $\tilde\phi_i$ agree with ϕ_i on a subset S of Σ then the solution $\tilde\phi_i$ agree with ϕ_i on $\mathcal{O} \cap D^+(S)$.



Also, ϕ_i depend continuously on initial data

We wish to convert Einst. eqs. into the form (E III)

By direct calculations one finds (in vacuum)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R =$$

all terms linear in $\partial\partial g_{..}$

$$\left\{ \begin{aligned} &= -\frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \Big[-2\, \partial_\beta \partial_{(\nu} g_{\mu)\alpha} + \partial_\alpha \partial_\beta g_{\mu\nu} \\ &\qquad + \partial_\mu \partial_\nu g_{\alpha\beta} \Big] \\ &+ \frac{1}{2} \sum_{\alpha,\beta,\rho,\sigma} g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} \Big[-\partial_\beta \partial_\rho g_{\sigma\alpha} + \partial_\alpha \partial_\beta g_{\rho\sigma} \Big] \end{aligned} \right. \qquad (\sim)$$

$$+ \tilde{F}_{\mu\nu}(g, \partial g) = 0$$

$\tilde{F}_{\mu\nu}$ are nonlinear functions of $g_{\alpha\beta}$ and $\partial g_{..}$ but independent of "$\partial^2 g_{\alpha\beta}$"; (of course with $\partial_\beta \partial_\nu g_{\mu\alpha}$ above there is $g^{\alpha\beta}$ which is nonlinear in $g_{\alpha\beta}$)

But the system ($\sim$) above is <u>not</u> in the form (E III), p. ($\mathcal{E}_g$ 3) because it involves (contains) also 4 <u>constraints</u>

$$\sum_\nu G_{\mu\nu} n^\nu = 0, \quad \text{or } G_{\mu 0} = 0 \quad {\scriptstyle (\text{with } n^\nu = 1,0,0,0)}$$

where n^ν is normal to Σ (given in special coordinates by $t =$ const); so in these coordinates $\sum G_{\mu\nu} n^\nu = 0$ follows from ($\sim$). These constraints do <u>not</u> contain $g_{\alpha\beta,tt}$ – whereas (E III) does contain $g^{00}(\dots)\nabla_0\nabla_0 g_{\alpha\beta}$

Hence, $H^\mu = g^{\alpha\beta} \nabla_\alpha \nabla_\beta x^\mu = \frac{1}{\sqrt{-g}} \Big(\sqrt{-g}\, g^{\alpha\beta} \underbrace{\frac{\partial x^\mu}{\partial x^\beta}}_{=\delta^\mu_\beta}\Big)_{,\alpha}$

$$\Rightarrow H^\mu = 0 \quad \rightarrow \boxed{\left(\sqrt{-g}\, g^{\alpha\mu}\right)_{,\alpha} = 0} \qquad (H)$$

usual form of conditions for harmonic gauge

In the <u>linear</u> theory the harmonic gauge conditions imply „de Donder/Lorenz gauge conditions":

$$\gamma^{\alpha\mu}{}_{,\mu} = 0, \quad \text{where } \gamma_{\alpha\mu} = h_{\alpha\mu} - \tfrac{1}{2}\eta_{\alpha\mu} h$$

Indeed, $g_{\alpha\mu} = \eta_{\alpha\mu} + h_{\alpha\mu}$, $g^{\alpha\mu} = \eta^{\alpha\mu} - h^{\alpha\mu}$

the determinant $-g = 1 + h$, $h = h^\sigma_\sigma$ (cp. „skripta")

$$\sqrt{-g}\, g^{\alpha\mu} = \sqrt{1+h}\,(\eta^{\alpha\mu} - h^{\alpha\mu}) = (1 + \tfrac{1}{2}h)\eta^{\alpha\mu} - h^{\alpha\mu}$$

$$= \eta^{\alpha\mu} - (h^{\alpha\mu} - \tfrac{1}{2} h \eta^{\alpha\mu}) = \eta^{\alpha\mu} - \gamma^{\alpha\mu}$$

Writing explicitly (H) in terms of "sums":

$$0 = H^\mu = \sum_\alpha \Big(\partial_\alpha g^{\alpha\mu} + \tfrac{1}{2} g^{\alpha\mu} \sum_{\rho,\sigma} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma}\Big) \qquad (\tilde{H})$$

Notice that H^μ contains only 1st. derivatives of $g_{\rho\sigma}$ but $\partial_\alpha H^\mu$ contains $\partial^2 g_{\rho\sigma}$ – will be used later – now!

<u>Intermezzo</u>:

Vladimir Alexandrovich Fo(c)k 1898 – 1974 in St Peterburgh

fundamental work in Quantum Mechanics (Hartree-Fock eq.), Klein-Gordon-Fock eq., Quantum-field theory – Fock space ··· founder of the school of theor. physics in Leningrad – the book: „The Theory of Space, Time and Gravitation" Defender of GR in USSR against Marxists „Propagandist" for using harmonic coordinat...

Reduced Einstein's Eqs (using ∂H^μ) $\mathcal{E}_g 9$

vacuum

$$R^{\text{harm.}}_{\mu\nu} \overset{\text{def}}{\equiv} R_{\mu\nu} + \sum_\alpha g_{\alpha(\mu}\partial_{\nu)} H^\alpha$$

„full", standard $R_{\mu\nu}$ containing various $\partial^2 g_{\alpha\beta}$

Contains $\partial^2 g_{\alpha\beta}$ which combine with those in $R_{\mu\nu}$

indeed, these two „parts" lead to

$$\boxed{R^{\text{harm}}_{\mu\nu} = -\frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \hat{F}_{\mu\nu}(g, \partial g)}$$

this expression for Ricci tensor valid only in harmonic coordinates

$\Rightarrow \quad R_{\mu\nu} = 0 \iff R^{\text{harm}}_{\mu\nu} = 0$ & harmonic conditions $(H), (\tilde{H})$

Nice result is that $R^{\text{harm}}_{\mu\nu} = 0$, i.e.,

$$-\frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \hat{F}_{\mu\nu}(g, \partial g) = 0$$

is in the form for which Leray theorem can be applied!

So given initial conditions $g_{ab} = h_{ab}$, $a,b = 1,2,3$, on init. $t = t_0$ (Σ), and $\frac{\partial h_{ab}}{\partial t}$

the local existence of solutions is guaranteed, $\frac{\partial g_{0a}}{\partial t}$ can be given freely so that $H^\mu = 0$ on Σ

In harmonic coordinates we saw that

$$0 = R^{harm.}_{\mu\nu} = R_{\mu\nu}$$

In addition one can show that for the Ricci scalar (scalar curvature) we obtain

$$\boxed{R^{harm.} = g^{\mu\nu} R^{harm}_{\mu\nu} = R + \frac{1}{2} g^{\mu\nu} g_{\alpha\mu} \partial_\nu H^\alpha + \frac{1}{2} g^{\mu\nu} g_{\alpha\nu} \partial_\mu H^\alpha = R + \frac{1}{2}\partial_\alpha H^\alpha + \frac{1}{2}\partial_\alpha H^\alpha = R + \partial_\alpha H^\alpha} \qquad (+)$$

