Today's approach

$\hat{\tau}^{\alpha}$ is not unique

If we introduce $\hat{\tau}'^{\alpha} = \hat{\tau}^{\alpha} + \hat{V}^{\alpha\beta}{}_{,\beta}$,

where $\hat{V}^{\alpha\beta} = -\hat{V}^{\beta\alpha}$, so conservation law (II) is still true, i.e. cp. p. 8

$$\partial_{\alpha}(\hat{T}^{\alpha} + \hat{\tau}'^{\alpha}) = 0$$

but now already $\hat{T}^{\alpha} + \hat{\tau}'^{\alpha} \neq 0$

$\hat{V}^{\alpha\beta}$ is called „superpotential", $\hat{V}^{\alpha\beta} = \hat{V}^{\alpha\beta}(g_{..}, \partial g_{..}, \xi)$

$\hat{V}^{\alpha\beta}$ is not in general tensor, it does not depend on the 2nd derivatives $\partial\partial g_{..}$

One requires

$$\int \hat{\tau}^{0}\, dV = M \qquad (c=1)$$

for Schwarzschild solution

(can have regular matter inside -)

Identities above contain arbitrary vector field ξ^{α} but we can consider 4 independent

vector fields $\xi^{\alpha}_{(b)} = \delta^{\alpha}_{b}$, $b = 0, 1, 2, 3$

and consider $\boxed{\hat{\tau}^{\alpha}_{\beta} = \hat{\tau}^{\alpha}(\xi_{(b)})}$

$\hat{\tau}^{\alpha}_{\beta}$ depends on $g_{\alpha\beta}$, $\partial g_{\alpha\beta}$ ... but not on ξ^{α}

One calls $\hat{\tau}^{\alpha}_{\beta}$ energy-momentum

complex(es) - or, pseudotensor(s)

Best known examples:

- Einstein complex ("unique" of the form $\sqrt{-g}\,\partial g_{..}\,\partial g_{..}$)

$$16\pi \underset{\text{Einst.}}{\hat{\tau}_{\alpha}{}^{\beta}} = -\delta_{\alpha}{}^{\beta}\hat{G} + g_{\kappa\lambda,\alpha}\frac{\partial \hat{G}}{\partial g_{\kappa\lambda,\beta}}$$

where
$$\hat{G} = \hat{R} - \left(g_{\alpha\beta,\kappa}\frac{\partial \hat{R}}{\partial g_{\alpha\beta,\kappa\lambda}}\right)_{,\lambda}$$

R ... Ricci scalar, $\hat{R}$ scalar density

advantages: simple, "good" density $\sim\sqrt{-g}$, does not contain the 2nd derivatives $\partial^2 g_{..}$ ("integral" of motion). But $\hat{\tau}^{\alpha\beta}$, or $\hat{\tau}_{\alpha\beta}$ is not symmetric so conservation of angular momentum is problem

Freud (von P., not Sigmund) superpotential leading to the Einstein complex:

$$\underset{E}{\hat{V}_{\kappa}{}^{\alpha\beta}} = \frac{1}{16\pi\sqrt{-g}}\, g_{\kappa\lambda}\left[g\left(g^{\alpha\delta}g^{\beta\lambda} - g^{\beta\delta}g^{\alpha\lambda}\right)\right]_{,\delta}$$

The total strongly conserved

$$\underset{\text{Eins.}}{\hat{\Theta}_{\alpha}{}^{\beta}} \equiv \underset{E}{\hat{V}_{\alpha}{}^{\beta\gamma}{}_{,\gamma}} = \underset{\text{Einst.}}{\hat{\tau}_{\alpha}{}^{\beta}} + \hat{T}_{\alpha}{}^{\beta}$$

matter

$$\underset{\text{Eins.}}{\hat{\Theta}_{\alpha}{}^{\beta}{}_{,\beta}} \equiv 0$$

gravity $\quad = \sqrt{-g}\,T_{\alpha}{}^{\beta}$

partial derivatives $\Rightarrow$ global conserv. law -

- <u>Landau - Lifšic complex</u>

$$\underset{LL}{\hat{\Theta}}^{\alpha\beta} \equiv \left(\sqrt{-g}\, g^{\alpha\delta}\, \hat{V}_{\delta}{}^{\beta\gamma}\right)_{,\gamma} = \underset{LL}{\hat{\tau}}^{\alpha\beta} + \sqrt{-g}\, T^{\alpha\beta}$$

