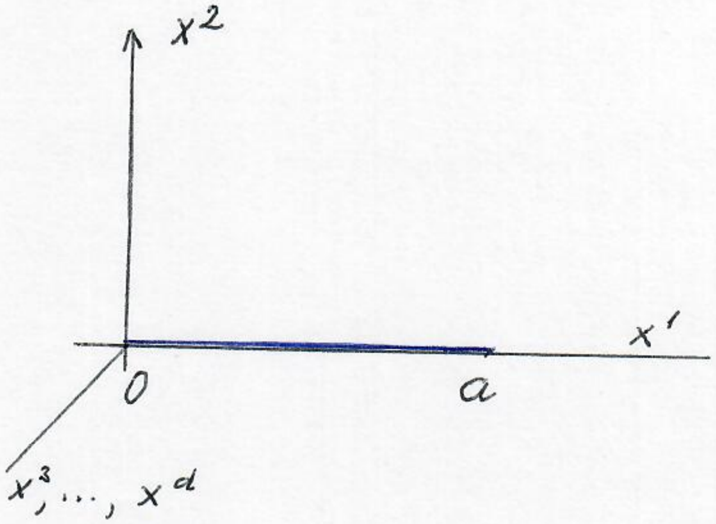
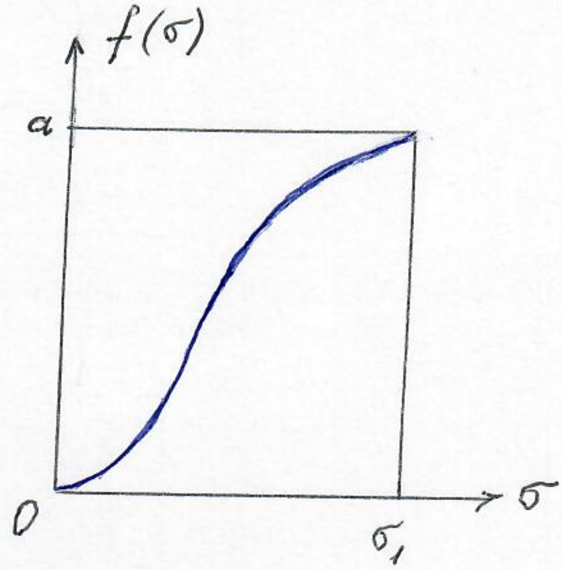
Stretched string





the endpoints of the open string are fixed

$$\text{at } x^1 = 0 \quad \text{and} \quad x^1 = a, \quad x^2, \dots x^d = 0$$

use static gauge, so $X^0 = c\tau$, the string is at rest, so X^1 is independent of time, we take

$$X^1 = X^1(t, \sigma) = f(\sigma), \quad f(0) = 0, \; f(\sigma_1) = a$$

See Figs. above

$f(\sigma)$ increasing, each point has a unique σ

$$\Rightarrow \quad \dot{X}^\mu = (c, 0, \vec{0})$$

$$X'^\mu = (0, f', \underbrace{\vec{0}}_{\text{for } x^2, \dots, x^d}), \quad f' = \frac{df}{d\sigma} > 0$$

$$\Rightarrow (\dot{X})^2 = -c^2, \quad (X')^2 = (f')^2,$$

$$\dot{X} \cdot X' = 0$$

$\Rightarrow$ The Nambu-Goto action (NG), 59 becomes

$$\underline{S} = -\frac{T_0}{c}\int_{t_i}^{t_f} dt \int_0^{\sigma_1} d\sigma \sqrt{0-(-c^2)f'^2} =$$

$$= -T_0 \int_{t_i}^{t_f} dt \underbrace{\int_0^{\sigma_1} d\sigma\, f'}_{= \underbrace{f(\sigma_1)}_{=a} - \underbrace{f(0)}_{0}} \quad = \int_{t_i}^{t_f} dt\,(-T_0 a) \quad (\sim)$$

The value of S does not depend on $f(\sigma)$
(cf. reparameterization invariance)

Commonly $S = \int L\,dt$ $\quad L = T - V$, here $T = 0$
kinetic energy

comparing with $(\sim) \Rightarrow$ $\boxed{V = T_0 a}$

Potential energy of the stretched string is $T_0 a$.
„Creating" the string of length a by stretching it out one does work – in this case one creates string's rest energy / mass – rest mass per unit length:

$$\mu_0 c^2 = \frac{V}{a} \Rightarrow \mu_0 = \frac{T_0}{c^2}$$

it arises because the string has a tension.
Also, „–" sign in front of the action appears OK because the potential energy of the stretched string is positive.