How does harmonic gauge condition $H^\mu = 0$ "propagates" in time

Choose Σ, $t = t_0$, assume constraints $G_{\mu\nu} n^\nu = 0$ satisfied on Σ, and assume $H^\mu = 0$ on Σ

we have

$$0 = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu} = R^{harm}_{\mu\nu} - \frac{1}{2} R^{harm} g_{\mu\nu} - \sum_\alpha \Big[g_{\alpha(\mu} \partial_{\nu)} H^\alpha - \frac{1}{2} g_{\mu\nu} \partial_\alpha H^\alpha \Big]$$

multiply the last equation by n^ν,

$\sum R_{\mu\nu} n^\nu - \frac{1}{2} R n_\mu = 0$ and assuming $R^{harm}_{\mu\nu} = 0$ on Σ and using $(+)$ above, we get

$$- \sum_{\nu\alpha} g_{\alpha(\mu} \partial_{\nu)} H^\alpha n^\nu + \frac{1}{2} n_\mu \sum_{\rho,\sigma} g^{\rho\sigma} \sum_\alpha g_{\alpha(\rho} \partial_{\sigma)} H^\alpha = 0$$

From here it follows:

if $H^\alpha = 0$ on Σ, so also $\partial_i H^\alpha = 0$

then also $\partial_t H^\alpha = 0$

Now turn to Bianchi identities:

$$0 = \nabla^\mu G_{\mu\nu} = \nabla^\mu \left(R^{harm}_{\mu\nu} - \frac{1}{2} R^{harm} g_{\mu\nu} \right)$$
$$- \nabla^\mu \sum_\alpha \left[g_{\alpha(\mu} \partial_{\nu)} H^\alpha - \frac{1}{2} g_{\mu\nu} \partial_\alpha H^\alpha \right]$$

assume $R^{harm}_{\mu\nu} - \frac{1}{2} R^{harm} g_{\mu\nu} = 0$ is satisfied

$$\Rightarrow \quad 0 = - \sum_{\rho,\mu,\alpha} \frac{1}{2} g_{\alpha\nu} g^{\rho\mu} \partial_\rho \partial_\mu H^\alpha + (\text{terms without } \partial^2 H^\alpha)$$

$$/ \times g^{\lambda\nu}$$

$$\Rightarrow \quad \boxed{0 = - \sum_{\rho,\mu} g^{\rho\mu} \partial_\rho \partial_\mu H^\lambda + (\dots)}$$

This is in <u>the form of the Leray system</u>

With initial conditions $H^\lambda = 0, \; \partial_t H^\lambda = 0$

at $t = t_0$ (on Σ)

Leray $\Rightarrow$ $H^\lambda = 0$ at $t \geq t_0$

In harmonic coordinates the uniqueness of solutions
Yvonne Choquet-Bruhat (97) well-posedness of EFEs
using harmonic coordinates - from 1956 - (first paper 1948)
1979 French Academy (from 1666) - first woman
tradition exchange the same field - from 1987 more
flexible - Lichnerowicz, Darmois, <u>Leray</u>
short history arxiv 1410.3490, 13. 10. 2014
„new" books, visit of Prague

Using 3+1 splitting of the metric (lecture notes by OS), (Eq 12) or "Looking" at the constraints "geometrically"

Gauss-Codazzi equations can be used to write down constraints in terms of quantities "living" just on 3-dim. hypersurface Σ and in the form invariant under coordinate transformations:

$$\underline{0 =} h_\alpha^{\ \beta} G_{\beta\gamma} n^\gamma = h_\alpha^{\ \beta}\left(R_{\beta\gamma} - \tfrac{1}{2} g_{\beta\gamma} R\right) n^\gamma$$

$$(C1) \qquad = h_\alpha^{\ \beta} R_{\beta\gamma} n^\gamma - \tfrac{1}{2} R \underbrace{h_\alpha^{\ \beta} n_\beta}_{=0} =$$

$$= \underline{K_{\alpha|\beta}^{\ \beta} - K_{\beta|\alpha}^{\ \beta}} \qquad \text{cp. Eq. (24.38) in (S)}$$

where $|$ is covariant derivative intrinsic to Σ

if there is a source, $\varkappa h_\alpha^{\ \beta} T_{\beta\gamma} n^\gamma$ stands on l.h.s.

In addition

$${}^{(3)}R_{\alpha\beta\gamma}^{\ \ \ \ \delta} = h_\alpha^{\ \varkappa} h_\beta^{\ \lambda} h_\gamma^{\ \mu} h_\nu^{\ \delta} R_{\varkappa\lambda\mu}^{\ \ \ \ \nu} - K_{\alpha\gamma} K_\beta^{\ \delta} + K_{\beta\gamma} K_\alpha^{\ \delta},$$

contracting in $(\alpha\gamma)$ and $(\beta\delta)$:

$$\Rightarrow \quad {}^{(3)}R = \underbrace{h^{\varkappa\mu} h_\nu^{\ \lambda} R_{\varkappa\lambda\mu}^{\ \ \ \ \nu}} - (K_\alpha^{\ \alpha})^2 + K_{\alpha\beta} K^{\alpha\beta}$$

$$= R_{\varkappa\lambda\mu\nu}\left(g^{\varkappa\mu} + n^\varkappa n^\mu\right)\left(g^{\nu\lambda} + n^\nu n^\lambda\right)$$

$$= R + 2 R_{\alpha\gamma} n^\alpha n^\gamma = 2 G_{\alpha\gamma} n^\alpha n^\gamma$$

so (C2) $$\underline{0 = G_{\alpha\beta} n^\alpha n^\beta = \tfrac{1}{2}\left[{}^{(3)}R + (K_\alpha^{\ \alpha})^2 - K_{\alpha\beta} K^{\alpha\beta}\right]}$$

if not vacuum, source is $\varkappa T_{\alpha\beta} n^\alpha n^\beta$ = "$\varkappa T_{00}$"

cp. Eq. (24.39) in (S)

$\mathcal{C}_g$ 13
$\mathcal{E}_{in}$ (-1)

Wald: General Relativity

 Fundamental theorem

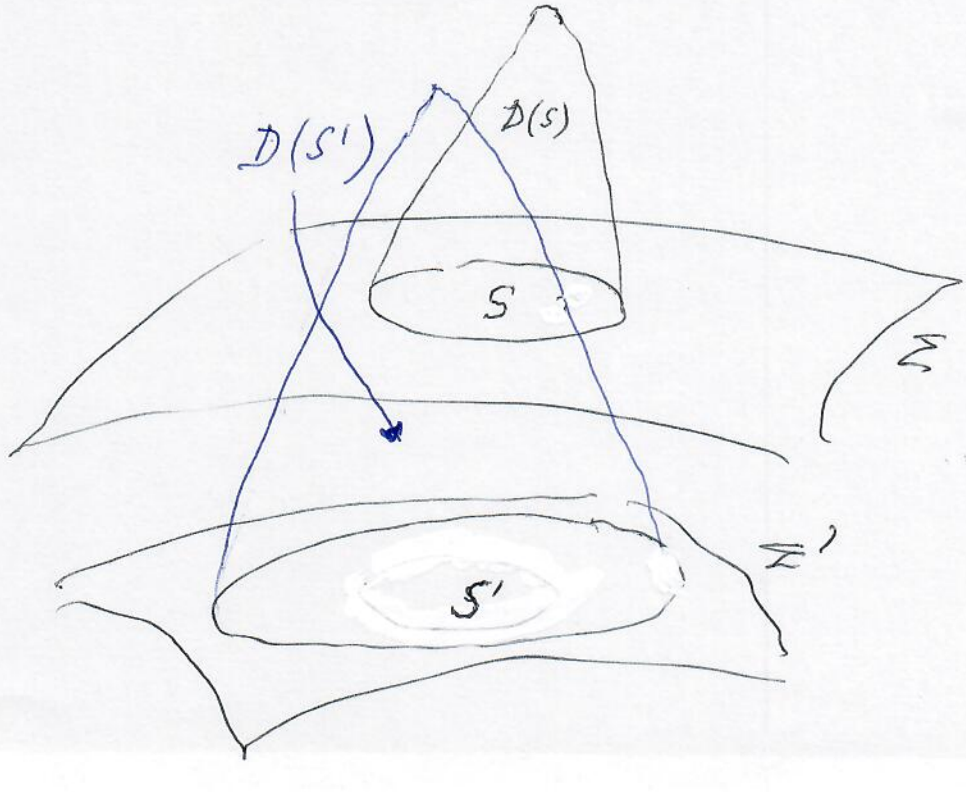
Thus, putting together all the results proven or outlined above, we arrive at the following theorem.