$$\underset{LL}{\hat{\Theta}}^{\alpha\beta}{}_{,\beta} = \left(\sqrt{-g}\, g^{\alpha\delta}\right)_{,\underline{\gamma\beta}} \hat{V}_{\delta}{}^{\beta\gamma} + \left(\sqrt{-g}\, g^{\alpha\delta}\right) \hat{V}_{\delta}{}^{\beta\gamma}{}_{,\underline{\gamma\beta}}$$

$$= 0$$

Under linear coordinate transformations, $x'^{\mu} = A^{\mu}{}_{\nu} x^{\nu} + B^{\mu}$, but $A^{\mu}{}_{\nu}$ need <u>not</u> satisfy „normalization" conditions like $\Lambda^{\mu}{}_{\nu}$ in Lorentz group,

$\underset{EINS}{\hat{\Theta}}_{\alpha}{}^{\beta}$ transform like tensor dentities of weight 1, $\sim \sqrt{-g}$

$\underset{LL}{\hat{\Theta}}^{\alpha\beta}$ —— " —— —— " —— 2, $\sim g$

$\Rightarrow$ $\underset{Eins}{\hat{\Theta}}_{\alpha}{}^{\beta}$ is slightly preferable

But $\underset{LL}{\hat{\Theta}}^{\alpha\beta}$ one important advantage: $\underset{LL}{\hat{\Theta}}^{\alpha\beta} = \underset{LL}{\hat{\Theta}}^{\beta\alpha}$

$\Rightarrow$ possibility of the formulation of angular momentum conserv. law — see seminar by David V and Richard S this afternoon

In general, one gets <u>reasonable results</u> for the total quantities like $P^{\mu} = \int\limits_{t=\text{const.}} (-g)\left(\underset{LL}{\tau}^{\mu 0} + T^{\mu 0}\right) d^3x$

this can be expressed as

$\oint\limits_{r\to\infty} \ldots$ using superpotentials, but one has to keep asymptotically Minkowskian coordinates

There exist a covariant superpotential

- Komar superpotential expressed with the help of a vector field ξ^α:

$$\underset{K}{\hat{U}}{}^{\alpha\beta} = \frac{1}{8\pi}\sqrt{-g}\left(\xi^{\beta;\alpha} - \xi^{\alpha;\beta}\right)$$

$$\Rightarrow \underset{K}{\hat{T}^\alpha + \hat{\tau}^\alpha} = \underset{K}{\hat{U}}{}^{\alpha\beta}{}_{;\beta} = \nabla_\beta\left(\nabla^{[\alpha}\sqrt{-g}\,\xi^{\beta]}\right)$$

We shall discuss this and use it for the calculation of Schwarzschild mass in the following part.

Use of Komar expression when $\exists$ Killing vector

„Prequel" on Killing vectors:

For <u>any</u> vector ξ_α:

(·) $$\nabla_\alpha \nabla_\beta \xi_\gamma - \nabla_\beta \nabla_\alpha \xi_\gamma = R_{\alpha\beta\gamma}{}^{\delta} \xi_\delta$$

(convention as in MTW)

For Killing vector

$$\nabla_\alpha \xi_\gamma = -\nabla_\gamma \xi_\alpha$$

substituting into (·)

(1) $$\nabla_\alpha \nabla_\beta \xi_\gamma + \nabla_\beta \nabla_\gamma \xi_\alpha = R_{\alpha\beta\gamma}{}^{\delta} \xi_\delta$$

now write down the same equation with $\alpha\beta\gamma \to \beta\gamma\alpha$
and the third eq. with $\alpha\beta\gamma \to \gamma\alpha\beta$ but this $\times(-1)$
and add the eqs.