Let us note that the eqs. of motion are indeed satisfied. From (I), p. S11 it follows that in this case

$$\frac{\partial P^{\sigma}_{\mu}}{\partial \sigma} = 0 \qquad \text{since} \qquad \frac{\partial P^{\tau}_{\mu}}{\partial \tau} = 0$$

From (IV), p. S12 it indeed follows

$$P^{\sigma}_{\mu} = -T_0 \frac{X'_{\mu}}{f'} \quad \text{nonvanishing for } \mu=1,$$

when $X'_1 = f'$, so $\dfrac{\partial P^{\sigma}_1}{\partial \sigma} = 0$. Since we have Dirichlet boundary conditions for the string, boundary conditions (p. S12) are also satisfied. And the free boundary condition $P^{\sigma}_0 = 0$ is satisfied - see $P^{\sigma}_0 = -T_0 \dfrac{X'_0}{f'}$

Wish analogy with a particle ⇒

Action in terms of transverse velocity

$\left.\dfrac{\partial \vec{X}}{\partial t}\right|_{\sigma}$ goes along σ-lines but these may be quite arbitrary, so $\dfrac{\partial \vec{X}}{\partial t}$ cannot be physical velocity.

String has no substructure - points on string would be meaningful only if there would preferred σ

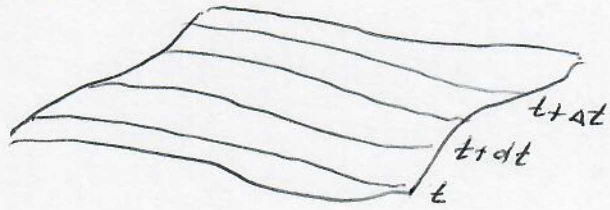
But reparameterization invariance.

Longitudinal motion on string not physically meaning.

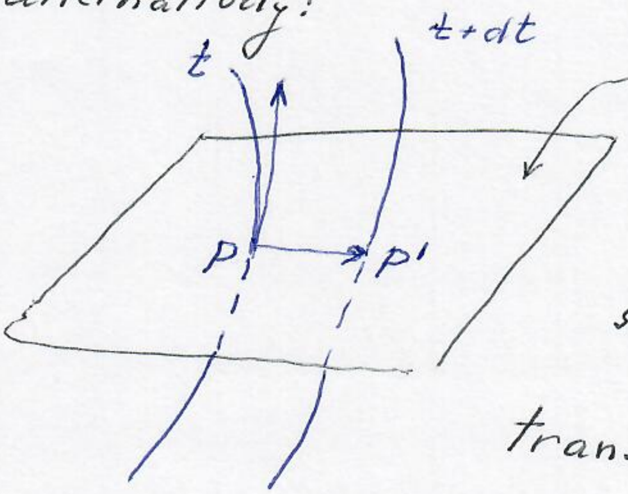
There is a transverse velocity.

In spacetime are two 2-surfaces associated with string:

- worldsheet in ST
- spatial surface as we observe the string at each time t



alternatively:



the plane orthogonal to the string at P

Later at $t+dt$ the string intersects the plane at P'

transverse velocity $\vec{v}_\perp$ is orthogonal to the string

Detailed definition:

Introduce s measuring length along the string at a fixed time; $s(\sigma)$ length of string in the interval $[0,\sigma]$

$s(0)=0$, $s(\sigma_1)$ – the length of open string

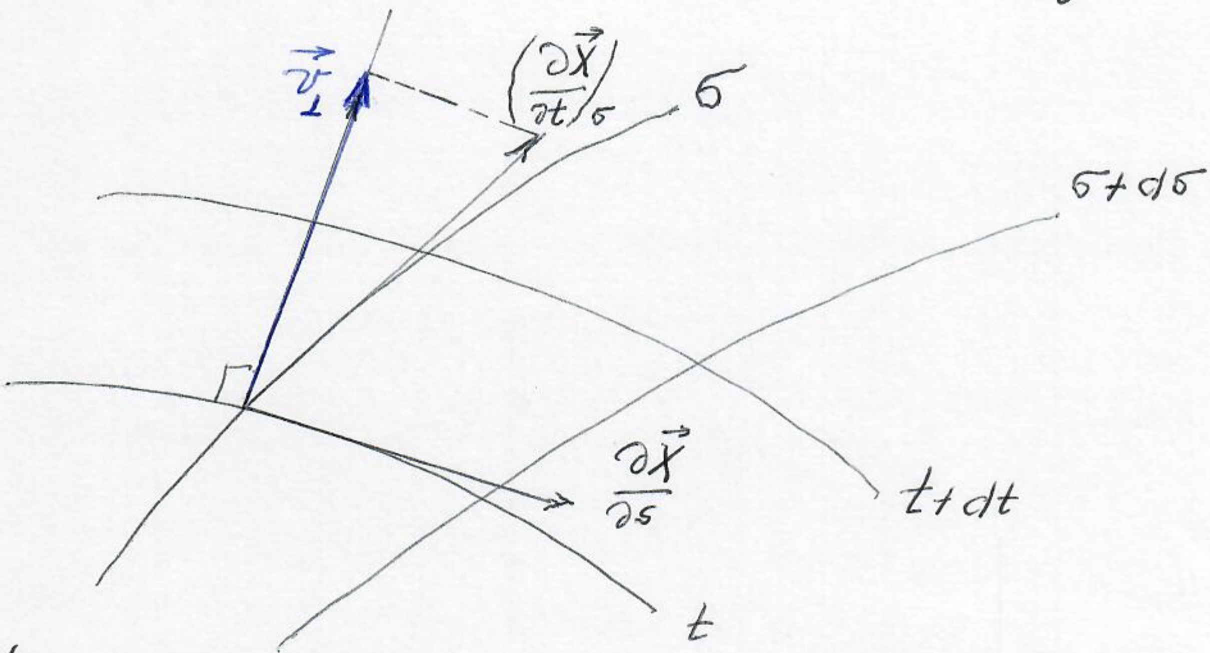
$$ds = |d\vec{X}| = \left|\frac{\partial \vec{X}}{\partial \sigma}\right| |d\sigma|$$