Theorem 10.2.2. *Let Σ be a three-dimensional C^∞ manifold, let h_{ab} be a smooth Riemannian metric on Σ, and let K_{ab} be a smooth symmetric tensor field on Σ. Suppose h_{ab} and K_{ab} satisfy the constraint equations (10.2.28) and (10.2.30).* *) *Then there exists a unique C^∞ spacetime, (M, g_{ab}), called the maximal Cauchy development of (Σ, h_{ab}, K_{ab}), satisfying the following four properties:* (i) *(M, g_{ab}) is a solution of Einstein's equation.* (ii) *(M, g_{ab}) is globally hyperbolic with Cauchy surface Σ.* (iii) *The induced metric and extrinsic curvature of Σ are, respectively, h_{ab} and K_{ab}.* (iv) *Every other spacetime satisfying* (i)–(iii) *can be mapped isometrically into a subset of (M, g_{ab}). Furthermore, (M, g_{ab}) satisfies the desired domain of dependence property in the following sense. Suppose (Σ, h_{ab}, K_{ab}) and $(\Sigma', h'_{ab}, K'_{ab})$ are initial data sets with maximal developments (M, g_{ab}) and (M', g'_{ab}). Suppose there is a diffeomorphism between $S \subset \Sigma$ and $S' \subset \Sigma'$ which carries (h_{ab}, K_{ab}) on S into (h'_{ab}, K'_{ab}) on S'. Then $D(S)$ in the spacetime (M, g_{ab}) is isometric to $D(S')$ in the spacetime (M', g'_{ab}). Finally, the solution g_{ab} on M depends continuously on the initial data (h_{ab}, K_{ab}) on Σ.* (A precise definition of the topologies on initial data and solutions which makes this map continuous is given in Hawking and Ellis 1973.)

to get our notation, change ab to $\alpha\beta$

*) These two constraint equations are

$$K_{\alpha}{}^{\beta}{}_{|\beta} - K^{\beta}{}_{\beta|\alpha} = 0 \quad \text{and} \quad {}^{(3)}R - (K^{\alpha}{}_{\alpha})^2 - K_{\alpha\beta}K^{\alpha\beta} = 0$$



C_{in} 1

Solving initial value constraint equations or constructing initial data

Constraint equations, including sources („$T_{\mu\nu}$"):

$$D^\alpha (K_{\alpha\beta} - K^\gamma_\gamma h_{\alpha\beta}) = -8\pi J_\beta$$

$$^{(3)}R + (K^\alpha_\alpha)^2 - K_{\alpha\beta} K^{\alpha\beta} = 16\pi\rho$$

here

$$\rho = T_{\alpha\beta} n^\alpha n^\beta, \quad J_\beta = -h_\beta^\gamma T_{\gamma\alpha} n^\alpha$$

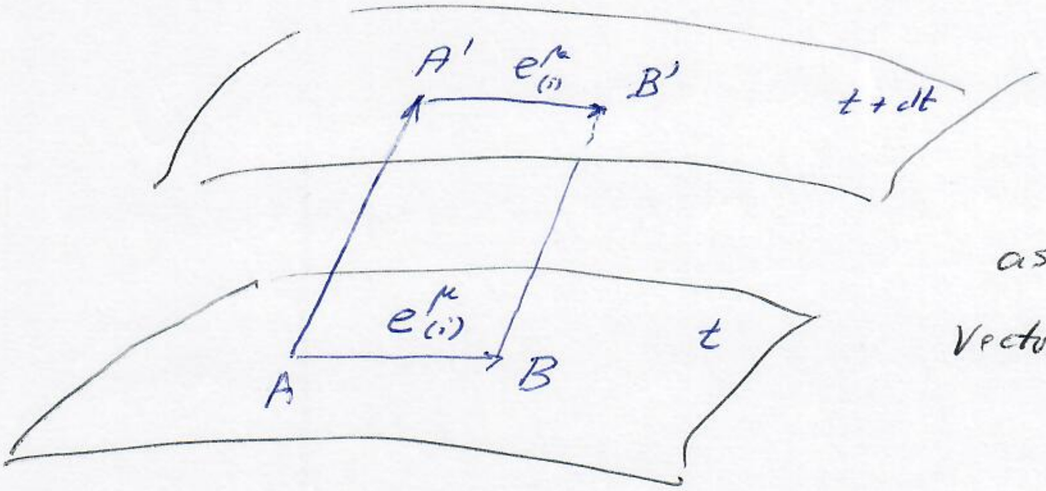
ρ and J^μ are energy and momentum densities seen by „normal" (4-velocity $= n^\alpha$) observers („Eulerian").

From now (hereafter) on choose "adapted" coordinates associated with 3+1 splitting: each hypersurface Σ denoted by some $t = $ const, coordinates x^i $(i = 1,2,3)$ label points in each Σ_t, 1-form $\Omega_\alpha = \nabla_\alpha t = (1,0,0,0)$

choosing $\underline{\text{3 spatial vectors } e^\mu_{(i)}}$ $(i=1,2,3)$ on each that are $\underline{\text{in}}$ a particular slice Σ,

$$\Omega_\alpha e^\alpha_{(i)} = 0$$

and let other $e^\alpha_{(i)}$ on other Σ are given by dragging along t^α, $\mathcal{L}_{t^\alpha} e^\mu_{(i)} = 0$.



as the 4th basis vector $e^\alpha_{(0)} = t^\alpha$

duality condition $t^\alpha \Omega_\alpha = 1 \Rightarrow t^\alpha = (1,0,0,0)$

since $n_\alpha \sim \Omega_\alpha = \nabla_\alpha t \quad \Rightarrow \quad n_i = 0$

the 4-dim metric $g_{\alpha\beta} = h_{\alpha\beta} - n_\alpha n_\beta$

spatial metric $(h_{\alpha\beta} n^\beta = 0)$

expressed in general coordinates

in adapted coordinates

$$h_{ij} = g_{ij}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}$$

$$h^{ij} h_{jk} = \delta^i_k$$

In <u>numerical relativity</u> (and elsewhere) it is common to denote $h_{ij} \equiv \gamma_{ij}$, lapse $N \equiv \alpha$, shift $N^i \equiv \beta^i$

$$h^{ij} \equiv \gamma^{ij} \qquad \beta_i \equiv \gamma_{ij} \beta^j$$

So the metric reads

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix} \quad \text{.... see O.S. (24.6)}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix} \quad \text{... OS (24.7)}$$