(2) $$\nabla_\beta \nabla_\gamma \xi_\alpha + \nabla_\gamma \nabla_\alpha \xi_\beta = R_{\beta\gamma\alpha}{}^{\delta} \xi_\delta$$

(3) $$-\nabla_\gamma \nabla_\alpha \xi_\beta - \nabla_\alpha \nabla_\beta \xi_\gamma = -R_{\gamma\alpha\beta}{}^{\delta} \xi_\delta$$

taking (1) + (2) + (3):

$$2 \nabla_\beta \nabla_\gamma \xi_\alpha = \Big(\underbrace{R_{\alpha\beta\gamma}{}^{\delta} + R_{\beta\gamma\alpha}{}^{\delta}}_{= -R_{\gamma\alpha\beta}} - R_{\gamma\alpha\beta}{}^{\delta} \Big) \xi_\delta$$

$$\Rightarrow \quad \nabla_\alpha \nabla_\beta \xi_\gamma = -R_{\beta\gamma\alpha}{}^{\delta} \xi_\delta = +R_{\gamma\beta\alpha}{}^{\delta} \xi_\delta$$

useful relation for a Killing vector

GE 15

Contracting in α, β,

$$\nabla^{\alpha} \nabla_{\alpha} \xi_{\gamma} = - R_{\gamma}{}^{\delta} \xi_{\delta}$$

So, in vacuum, where $R_{\gamma}{}^{\delta} = 0$, the Killing vector satisfies the wave equation

$$\nabla^{\alpha} \nabla_{\alpha} \xi_{\gamma} = g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} \xi_{\gamma} = 0$$

$$\underset{\kappa}{\Theta}^{\alpha} = \underset{\kappa}{\hat{U}}^{\alpha\beta}{}_{;\beta} = \frac{1}{8\pi} \sqrt{-g}\, \nabla_{\beta} \underbrace{\left(\xi^{\beta;\alpha} - \xi^{\alpha;\beta} \right)}_{= P^{\alpha}}$$

analogous to Maxwell

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, F^{\alpha\beta} \right)_{,\beta} = \left(A^{\beta;\alpha} - A^{\alpha;\beta} \right)_{;\beta} = J^{\alpha}$$

$$P_{(\xi)} = \int_{x^0 = \text{const.}} P^0 \sqrt{-g}\, dx^1 dx^2 dx^3 = \text{const}$$

like $Q = \frac{1}{c} \int_{x^0 = \text{const.}} J^0 \sqrt{-g}\, dx^1 dx^2 dx^3 = \text{const}$

total charge

total Komar mass defined as (using P^0)

$$M = \frac{2}{\kappa c^2} \int_S \sqrt{-g}\, \nabla^i \xi^0 \underbrace{df_i}_{\substack{\text{vector} \\ \text{surface} \\ \text{element (spatial)}}}$$

$\kappa = \frac{8\pi G}{c^4} = 8\pi$ using geom. units

using Gauss' theor.

GE16

In case of Schwarzschild solution

$$ds^2 = -\left(1-\frac{2M_S}{r}\right)dt^2 + \frac{1}{1-\frac{2M_S}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$M_S \equiv M$ schwarzschild

Calculate $\nabla^i \xi^0$ only for $i=r$ is non-vanishing

$$\nabla^i \xi^0 = g^{i\mu} \nabla_\mu \xi^0 = g^{i\mu}\left(\partial_\mu \xi^0 + \Gamma^0_{\mu\sigma}\xi^\sigma\right)$$

timelike Killing: $\xi^\sigma = (1,0,0,0)$, only $\xi^0 \neq 0$

$$\Rightarrow \nabla^r \xi^0 = g^{rr}\left(\cancel{0} + \Gamma^0_{r0}\cdot 1\right)$$

$$= \left(1-\frac{2M}{r}\right)\Gamma^0_{r0}$$

$$\Gamma^0_{r0} = \frac{1}{2}g^{0\mu}\left(g_{\mu r,0} + \underbrace{g_{0\mu,r}}_{\text{only this} \neq 0} - g_{0r,\mu}\right) =$$