With the parameterization by s, $\frac{\partial \vec{X}}{\partial s}$ is unit vector tangent to the string at given t:

$$\frac{\partial \vec{X}}{\partial s}\cdot\frac{\partial \vec{X}}{\partial s} = \frac{\partial \vec{X}}{\partial \sigma}\cdot\frac{\partial \vec{X}}{\partial \sigma}\left(\frac{d\sigma}{ds}\right)^2 = \underbrace{\left|\frac{\partial \vec{X}}{\partial \sigma}\right|^2\left(\frac{d\sigma}{ds}\right)^2 = 1}$$

$$\Rightarrow \frac{ds}{d\sigma} = \left|\frac{\partial \vec{X}}{\partial \sigma}\right|$$

$\vec{v}_\perp$ is defined as the component of the velocity $\dfrac{\partial \vec{X}}{\partial t}$ in the direction perpendicular to the string:



Noting that

for any vector $\vec{u}$ its component $\perp$ to a unit $\vec{n}$ is $\vec{u} - (\vec{u}\cdot\vec{n})\vec{n}$

we find (recalling that $\partial \vec{X}/\partial s$ is unit)

$$\boxed{\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t} - \left(\frac{\partial \vec{X}}{\partial t}\cdot\frac{\partial \vec{X}}{\partial s}\right)\frac{\partial \vec{X}}{\partial s}}$$

Indeed, we get $\vec{v}_\perp \cdot \dfrac{\partial \vec{X}}{\partial s} = \dfrac{\partial \vec{X}}{\partial t}\cdot\dfrac{\partial \vec{X}}{\partial s} - \left(\dfrac{\partial \vec{X}}{\partial t}\cdot\dfrac{\partial \vec{X}}{\partial s}\right)\underbrace{\dfrac{\partial \vec{X}}{\partial s}\cdot\dfrac{\partial \vec{X}}{\partial s}}_{=1} = 0$

$\Rightarrow$

$$v_\perp^2 = \left(\frac{\partial \vec{X}}{\partial t}\right)^2 - \left(\frac{\partial \vec{X}}{\partial t}\cdot\frac{\partial \vec{X}}{\partial s}\right)^2$$

we wish to rewrite action in terms of $v_\perp$

In static gauge $\tau = t$

$$\dot{X}^2 = -c^2 + \left(\frac{\partial \vec{X}}{\partial t}\right)^2, \quad (X')^2 = \left(\frac{\partial \vec{X}}{\partial \sigma}\right)^2, \quad \dot{X}\cdot X' = \frac{\partial \vec{X}}{\partial t}\cdot\frac{\partial \vec{X}}{\partial \sigma}$$

since before we had $\dfrac{\partial X^\mu}{\partial \tau} = \left(\dfrac{\partial X^0}{\partial t}, \dfrac{\partial \vec{X}}{\partial t}\right) = \left(c, \dfrac{\partial \vec{X}}{\partial t}\right)$

From here

$$(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2=\dots=\left(\frac{ds}{d\sigma}\right)^2\left[\left(\frac{\partial\vec{X}}{\partial t}\frac{\partial\vec{X}}{\partial s}\right)^2+c^2-\left(\frac{\partial\vec{X}}{\partial t}\right)^2\right]$$

$$\Rightarrow\quad \underbrace{(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2}_{\sqrt{\cdots}\ \text{enters the action}}=\left(\frac{ds}{d\sigma}\right)^2\left(c^2-v_\perp^2\right)$$

$\Rightarrow$ string action

$$\boxed{S=-T_0\int dt\int_0^{\sigma_1}d\sigma\underbrace{\left(\frac{ds}{d\sigma}\right)}_{\substack{\text{not cancelled because}\\ \text{fixed boundaries }(0,\sigma_1)\text{ "in }\sigma\text{"}}}\sqrt{1-\frac{v_\perp^2}{c^2}}}$$

S true for both open and closed strings

Associated Lagrangian is

$$\boxed{L=-T_0\int ds\sqrt{1-\frac{v_\perp^2}{c^2}}}\qquad(+)$$

$T_0\,ds$ ... rest energy of "infinitesimal part" of string

(+) - a clear („telling") generalization of the Lagrangian of the relativistic particle.