Ein 3

$$n^\mu = (1/\alpha, -\beta^i/\alpha) \qquad n_\mu = (-\alpha, 0)$$

$$\sqrt{-g} = \alpha\sqrt{\gamma}$$ O.S. p. 380

In numerical relativity (and elsewhere) is is common to define the extrinsic curvature as the negative expansion of normals to Σ, so with opposite sign than in 'Scripta', so

$$K_{\alpha\beta} = -h_\alpha{}^\gamma h_\beta{}^\delta \nabla_\gamma n_\delta = -\frac{1}{2}\mathcal{L}_{\vec{n}} h_{\alpha\beta}$$

cp O.S. (24.12) has opposite sign
"no change" in vacuum (constraints)

Note:

In "adapted coordinate system",

$n_i = 0$, so since $n^\mu K_{\mu\nu} = 0 \Rightarrow n_\mu K^{\mu\nu} = 0$

for $n_i = 0 \Rightarrow n_0 K^{0\nu} = 0 \Rightarrow K^{00} = K^{0i} = 0$,

in general, however, K_{00} and K_{0i} are not $= 0$

$\Rightarrow$ usually only spatial components K_{ij} used.

In coordinate system adapted to the foliation the constrainst finally read:

$$\boxed{\begin{aligned} {}^{(3)}R + K^2 - K_{ij}K^{ij} &= 16\pi\rho \qquad (1) \\ D_j(K^{ij} - \gamma^{ij}K) &= 8\pi j^i \qquad (2) \end{aligned}}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}, \quad j^i \equiv -P^i_\mu n^\nu T^\mu_\nu$$

projector $P^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta$

(Note that the change of sign of K does leave the constraints unchanged in vacuum.)

Constraints represent 4 coupled elliptic PDEs
Solutions in general cannot be found analytically.
Two approaches: 1) conformal decomposition
2) conformal thin-sandwich approach

1) Conformal decomposition

(York, Licherowicz 1970s)
Y L

$\{\gamma_{ij}, K_{ij}\}$ 12 quantities – in general unclear which 8 to take as „free" data and solve constraints for remaining 4. Y&L consider a conformal transformation of the 3-metric γ_{ij}:

(3)
$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

$\bar{\gamma}_{ij}$ taken as given. The constraint (1) then reads

(4)
$$8\bar{D}^2\psi - \bar{R}\psi + \psi^5(K_{ij}K^{ij} - K^2) + 16\pi\psi^5\rho = 0$$

$\bar{D}^2 \equiv \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j$, $\bar{R}$ ... Ricci from $\bar{\gamma}_{ij}$

Next, one "extracts" trace from the extrinsic curvature: introducing

$$A^{ij} = K^{ij} - \frac{1}{3}\gamma^{ij}K \qquad (\text{so } \gamma_{ij}A^{ij} = 0!)$$

and writes the eq. for ψ in the form

$$8\bar{D}^2\psi - \bar{R}\psi + \psi^5\left(A_{ij}A^{ij} - \frac{2}{3}K^2\right) + 16\pi\psi^5\rho = 0$$

this is the „Hamiltonian constraint"

Momentum constraints become

$$D_j A^{ij} - \frac{2}{3} D^i K - 8\pi j^i = 0$$

We wish to transform them into 3 equations for 3 unknowns

- just "sketching" the procedure

useful result (general) first: any symmetric, tracefree tensor S^{ij} can be split into "transverse" and longitudinal parts

$$S^{ij} = S_*^{ij} + (\underset{\text{operator}}{\mathbb{L}} \underset{\text{vector}}{W})^{ij} \qquad (5)$$

S_*^{ij} is symmetric, traceless and transverse, i.e., its divergence vanishes, $D_j S_*^{ij} = 0$

W^j is vector,

$$\underbrace{(\mathbb{L} W)^{ij}}_{} \equiv D^i W^j + D^j W^i - \frac{2}{3} \gamma^{ij} D_k W^k \qquad (6)$$

longitudinal part of S^{ij} ... called conformal Killing form

If $(\mathbb{L} W)^{ij} = 0$, W^i is the conformal Killing vector (Killing if $D_k W^k = 0$). Then

$$\mathcal{L}_{\vec{W}} (\gamma^{-1/3} \gamma_{ij}) = D_i W_j + D_j W_i - \frac{2}{3} \gamma_{ij} D_k W^k = (\mathbb{L} W)_{ij} = 0$$

Conformal transformation

$$\bar{A}^{ij} = \psi^{10} A^{ij}$$ ("10" leads to simplifications

raise and lower indices with conformal metric $\bar{\gamma}_{ij}$
so that $\bar{A}_{ij} = \psi^2 A_{ij}$. Similarly to (5) p. $\mathcal{E}_{in}5$

decompose
$$\underbrace{\bar{A}^{ij} = \bar{A}^{ij}_*}_{\text{transverse, traceless}} + (\bar{\mathbf{L}} \bar{W})^{ij} \qquad (7)$$

$\bar{\mathbf{L}}$ ... using $\bar{\gamma}_{ij}$

one finds momentum constraints as follows

$$\bar{\Delta}_{\mathbf{L}} \bar{W}^i - \frac{2}{3}\psi^6 \bar{D}^i K - 8\pi \psi^{10} j^i = 0,$$

where
$$\bar{\Delta}_{\mathbf{L}} \bar{W}^i \equiv \bar{D}_j (\bar{\mathbf{L}} \bar{W})^{ij} = \bar{D}^2 \bar{W}^i + \bar{D}_j \bar{D}^i \bar{W}^j \qquad (8)$$
$$- \frac{2}{3} \bar{D}^i \bar{D}_j \bar{W}^j = \bar{D}^2 \bar{W}^i + \frac{1}{3} \bar{D}^i \bar{D}_j \bar{W}^j + \bar{R}^i_j \bar{W}^j$$

Ricci from commuting $\bar{D}_i$

How to find <u>transverse, traceless $\bar{A}^{ij}_*$</u> in (7)?