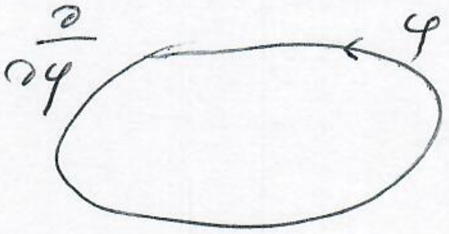
$$= \frac{1}{2}g^{00}g_{00,r} = \frac{1}{2}\left(-\frac{1}{1-\frac{2M_S}{r}}\left(-\frac{2M_S}{r^2}\right)\right)$$

$$\Rightarrow \nabla^r \xi^0 = \frac{M_S}{r^2}$$

$$M = \frac{2}{8\pi}\underbrace{\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\, \cancel{r^2}\sin\theta\,\frac{M_S}{\cancel{r^2}}\,d\theta\, d\varphi}_{4\pi M_S} =$$

$$= M_S$$

Axially symmetric spacetimes (like Kerr) contain also a spacelike Killing vector whose integral curves are closed lines



$$\xi_{(\varphi)} = (0, 0, 0, 1) \text{ in Schw. coordinates}$$

In terms of this Killing vectors one can define total angular momentum of asympt. flat spacetime – similarily to total mass:

$$J = \frac{-1}{\kappa c} \int_S \sqrt{-g}\, \eta^{0i}\, df_i$$

8π in geom. units

× factor 2

For Kerr $J = Ma$

(GE18)

Motivation for Komar expression as the mass-energy of a stationary spacetime without using superpotential

(see R. Wald, GR 11.2)

Consider stationary, asymptotically flat spacetime which is vacuum near infinity.

There exists Killing vector ξ^α which is <u>timelike</u> and normalized so that its magnitude

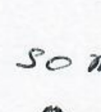
$$V = \sqrt{-\xi_\alpha \xi^\alpha} \to 1 \text{ at } \infty$$

In natural adapted coordinates $\xi^\alpha = (1, 0, 0, 0)$, so $V = \sqrt{-g_{00}}$ is the redshift factor (see lectures by OS)

Let at some fixed x^α there is an observer at rest in the natural coordinates, so with worldline tangent to ξ^α. Its 4-velocity reads

$$U^\alpha = \xi^\alpha / V \quad \text{... unit timelike vector}$$

Such observer must be „supported" (say sit in a „rocket") by some force in order not to fall to the center



The 4-acceleration of the observer is

$$a^\beta = U^\alpha U^\beta{}_{;\alpha} = \frac{\xi^\alpha}{V} \nabla_\alpha \left(\frac{\xi^\beta}{V} \right)$$

$$= \frac{\xi^\alpha}{V^2} \nabla_\alpha \xi^\beta + \frac{\xi^\alpha \xi^\beta}{V} \left(-\frac{1}{V^2} \right) \underbrace{V_{;\alpha}}_{= \frac{1}{2} \frac{1}{V} \left(-2 \xi_{\rho;\alpha} \xi^\rho \right)}$$

recall

$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

for Killing v.

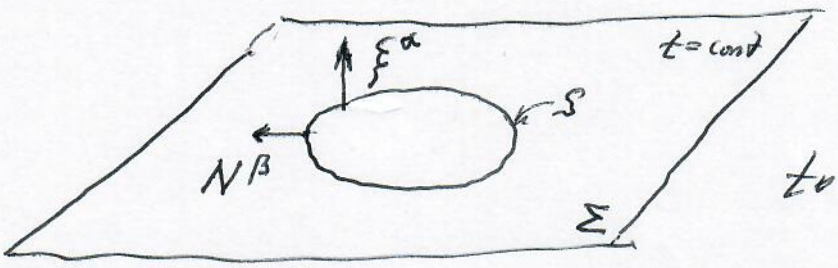
but $\xi_{\rho;\alpha} \xi^\rho \xi^\alpha = 0$ since $\xi_{\rho;\alpha} = -\xi_{\alpha;\rho}$

This $a^\beta = \frac{\xi^\alpha}{V^2} \nabla_\alpha \xi^\beta$ determines the force with which it is necessary to support a particle with mass $m=1$ from falling "down".