The results for „$P^{\cdot\cdot}_{s}$":

$$P^{\sigma\mu}=-\frac{T_0}{c}\,\frac{\left(\frac{\partial\vec{X}}{\partial\sigma}\cdot\frac{\partial\vec{X}}{\partial t}\right)\dot{X}^\mu-\left[-c^2+\left(\frac{\partial\vec{X}}{\partial t}\right)^2\right]X^{\mu\prime}}{c\,\frac{ds}{d\sigma}\sqrt{1-\frac{v_\perp^2}{c^2}}}=$$

putting $\frac{ds}{d\sigma}$ from denominator up to numerator:

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\frac{\left(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t}\right)\dot{X}^{\mu} + \left[c^2 - \left(\frac{\partial\vec{X}}{\partial t}\right)^2\right]\frac{\partial X^{\mu}}{\partial s}}{\sqrt{1-\frac{v_\perp^2}{c^2}}} \qquad (\sim)$$

For $\mu = 0$ simplification since $\dot{X}^0 = c$
and $\partial X^0/\partial s = c\,\partial t/\partial s = 0$:

$$\mathcal{P}^{\sigma 0} = -\frac{T_0}{c}\frac{\left(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t}\right)}{\sqrt{1-\frac{v_\perp^2}{c^2}}} \qquad (*)$$

Similar calculations leads to

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{ds}{d\sigma}\frac{\dot{X}^{\mu} - \left(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t}\right)\frac{\partial X^{\mu}}{\partial s}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}$$

Open string endpoints

1) Move transversally to the string, i.e. their velocity is $\perp$ to the tangent to the string at the endpoint
2) Move with the velocity of light

ad 1) at the endpoint $\mathcal{P}^{\sigma 0} = 0$. From (*) above

$$\Rightarrow \underbrace{\frac{\partial\vec{X}}{\partial s}}_{\text{unit, tangent to string}}\cdot\underbrace{\frac{\partial\vec{X}}{\partial t}}_{\text{velocity of the endpoint}} = 0 \qquad \text{qed}$$

2) because of $\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t} = 0$, $(\sim)$ implies

$$\mathcal{P}^{\sigma\mu} = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial X^{\mu}}{\partial s}, \quad \text{for } \mu = 1,\dots d: \vec{\mathcal{P}}^{\sigma} = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial\vec{X}}{\partial s} = 0$$

at the endpoint $\Rightarrow v^2 = c^2$

σ - parameterization

Lines of constant σ are constructed so that they are always perpendicular to the lines of constant t, i.e. the tangents $\frac{\partial \vec{X}}{\partial \sigma}$ to the strings are at any point perpendicular to the lines of constant σ with tangents $\left.\frac{\partial \vec{X}}{\partial t}\right|_\sigma$:

$$\boxed{\frac{\partial \vec{X}}{\partial \sigma} \cdot \frac{\partial \vec{X}}{\partial t} = 0}$$

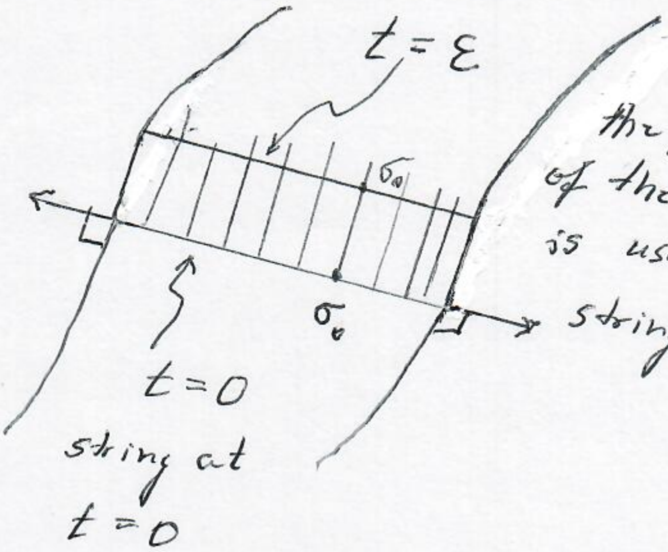
Also, since $\frac{\partial \vec{X}}{\partial t}$ is perpendicular to the string,

$$\underline{\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t}} \quad \text{everywhere}$$

(recall the expression for $\vec{v}_\perp$ on p. S21 — there $\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} = 0$)