<u>Start</u> from <u>arbitrary</u> symmetric-traceless <u>$\bar{M}^{ij}$</u> (but not transverse)
Transverse part can be written in the form

$$\bar{M}^{ij}_* = \bar{M}^{ij} - (\bar{\mathbf{L}} \bar{Y})^{ij}, \qquad (9)$$

where $\bar{Y}^i$ is vector which can be determined from

$$0 = \bar{D}_j \bar{M}^{ij}_* = \bar{D}_j \bar{M}^{ij} - \underbrace{\bar{D}_j (\bar{\mathbf{L}} \bar{Y})^{ij}}_{= \bar{\Delta}_{\mathbf{L}} \bar{Y}^i \text{ see (8)}}$$

this is necessary to solve for $\bar{Y}^i$ with $\bar{M}^{ij}$ given

Apply this construction for $\bar{A}^{ij}$, take

$$\bar{A}^{ij}_{*} = \bar{M}^{ij}_{*}$$

Then
$$\bar{A}^{ij} = \bar{A}^{ij}_{*} + (\bar{\mathbb{L}}\bar{W})^{ij} \qquad \text{(see (7))}$$
$$= \bar{M}^{ij}_{*} + (\bar{\mathbb{L}}\bar{W})^{ij} = \underbrace{\bar{M}^{ij} - (\bar{\mathbb{L}}\bar{Y})^{ij}} + (\bar{\mathbb{L}}\bar{W})^{ij} =$$

express from (9)

$$= \bar{M}^{ij} + (\bar{\mathbb{L}}\bar{V})^{ij}, \quad \text{where } \bar{V}^i = \bar{W}^i - \bar{Y}^i$$

The complete form of the constraints in terms of $\bar{A}^{ij}$, $\bar{V}^i$, $\bar{M}^{ij}$ reads as follows:

$$8\bar{D}^2\psi - \bar{R}\psi + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - \frac{2}{3}\psi^5 K^2 + 16\pi\psi^5\rho = 0 \qquad (C\,I)$$
$$\bar{\Delta}_{\mathbb{L}}\bar{V}^i + \bar{D}_j\bar{M}^{ij} - \frac{2}{3}\psi^6\bar{D}^i K - 8\pi\psi^{10}j^i = 0 \qquad (C\,II)$$

These equations determine ψ and $\bar{V}^i$ after free data for $\bar{\gamma}_{ij}$, $\bar{M}^{ij}$, K and sources ρ, j^i are given. (It is usual to define $\bar{\rho} = \psi^8\rho$, $\bar{j}^i = \psi^{10}j^i$ as conformally rescaled ρ, j^i.)

(C I), (C II) is usual form of the constraints in York-Lichnerowicz conformal decomposition.

From the solution of (C I), (C II) the original geometrical/physical quantities are:

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij}, \quad K^{ij} = \psi^{-10}\bar{A}^{ij} + \frac{1}{3}\gamma^{ij}K$$

here $\bar{A}^{ij} = (\bar{\mathbb{L}}\bar{V})^{ij} + \bar{M}^{ij}$

2) Conformal thin-sandwich approach

York, J. W., „Conformal 'thin-sandwich' data for the initial-value problem of general relativity
Phys. Rev. Lett. 82, 1350 (1999)

Prescribe the conformal metric on each of two nearby spatial hypersurfaces („thin-sandwich") or, equivalently prescribe

$$\bar{\gamma}_{ij} = \psi^{-4}\gamma_{ij} \quad \text{and} \quad \bar{u}_{ij} = \partial_t \bar{\gamma}_{ij}$$

next require $\quad \bar{\gamma}^{ij}\,\bar{u}_{ij} = 0 \quad$ (fixed vol. elem. of $\bar{\gamma}$)

trace-free part of the evolution eq. for

$$u_{ij} := \partial_t \gamma_{ij} - \tfrac{1}{3}\gamma_{ij}\left(\gamma^{mn}\partial_t \gamma_{mn}\right)$$

$$= -2\alpha A_{ij} + (\mathbb{L}\beta)_{ij} \qquad \alpha \cdots \text{lapse}, \beta^i \text{shift}$$

$$\bar{\gamma}^{ij}\bar{u}_{ij} = 0 \Rightarrow \partial_t \ln\psi = \partial_t \ln \gamma^{1/12}$$

$$\Rightarrow \bar{u}_{ij} = \partial_t(\psi^{-4}\gamma_{ij}) = \psi^{-4}\left(\partial_t\gamma_{ij} - \tfrac{1}{3}\gamma_{ij}\partial_t \ln\gamma\right) = \psi^{-4}u_{ij}$$

taking $\bar{A}^{ij} = \psi^{10}A^{ij}$ as earlier we get

$$\bar{A}^{ij} = \frac{1}{2\bar{\alpha}}\left[(\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij}\right],$$ it is natural to take

$$\beta^i = \bar{\beta}^i;\ \bar{\alpha} = \psi^{-6}\alpha$$

Hamiltonian constraint unchanged,

$$8\bar{D}^2\psi - \bar{R}\psi + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - \tfrac{2}{3}\psi^5 K^2 + 16\pi\psi^5\rho = 0 \tag{10}$$

Momentum changed:

$$\bar{D}_j\left[\frac{1}{2\bar{\alpha}}(\bar{\mathbb{L}}\beta)^{ij}\right] - \bar{D}_j\left[\frac{1}{2\bar{\alpha}}\bar{u}^{ij}\right] - \tfrac{2}{3}\psi^6\bar{D}^i K - 8\pi\psi^{10}j^i = 0 \tag{11}$$

In this case one has to solve last two equations for the
 - conformal factor ψ
 - shift vector β^i (Cin 9)

(i.e. 4 equations for 4 variables)

as free data one has

- conformal metric $\bar{\gamma}_{ij}$
- its time derivative $\bar{u}^{ij}$
- the trace of extrinsic curvature K
- conformal lapse (densitized - see above)
- matter densities $\bar{\rho}, \bar{j}^i$

The result for the physical quantities then reads:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \qquad (12a)$$

$$K^{ij} = \psi^{-10} \bar{A}^{ij} + \frac{1}{3}\gamma^{ij} K \qquad (12b)$$

where

$$\bar{A}^{ij} = \frac{1}{2\bar{\alpha}} \left[(\bar{\mathbf{L}}\beta)^{ij} - \bar{u}^{ij} \right] \qquad (12c)$$

With this approach one finds not only metric γ_{ij}, extrinsic curv. K^{ij} which satisfy constraints but also the shift vector β^i from the momentum constraint and physical lapse $\alpha = \psi^6 \bar{\alpha}$ (see above)

$\Rightarrow$ initial data for both γ_{ij} and K_{ij} but also for the gauge functions α, β^i

Multiple black hole initial data

Assume vacuum, ρ, j^i vanish. Astrophysically important especially for binary black hole data ⇒ the merger leads to strong source of gravitational waves.

1. Data with $K_{ij} = 0$, i.e. time-symmetric initial data

recall $K_{ij} \sim -\frac{1}{2} \mathcal{L}_n h_{ij}$

⇒ Momentum constraints automatically satisfied, Hamiltonian constraint (CII) implies

$$8 \bar{D}^2 \psi - \bar{R} \psi = 0 \qquad (13) \quad (\text{viz } \rho = 0)$$

To have really simple problem choose $\bar{\gamma}_{ij} = \eta_{ij}$
⇒ $\bar{R} = 0$ and ( ) above leads to flat-space Laplace equation

$$D^2_{flat} \psi = 0 \qquad (14)$$

Want asympt. flat soltn. ⇒ at ∞ $\psi = 1$; if $\psi = 1$ everywhere then $dl^2 = dx^2 + dy^2 + dz^2$, so we found initial data for Minkowski space

More interesting: (Em 11)

$$\psi = 1 + \frac{k}{r}, \quad k = \text{const}$$

here $\psi \to 1$ -- OK

this gives $$dl^2 = \left(1 + \frac{k}{r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right), \qquad (15)$$

so spatial Schwarzschild metric in isotropic coordinates with $M = 2k$.

This only is spatial metric. The time components of $g_{0\mu}$ correspond to choosing lapse and shift.