However, from the point of view of an observer "at ∞") this force is smaller by a redshift factor V ($V = \sqrt{-g_{00}} = \sqrt{1-\frac{2M}{r}}$ in Schw.), so it is

$$\frac{\xi^\alpha}{V} \nabla_\alpha \xi^\beta$$

Consider a (topologically) spherical massive shell S which is in the (spatial) hypersurface Σ ($t=$ const) perpendicular to timelike ξ^α



Let N^β is unit outw. normal perpendicular to S, $N^\beta N_\beta = 1$, $N_\beta \xi^\beta = 0$

The total force acting on S (preventing it from falling) with unit surface mass density is

(F) $$F = \int_S N^\beta \frac{\xi^\alpha}{V} \nabla_\alpha \xi_\beta \, dS$$ ← inv. surface element on S

Introduce a normal „bivector" from ξ^α and N^β in the form

(N) $$N^{\alpha\beta} = \frac{2}{V} \xi^{[\alpha} N^{\beta]}$$

one can check that $|N_{\alpha\beta} N^{\alpha\beta}| = 2$

Using the Killing v. property $\nabla_\alpha \xi_\beta = \nabla_{[\alpha} \xi_{\beta]}$ (GE20)
it is easy to see that the force (F) can be written as

$$F = \frac{1}{2} \int_S N^{\alpha\beta} \nabla_\alpha \xi_\beta \, dS \qquad (F^*)$$

(substituting for $N^{\alpha\beta}$ from (N) one indeed arrives at (F))
Now, after showing that $\mathcal{E}_{\alpha\beta\gamma\delta} = -6 N_{[\alpha\beta} \mathcal{E}_{\gamma\delta]}$, which requires „more calculations")

(complete antisym. $\mathcal{E}_{\gamma\delta}$ – „volume elem" on S)

one can rewrite (F^*) as

$$\boxed{F = -\frac{1}{2} \int_S \mathcal{E}_{\alpha\beta\gamma\delta} \nabla^\gamma \xi^\delta \, dx^\alpha \wedge dx^\beta}$$

where the integrand can be understood as 2-form

$$\alpha_{\alpha\beta} \, dx^\alpha \wedge dx^\beta$$

where

$$\alpha_{\alpha\beta} = \mathcal{E}_{\alpha\beta\gamma\delta} \nabla^\gamma \xi^\delta$$

In vacuum this 2-form is closed so $d\alpha = 0$

Proof (indication):
multiply α by $\mathcal{E}^{\cdots}$ which being antisym. produces $d\alpha$:

$$\mathcal{E}^{\mu\nu\alpha\beta} \nabla_\nu \underbrace{\left(\mathcal{E}_{\alpha\beta\gamma\delta} \nabla^\gamma \xi^\delta \right)}_{\alpha}$$

$$= \underbrace{\mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}_{\alpha\beta\gamma\delta}}_{\delta\delta - \delta\delta} \nabla_\nu \nabla^\gamma \xi^\delta = -4 \nabla_\nu \nabla^{[\mu} \xi^{\nu]} =$$

$$4 \nabla_\nu \nabla^\nu \xi^\mu = -4 R^\mu{}_\nu \xi^\nu$$

if $R^\mu_\nu = 0$ this $= 0$

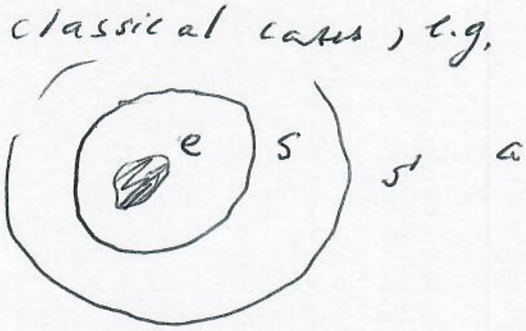
see p. (GE17)

Since in vacuum $d\alpha = 0$,
by using Stoke's theorem $\left(\int_V d\alpha = \int_{\partial V} \alpha \right)$
it follows that in vacuum

$$F = -\frac{1}{2} \int_S \varepsilon_{\alpha\beta\gamma\delta} \nabla^{\gamma} \xi^{\delta} dx^{\alpha} \wedge dx^{\beta}$$

does <u>not</u> depend on the choice of surface S

like in classical cases, e.g. elstatics - charge e



using Gauss theorem

$$e = \frac{1}{4\pi} \int \vec{E} \, d\vec{S}$$

indep. of S (in vacuum)

One then defines, in GR case, the total mass surrounded by S in asympt. flat stationary spacetime by

$$M = -\frac{1}{8\pi} \int_S \varepsilon_{\alpha\beta\gamma\delta} \nabla^{\gamma} \xi^{\delta} dx^{\alpha} \wedge dx^{\beta}$$

But this gives Komar mass and
in Schwarzschild case $M = M_{Schw.}$