Expressions for P's simplify as follows:

$$\text{(E)} \quad \left| \; \mathcal{P}^{\tau\mu} = \frac{T_0}{c^2} \frac{\frac{ds}{d\sigma}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \frac{\partial X^\mu}{\partial t}, \quad \mathcal{P}^{\sigma\mu} = -T_0 \sqrt{1 - \frac{v_\perp^2}{c^2}} \frac{\partial X^\mu}{\partial s} \; \right|$$



the parameterization of the $t = 0$ string is used to parameterize the string at $t = \varepsilon$

Equation of motion in σ parameterization

so we now have σ and $t=\tau$ so EOM reads

$$(H) \qquad \frac{\partial P^{\tau\mu}}{\partial t} = -\frac{\partial P^{\sigma\mu}}{\partial \sigma}$$

Take $\mu = 0$. From previous page 524

$$P^{\sigma 0} \sim \frac{\partial X^0}{\partial s} = 0, \qquad P^{\tau 0} = \frac{T_0}{c}\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}} \qquad \left(\text{viz } \frac{\partial X^0}{\partial t} = c\right)$$

⇒ EOM becomes

$$\frac{\partial P^{\tau 0}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{T_0 \frac{ds}{d\sigma}}{c\sqrt{1-\frac{v_\perp^2}{c^2}}}\right) = 0 \qquad (\cdot)$$

consider a small piece given by $d\sigma$ fixed – but ds can be time-dependent – length of the piece $d\sigma$. $(\cdot)$ ⇒

$$\frac{T_0\, ds}{\sqrt{1-\frac{v_\perp^2}{c^2}}} = \text{const}$$

relativistic energy of the piece of the string

so the energy of each piece $d\sigma$ is conserved

From (E) on previous page

$$\vec{P}^{\tau} = \frac{T_0}{c^2}\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\underbrace{\frac{\partial \vec{X}}{\partial t}}_{=\vec{v}_\perp}, \qquad \vec{P}^{\sigma} = -T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\,\frac{\partial \vec{X}}{\partial s}$$

$P^{\tau\mu}$ for $\mu = 1, \ldots d$

Equations of motion (I) from previous page become

$$\frac{\partial}{\partial\sigma}\left[T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\,\frac{\partial\vec{X}}{\partial s}\right]=\frac{\partial}{\partial t}\left[\frac{T_0}{c}\,\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,\vec{v}_\perp\right]=$$

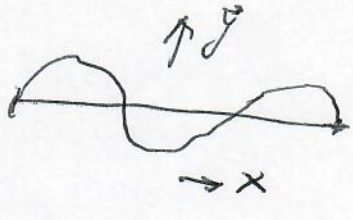
using (*) from previous page

$$=\frac{T_0}{c^2}\,\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,\frac{\partial\vec{v}_\perp}{\partial t}\quad\Big/\ *\ \frac{d\sigma}{ds}$$

(RS) $$\Rightarrow\ \boxed{\frac{T_0}{c^2}\,\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\,\frac{\partial\vec{v}_\perp}{\partial t}=\frac{\partial}{\partial s}\left[T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\,\frac{\partial\vec{X}}{\partial s}\right]}$$

Compare with equation of motion of a non-relativistic string

(CS) $$\boxed{\mu_0\frac{\partial^2\vec{y}}{\partial t^2}=T_0\frac{\partial^2\vec{y}}{\partial x^2}=\frac{\partial}{\partial x}\left(T_0\frac{\partial\vec{y}}{\partial x}\right)}$$



For small oscillations the length parameter s along the string $\approx x$ along static classical string. Then,

Comparing (RS) and (CS) $\Rightarrow$ RS has velocity-dependent effective tension

$$T_{eff}=T_0\sqrt{1-\frac{v_\perp^2}{c^2}}$$

at the endpoints $v_\perp^2=c^2\Rightarrow T_{eff}=0!$ intuitively OK

and velocity-dependent effective mass density

$$\mu_{eff}=\frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v_\perp^2}{\ldots}}}$$

$\mu_{eff}\to\infty$ at endpoint but $\int$ is finite, OK

Lorentz symmetry and currents

Lorentz transformations leave invariant the form $\eta_{\mu\nu} X^\mu X^\nu$.

Infinitesimal LT $\quad X^\mu \to X^\mu + \delta X^\mu$

$$\delta X^\mu = \varepsilon^{\mu\nu} X_\nu \qquad (LT)$$

matrix of infinitesimal constants

Lorentz invariance $\delta(\eta_{\mu\nu} X^\mu X^\nu) = 0$

$$\Rightarrow \quad 2\eta_{\mu\nu}\,\delta X^\mu X^\nu = 2\eta_{\mu\nu}(\varepsilon^{\mu\rho} X_\rho) X^\nu$$

$$= 2\varepsilon^{\mu\rho} X_\rho X_\mu = 0$$

for all X_μ's $\Rightarrow \varepsilon^{(\mu\rho)} = 0 \Rightarrow \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$

It is easy to proof explicitly that the Lagrangian density for the strings is Lorentz invariant.