Following the "thin-sandwich" we can ask $\partial_t K = 0$

this implies equation for lapse $\bar{\alpha}$ and ψ which can be solved and asking $\bar{\alpha} \to 1$ at $r \to \infty$ we find

(it is valid $D^2_{flat}(\bar{\alpha}\psi^7) = D^2_{flat}(\alpha\psi) = 0$)

that $$\alpha = \frac{1 - M/2r}{1 + M/2r}$$ (to get this we ask $\alpha \to 1$ at ∞ and $\to 0$ at horizon $M = 2r$)

$\Rightarrow$ complete Schwartzschild metric in isotropic coordinates -

Still more interesting

$$\psi = 1 + \sum_{i=1}^{N} \frac{m_i}{2|\vec{r} - \vec{r}_i|} \qquad (16)$$

... N black holes which are momentarily at rest located at $\vec{r}_i$.

m_i ... "bare masses" of individual black holes

total ADM mass (see what follows) $M_{ADM} = \sum_{i=1}^{N} m_i$

Intermezzo (going back to „energy problem")

ADM total mass and momentum of asymptotically flat spacetimes

Arnowitt & Deser & Misner ..... brief information

If weak gravity, $|h_{\mu\nu}| \ll 1$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

(17) $$M = \int \rho \, dV$$ ... total mass-energy

(18) $$P_i = \int j_i \, dV$$ ... total momentum

using constraints (1), (2), p. $\mathcal{C}_{in}$ 3

$$ {}^{(3)}R + \underbrace{K^2}_{\text{small}} - \underbrace{K_{ij}K^{ij}}_{\text{small}} = 16\pi\rho$$

$$D_j \left(K^{ij} - \gamma^{ij} K\right) = 8\pi j^i$$

$D_j \downarrow \partial_j$, $\gamma^{ij} \downarrow \delta^{ij}$

in lin. th. $= \partial_j \left(\partial_i h_{ij} - \partial_j h\right)$, $h = h_{ss}$

and substituting ρ and j^i into (17), (18) above

$$\Rightarrow \quad M = \frac{1}{16\pi} \int \partial_j \left(\partial_i h_{ij} - \partial_j h\right)$$

$$P_i = \frac{1}{8\pi} \int \partial_j \left(K_{ij} - \delta_{ij} K\right) dV$$

both integrands $\partial_j(\ \ )$ in the form of divergence

Gauss th. $\Rightarrow$

(19) $$\boxed{M = \frac{1}{16\pi} \oint_S \left(\partial_i h_{ij} - \partial_j h\right) dS^j}$$ ADM mass

(20) $$\boxed{P_i = \frac{1}{8\pi} \oint_S \left(K_{ij} - \delta_{ij} K\right) dS^j}$$ ADM momentum

In strong fields the total M and P_i cannot be given just by (17) and (18) volume integrals because the contribution of grav. field is not taken into account.

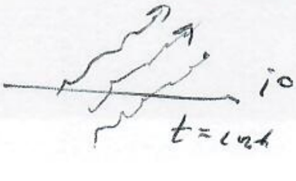
However, we can <u>define</u> the total mass and momentum by their gravit. effects on distant masses so the surface $\oint$ in (19), (20) still give ~~m~~ meaningful results if they are expressed at weak field regime at infinity. Therefore quite generally one can define the ADM mass and momentum of asympt. flat spacetimes as follows

$$\boxed{\begin{aligned} &(21) \quad M_{ADM} = \frac{1}{16\pi} \lim_{r\to\infty} \oint_S \left(\delta^{ij}\partial_i h_{jk} - \partial_k h\right) dS^k \\ &(22) \quad P^i_{ADM} = \frac{1}{8\pi} \lim_{r\to\infty} \oint_S \left(K_j^{\ i} - \delta_j^{\ i} K\right) dS^j \end{aligned}}$$

one can define also ADM angular momentum (cp „$\vec{r}\times\vec{j}$") by

$$(23) \quad \boxed{J^i_{ADM} = \frac{1}{8\pi} \lim_{r\to\infty} \oint_S \varepsilon^{ijk} x_j \left(K_{kl} - \delta_{kl} K\right) dS^l}$$

<u>Notes</u> – all defined at spatial infinity i^0, so these quantities are constant during evolution



– necessary to use asymptotically Cartesian coordinates [illegible] "

general by flat δ_{ij}

The bare mass is the standard mass of a Schwarz. black hole only if one black hole is present.

If more holes present, one can go to asympt. flat end associated with each of them and find the ADM mass there. Illustrate first on 1 black hole: choose spherical coordinates around the centre, use radial coordinate r in isotropic coord. and put

$$r = \left(\frac{\mathcal{M}}{2}\right)^2 \frac{1}{\hat{r}}, \qquad (24)$$

so that isotropic Schw. metric (15) becomes

$$dl^2 = \left(1 + \frac{\mathcal{M}}{2\hat{r}}\right)^4 \left(d\hat{r}^2 + \hat{r}^2 d\Omega^2\right) \qquad (25)$$

This metric evaluated at $\hat{r} = a$ $\equiv$ to the metric (15) at $r = a$.

$\Rightarrow$ The mapping (24) maps the metric into the same - isometry. The origin at $r = 0$ is mapped to $\hat{r} = \infty$ and vice versa. $\Rightarrow$ Einstein-Rosen bridge

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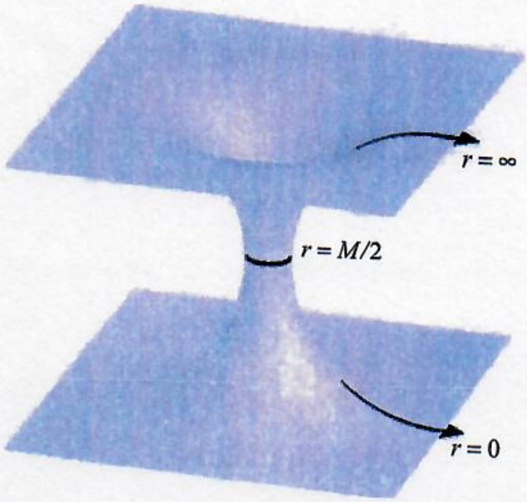

Einstein–Rosen bridge

Figure 3.1 Schematic embedding diagram of Schwarzschild geometry at a moment of time symmetry with one degree of rotational freedom suppressed ($\theta = \pi/2$). The spatial metric is given by equation (3.19). Here a spatial slice of Schwarzschild in the equatorial plane is embedded as a 2-dimensional surface (paraboloid) in a Euclidean 3-space.

The isotropic radius r corresponding to the smallest areal (or circumferential) radius R is $r = M/2$, which we refer to as the black hole *throat*. The throat is located in the horizontal symmetry plane on the circle of smallest circumference in the embedding diagram[9] in Figure 3.1. In Chapter 7.2 (equation 7.26) we will see that, for a single Schwarzschild black hole, the throat coincides with both the apparent and event horizons.

The isometry (3.20) maps points on the throat into themselves, and applying the isometry twice yields the identity transformation. We can therefore think of (3.20) as a reflection in the throat. In the embedding diagram in Figure 3.1 this is a reflection across the horizontal symmetry plane. The geometry close to the origin ($r \to 0$) is identical to the geometry near infinity ($r \to \infty$). We can therefore think of the geometry described by the solution (3.19) as two separate, identical universes, which are connected by a throat, or a so-called *Einstein–Rosen* bridge. Equivalently, a time-symmetric slice of Schwarzschild as depicted in Figure 3.1 corresponds to the $v = 0$ ($t = 0$) hypersurface in the Kruskal–Szekeres diagram in Figure 1.1. On this diagram the throat at areal radius $R = 2M$ connecting the two asymptotically flat universes is located at the origin, $(u, v) = (0, 0)$.

So far we have only rediscovered the vacuum Schwarzschild solution, and that alone would hardly justify the effort of having developed all this decomposition formalism. The formalism is very powerful, however, and allows for the construction of much more general solutions. In Chapters 12 and 15, for example, we will use this approach to construct binary black hole and neutron star initial data. To catch a glimpse of how useful this formalism is, we point out that it is almost trivial to generalize our *one* black hole solution (3.18) to an arbitrary number of black holes at a moment of time symmetry.[10] Since (3.16) is linear,

[9] See, e.g., Misner *et al.* (1973), Chapters 23.8 and 31.6 for the construction of embedding diagrams for Schwarzschild geometry.

[10] E.g., Brill and Lindquist (1963).

Em16

Figure 3.2 Schematic embedding diagram of the geometry described by metric (3.23) for two black holes at a moment of time symmetry. This is the three-sheeted topology, which does not satisfy an isometry across each throat.

$K_{ij} = 0$

we obtain the solution simply by adding the individual contribution of each black hole according to

$$\boxed{\psi = 1 + \sum_{\alpha} \frac{\mathcal{M}_\alpha}{2r_\alpha}.} \tag{3.23}$$

Here $r_\alpha = |x^i - C^i_\alpha|$ is the (coordinate) separation from the center C^i_α of the αth black hole. The total mass of the spacetime is the sum of the coefficients $\mathcal{M}_\alpha$. However, since the total mass will also include contributions from the black hole interactions, $\mathcal{M}_\alpha$ can be identified with the mass of the αth black hole only in the limit of large separations. Particularly interesting astrophysically and for the generation of gravitational waves is the case of binary black holes, in which case (3.23) reduces to

$$\boxed{\psi = 1 + \frac{\mathcal{M}_1}{2r_1} + \frac{\mathcal{M}_2}{2r_2}.} \tag{3.24}$$

This simple solution to the constraint equations for two black holes instantaneously at rest at a moment of time symmetry can be used as initial data for head-on collisions of black holes (see Chapter 13.2).

We can now define mappings equivalent to (3.20), which represent reflections through the αth throat. In general, the existence of other black holes destroys the symmetry that we found for a single black hole. Each Einstein–Rosen bridge therefore connects to its own asymptotically flat Universe. Drawing an embedding diagram for such a geometry yields several different "sheets", where each sheet corresponds to one Universe. A geometry containing N black holes may contain up to $N + 1$ different asymptotically flat universes (see Figure 3.2).

If desired, however, the isometry across the throats can be restored as follows. Recall that equation (3.16) is equivalent to the Laplace equation in electrostatics, so that we

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Figure 3.3 Schematic embedding diagram of a "symmetrized" two black hole solution. This is a two-sheeted topology, in which two Einstein–Rosen bridges connect two identical, asymptotically flat universes.

Figure 3.4 Illustration of a wormhole black hole solution.

can borrow the method of *spherical inversion images*[11] to analyze it. For each throat in (3.23) we can add terms inside that throat that correspond to images of the other black holes. Doing so, the solution (3.23) becomes "symmetrized" so that the reflection through each throat is again an isometry. In other words, each Einstein–Rosen bridge connects to the *same* asymptotically flat Universe, and the geometry consists of only two asymptotically flat universes, which are connected by several Einstein–Rosen bridges (see Figure 3.3).

For two equal-mass black holes we may also interpret this solution as a *wormhole* black hole solution. To see this, consider the solution illustrated in Figure 3.3 for two throats of equal mass. Cut off the bottom Universe at the two throats, which leaves two "open-ended" throats hanging down from the top Universe. We can now identify these two open ends with each other, effectively gluing them together. As illustrated in Figure 3.4, the two throats now form a "wormhole" that connects to a single, asymptotically flat (but multiply connected) Universe. Given the original isometry conditions across the throats, and given

[11] See Misner (1963); Lindquist (1963).

Wormhole

Choose time-symmetric initial value data at $t=0$:

$$\frac{\partial g_{\alpha\beta}}{\partial t}\bigg|_{t=0} = 0 \;, \quad g_{0i}\Big|_{t=0} = 0$$

($g_{0i} \sim dt\,dx^i$ changes sign if $t \to -t$)

hence $\underline{K_{\alpha\beta} = 0}$, we are in vacuum $\rho = J_\beta = 0$. The only constraint - only in. value equat.

(26) $$\boxed{{}^{(3)}R = 0}$$

(Notes: we already proved that Bianchi identities and dynamical Einstein's equations $G_{ij} = 0$, $i = 1,2,3$ guarantee that the constraints $G_{\alpha 0} = 0$ will be satisfied all the time if satisfied at $t=0$.)

Recall conformal transformation (3), or (12a) p. 4, i.e. p. 9

(27) $$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

$\leftarrow$ given "base" suitable metric

the constraint ${}^{(3)}R = 0$ thus implies (see Eq. (4), p. Ein 4) with $K_{ij} = 0 = \rho$

(28) $$\underset{\bar{\Delta}}{\overset{\parallel}{\bar{D}^2}}\psi - \frac{1}{8}\underset{\text{Ricci from } \bar{\gamma}}{\bar{R}}\psi = 0$$

A convenient choice of the background (base) metric is (Misner)

(29) $$d\bar{s}^2 = \bar{\gamma}_{ij}\,dx^i dx^j = d\mu^2 + d\eta^2 + \sin^2\eta\, d\varphi^2$$

μ, η, φ -- see above

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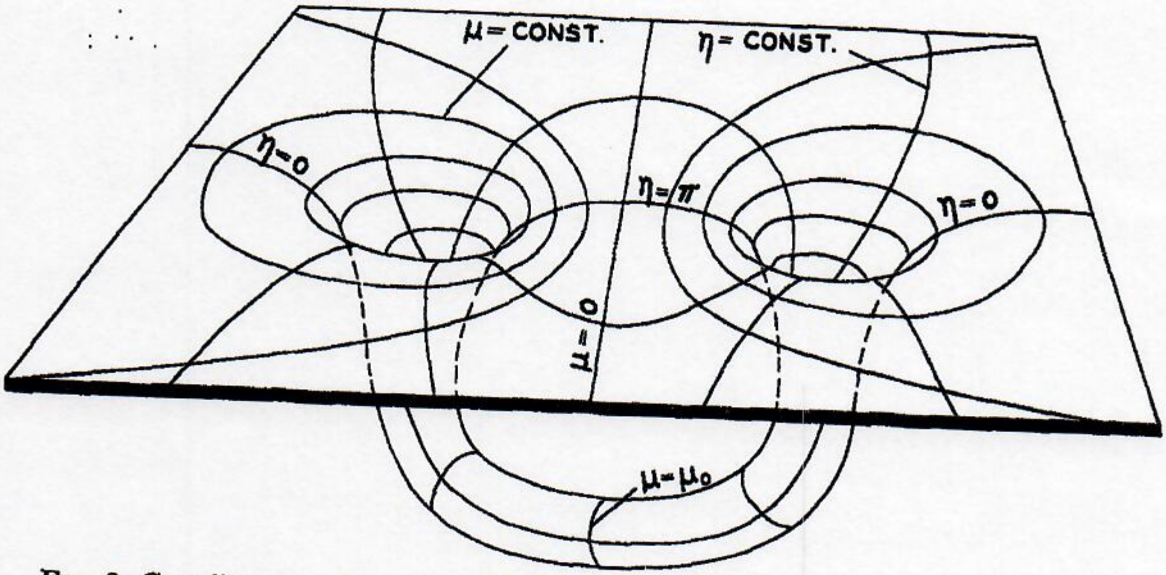
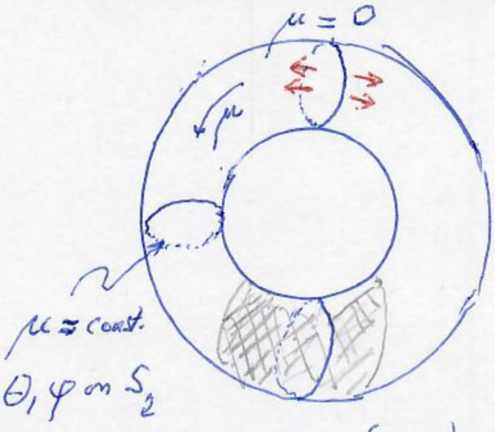