This invariance implies the existence of conserved currents. Writing the action in the form

$$S = \int d\xi^0 d\xi^1 \mathcal{L}(\partial_0 X^\mu, \partial_1 X^\mu) \qquad , (\xi^0, \xi^1) = (\tau, \sigma)$$

(recall (V) on p. $\boxed{S10}$) $\qquad$ derivatives w.r.t. ξ^0, ξ^1

To find conserved currents, do first variation δX^μ

$$\delta X^\mu = \varepsilon^\mu \ (\text{indep. of } \tau, \sigma) \ - \text{„}\underline{\text{translations}}\text{"}$$

and find (see the beginning of semester)

$$\cancel{\varepsilon^\mu} j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \cancel{\varepsilon^\mu}$$

$$\Rightarrow j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \Rightarrow (j^0_\mu, j^1_\mu) =$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \frac{\partial \mathcal{L}}{\partial X'^\mu} \right)$$

But these are just P^α_μ we saw on S12 before!

$$j^\alpha_\mu = P^\alpha_\mu, \text{ so } (j^0_\mu, j^1_\mu) = (P^\tau_\mu, P^\sigma_\mu)$$

and the equation for current conservation becomes

$$\partial_\alpha P^\alpha_\mu = \frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} = 0$$

which is just equations of motion (I), p. S11 for the relativistic string. However the currents associated with more general Lorentz transformation (not just translations) (LT) on previous page ($\delta X^\mu = \varepsilon^{\mu\nu} X_\nu$) are

$$\varepsilon^{\mu\nu} j^\alpha_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu = P^\alpha_\mu \varepsilon^{\mu\nu} X_\nu$$

Since $j^\alpha_{\mu\nu}$ is multiplied by $\varepsilon^{\mu\nu}$ we can write just antisymm. part of the r.h.s., so

$$\varepsilon^{\mu\nu} j^\alpha_{\mu\nu} = \left(-\tfrac{1}{2}\varepsilon^{\mu\nu}\right)\left(X_\mu P^\alpha_\nu - X_\nu P^\alpha_\mu\right)$$

Hence, we define the currents associated with Lorentz inv. as

$$\boxed{\mathcal{M}^{\alpha}_{\mu\nu} = X_\mu P^{\alpha}_{\nu} - X_\nu P^{\alpha}_{\mu}}$$

Clearly, $\mathcal{M}^{\alpha}_{\mu\nu} = -\mathcal{M}^{\alpha}_{\nu\mu}$

current conservation reads

$$\frac{\partial \mathcal{M}^{\tau}_{\mu\nu}}{\partial \tau} + \frac{\partial \mathcal{M}^{\sigma}_{\mu\nu}}{\partial \sigma} = 0$$

To get the total charges for the first case we define

$$P_\mu(\tau) = \int_0^{\sigma_1} P^{\tau}_{\mu}(\tau,\sigma)\, d\sigma$$

density of spacetime momentum

so over whole string with $\underline{\tau \text{ fixed}}$

spacetime momentum carried by this string. Satisfying the boundary condition one can show that

$$\frac{dp_\mu}{d\tau} = 0 \quad \ldots$$ The

In static gauge $\tau = t$, so $dp_\mu / dt = 0$ - the Lorentz observer sees momentum conserved.

Similarly, Lorentz charges $M_{\mu\nu}$ using τ= const line is

$$M_{\mu\nu} = \int_0^{\sigma_1} \mathcal{M}^{\tau}_{\mu\nu}(\tau,\sigma)\, d\sigma = \int (X_\mu P^{\tau}_{\nu} - X_\nu P^{\tau}_{\mu})\, d\sigma$$

In 4 dimensions there are 6 conserved charges.

M_{ij} determine the string angular momentum
by $L_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$, $L_1 = M_{23}$, $L_2 = M_{31}$, $L_3 = M_{12}$
the other components M^{0i} are associated with the boosts and position of the center of mass

$$\frac{c\,M^{0i}}{E} = t\frac{c^2 p^i}{E} - \frac{1}{E}\int d\sigma\, X^i\, c\, \mathcal{P}^{\tau 0}$$

E .. conserved energy of the string

Slope parameter α'

Alternative parameter to the tension T_0.
If one considers a rigidly rotating open string,

$$\alpha' E^2 = \frac{J}{\hbar}$$

J – angular momentum of the string

E – energy of the string

$\hbar$ – Planck – here convention

α' originally introduced in quantum theory of strings

$$[\alpha'] = \frac{1}{[E]^2}$$

Calculating for the rotating string

$$\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos\frac{\pi\sigma}{\sigma_1}\left(\cos\frac{\pi c t}{\sigma_1}, \sin\frac{\pi c t}{\sigma_1}\right)$$

one finds ... $M_{12} = \dfrac{\sigma_1^2 T_0}{2\pi c}$

Since $J = |M_{12}|$ and $\sigma_1 = E/T_0$ one finds $J = \frac{1}{2\pi T_0 c}E^2$

$$\Rightarrow \boxed{\alpha' = \frac{1}{2\pi T_0 \hbar c}\,, \quad T_0 = \frac{1}{2\pi \alpha' \hbar c}}$$