FIG. 3. Coordinate curves μ = const. and η = const. are drawn on a two-dimensional section of the wormhole manifold (cf. Fig. 2(c)). At large distances these coordinate curves become arcs of circles.

*) „The Two Body Problem in Geometrodynamics," Annals of Phys. 29, 304–331 (1964)

Ch. Misner: „Wormhole Initial Conditions," Phys. Rev. 118, 1110 [1960]

How to construct 3-d wormhole:



3-d Doughnut $D = S^1 \times S^2$

$$ds^2_{Dough.} = d\mu^2 + (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

cross-section of D is a sphere (μ = const)

part of the doughnut near $\mu = \pi \equiv -\pi$ (indicated by ▦) will become the tube connecting the mouths of the wormhole

The antipodal part near $\mu = 0$ $(\equiv 2\pi)$ must be ruptured („split") and spread out to become asympt. flat space at ∞.

Wormhole initial data

Consider standard E^3 with Cartesian coordinates x, y, z

introduce „bispherical coordinates" μ, η, φ by equations

$$\coth \mu = \frac{x^2+y^2+z^2+a^2}{2az} \qquad (30)$$

$$\cot \eta = \frac{x^2+y^2+z^2-a^2}{2a\sqrt{x^2+y^2}} \qquad (31)$$

$$\cot \varphi = \frac{x}{y} \qquad (32)$$

$\underline{a}$ -- parameter, $[a] = L$ -- fixes the scale of coordinates

Surface $\mu = \mu_0 =$ const is a sphere S:

$$x^2+y^2+(z-a\coth\mu_0)^2 = a^2 \underbrace{\operatorname{csch}^2\mu_0}_{\equiv (1/\sinh^2 x)}$$

so with the center on z axis at $z = a\coth\mu_0$ radius $R = a \operatorname{csch}\mu_0$

another sphere S' given by $\mu = -\mu_0$

points outside these spheres have $-\mu_0 < \mu < \mu_0$,

$0 \leq \eta \leq \pi$, $0 \leq \varphi < 2\pi$ (periodic coordinates (angular))

also one identifies the points (μ, η, φ) and $(\mu + 2n\mu_0, \eta, \varphi)$ for all integers n. In this way the match up of S and S'

$\Rightarrow$ topology of the wormhole

Eq. (28) for the conformal factor ψ then leads to

$$(33)\quad \left[\frac{\partial^2}{\partial\mu^2} + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\right]\psi = 0$$

Misner formulates appropriate boundary conditions for ψ by comparing with the flat metric (29) with metric in flat space in bispherical coordinates

$$ds^2_{flat} = \frac{a^2}{(\cosh\mu - \cos\eta)^2}\left[d\mu^2 + d\eta^2 + \sin^2\eta\, d\varphi^2\right]$$

then one requires that $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$ is asymptotically flat so that it goes to ds^2_{flat} as $\underbrace{\mu \to 0,\ \eta \to 0}_{\text{infinity}}$

$$\Rightarrow \quad \psi(\mu,\eta,\varphi) \xrightarrow[\substack{\mu\to 0\\ \eta\to 0}]{} \left(\frac{a}{\cosh\mu - \cos\eta}\right)^{1/2} \qquad (34)$$

$\psi(\mu,\eta,\varphi)$ analytic for $-\mu_0 \le \mu \le \mu_0$, $0 \le \eta \le \pi$, $0 \le \varphi < 2\pi$ but has a pole at $\mu = 0$, $\eta = 0$ in the form (34).

It must be also periodic in each argument with periods $2\mu_0$, 2π – rotational symmetric about the z axis joining two spheres at $\mu = \pm\mu_0$, $\Rightarrow$ $\partial\psi/\partial\varphi = 0$

Finally, there must be mirror symmetry in the $z = 0$ plane since the two masses are assumed to be identical $\Rightarrow$

$$\psi(-\mu,\eta,\varphi) = \psi(\mu,\eta,\varphi)$$

Misner has shown in Phys. Rev. 118, 1110 (1960) that there is the unique solution of Eq. (33) satisfying the described conditions of the following form

$$\psi = \sqrt{a} \sum_{n=-\infty}^{\infty} \left[\cosh(\mu + 2n\mu_0) - \cos\eta\right]^{-1/2}$$

The total ADM mass of the system is

$$M = 4a \sum_{n=1}^{\infty} \operatorname{csch} n\mu_0$$

Individual mass m associated with each mouth is given by

$$m = 2a \sum_{n=1}^{\infty} n \operatorname{csch} n\mu_0$$

The total mass M is not $= 2 \cdot m$ of bare individual masses because total M

The length* of the minimal closed 3-geodesic connecting the two mouths is given by

$$L_0 = 2a\Big[1 + 2\mu_0 \underbrace{\sum_{n=1}^{\infty} n \operatorname{csch} n\mu_0}_{= \frac{m}{2a}}\Big]$$

*) defined as the length of the curve $\eta = \pi$, $\varphi = 0$, $-\mu_0 \le \mu \le \mu_0$

On degrees of freedom of grav. field

- in standard mechanics: $2n$ positions + velocities

$$\frac{\text{number of required initial data}}{2} = n$$

- Klein-Gordon field:

ϕ, $n^\alpha \phi_{,\alpha}$ ($\phi_{,t}$ if Σ is $t = \text{const}$) 2 functions

$\Rightarrow$ KG field „one degree of freedom" for each point of space

- GR: h_{ab}, $\partial_t h_{ab} \leftrightarrow K_{ab}$

$\Rightarrow$ $6 + 6 = 12$ components but 4 constrains

$\Rightarrow$ 8 functions ...

but 3 free functions - coordinate transformations (diffeos) on initial hypersurface Σ

$\Rightarrow$ 5 indep. functions

+ specification of Σ: $f(x^\alpha) = \text{const}$

$\Rightarrow$ 4 functions

divided by 2 $\Rightarrow$ gravitational field has 2 degrees of freedom per point

(= degrees of freedom of linear spin 2 field in flat spacetime)

The difficulty: to specify which 2 quantities to choose (problem of quantizing gravity)