Light-cone relativistic strings, or light-cone gauge

Static gauge $X^0(\tau,\sigma) = c\tau$

now gauges in which τ is a linear combination of the string coordinates:

$$n_\mu X^\mu(\tau,\sigma) = \lambda \tau \qquad (G1)$$

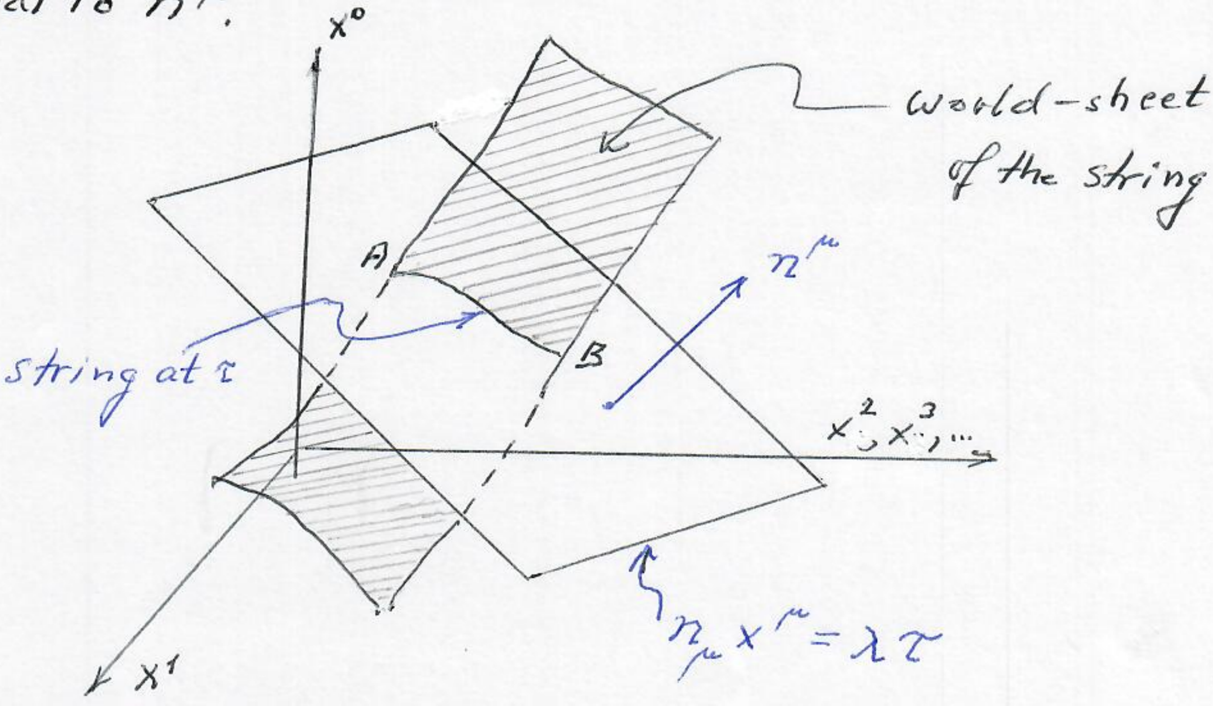
for $n_\mu = (1, 0, \dots 0)$ and $\lambda = c$ we get static gauge

another similar condition is

$$n_\mu x^\mu = \lambda \tau \qquad (G2)$$

general coordinates, not necessarily of the string

At τ (fixed) All points satisfying (G2) form a hyperplane normal to n^μ.



The gauge condition $n_\mu x^\mu = \lambda\tau$. The strings are the curves at the intersection of world-sheet with hyperplanes $\perp$ to n^μ.

The light-cone gauge

$$n^\mu = (1, 1, 0, \dots, 0) \quad \text{nullvector}$$

or $$n_\mu \equiv \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0\right)$$

So $$n_\mu X^\mu = \frac{X^0 + X^1}{\sqrt{2}} = X^+$$

denote $$X^- = \frac{X^0 - X^1}{\sqrt{2}}$$

$$X^I = (X^2, X^3, \dots) \quad \text{transversal parts}$$

From here just some „results of interest" without derivation – no requirements for exam

Equations of motion in the light cone gauge and suitable σ parameterization imply wave equations for string coordinates

$$\boxed{\ddot{X}^\mu - X''^\mu = 0}$$

but this „simplicity" is „compensated" by constrained equations

$$\boxed{(\dot{X} \pm X')^2 = 0}$$

this arises from gauge condition $n_\sigma P^\sigma = 0$ which implies

$$\dot{X} \cdot X' = 0$$

$$\dot{X}^2 + X'^2 = 0$$

Real dynamics is in the transversal modes (these will be quantized)

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty}\frac{1}{n}\,\alpha_n^I\,e^{-in\tau}\cos n\sigma$$

See more below

No dynamics in $X^{\perp}$

Postulates for the commutation relations ...

$$[X^I(\sigma), P^{\tau J}(\sigma')] = i\eta^{IJ}\delta(\sigma-\sigma')$$

$$[X^I(\sigma), X^J(\sigma')] = 0$$

$$[P^{\tau I}(\sigma), P^{\tau J}(\sigma')] = 0 \quad \leftarrow \text{Schrödinger picture (operators)}$$

Similarly for Heisenberg picture when $X^I(\tau,\sigma)$, $P^{\tau J}(\tau,\sigma)$

But there are also conserved currents due to Lorentz symmetry $\mathcal{M}^{\alpha}_{\mu\nu} = X_\mu P^\alpha_\nu - X_\nu P^\alpha_\mu$ satisfying

$$\frac{\partial \mathcal{M}^\tau_{\mu\nu}}{\partial\tau} + \frac{\partial \mathcal{M}^\sigma_{\mu\nu}}{\partial\sigma} = 0 \qquad \text{see page } \lfloor S29, 30)$$

Total charges $M_{\mu\nu} = \int(\mathcal{M}^\tau_{\mu\nu}\,d\sigma - \mathcal{M}^\sigma_{\mu\nu}\,d\tau)$

When calculating Lorentz charges at given time $\tau = \text{const.}$

$$\Rightarrow \quad M_{\mu\nu} = \int \mathcal{M}^\tau_{\mu\nu}(\tau,\sigma)\,d\sigma$$

Raising indices and substituting for $\mathcal{M}^{\tau}_{\mu\nu}$ calculations show that

$$M^{\mu\nu} = \underbrace{x_0^\mu p^\nu - x_0^\nu p^\mu}_{\text{const oper.}} - i \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu\right)$$

The string classical modes α_n^I become quantum operators when strings are quantized.

α_n^I — annihilation operators

α_{-n}^I $(n \geq 1)$ creation operators

Commutation relations nontrivial.

For example

$$[\alpha_m^I, \alpha_n^J] = m\,\eta^{IJ}\,\delta_{m+n,0}$$

so α_0^I commutes with all other oscillators

Charges in quantum regime generate Lorentz transformations

for particle $[L_x, L_y] = i L_z$

here

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho} M^{\nu\sigma} - i\eta^{\nu\rho} M^{\mu\sigma} + i\eta^{\mu\sigma} M^{\rho\nu} - i\eta^{\nu\sigma} M^{\rho\mu}$$

in the light-cone gauge it turns out the following commutation relations follow

$[M^{-I}, M^{-J}] = 0$ remember we introduced

$x^+ = \frac{x^0 + x^1}{\sqrt{2}}$, $x^- = \frac{x^0 - x^1}{\sqrt{2}}$, $x^I, x^J \ldots$ "transversal"

so e.g. M^{-I} — I denotes transversal part, $-$ denotes coordinate x^-

also $[M^{+-}, M^{+I}] = i M^{+I}$ $\qquad M^{+-} \equiv M^{10}$ generates boosts along x^1

In addition one should have the following commutator

$[M^{-I}, M^{-J}] = 0$ vanishing

Long calculations lead to the following result;

$[M^{-I}, M^{-J}] =$

α's from the decompositn of $X^I(\tau,\sigma)$ into cos no

see S 33

$$= \underbrace{\frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} \left(\alpha^I_{-m} \alpha^J_m - \alpha^J_{-m} \alpha^I_m \right) \times \left\{ m \left[1 - \frac{1}{24}(D-2) \right] + \frac{1}{m} \left[\frac{1}{24}(D-2) + a \right] \right\}}$$

this will vanish if

$1 - \frac{1}{24}(D-2) = 0$

$\Rightarrow D = 26$

$= 0$

$\Rightarrow$

$a = -\frac{1}{24}(D-2) = -1$

B. Zwiebach p. 262 A First Course in String Theory 2nd edition Cambridge Univ. Press 2009

There is much at stake in this calculation. It is in fact, one of the most important calculations in string theory. Our Lorentz charge has two undetermined parameters: the dimension D of spacetime, implicit in the sums over transverse directions, and the constant a affecting the mass of the particles. The calculation is long and uses many of our previously derived results, including the Virasoro commutation relations. We will not attempt to do it here, but the result is

$$\left[M^{-I}, M^{-J}\right] = -\frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} \left(\alpha^I_{-m}\alpha^J_m - \alpha^J_{-m}\alpha^I_m\right) \times \left\{m\left[1 - \frac{1}{24}(D-2)\right] + \frac{1}{m}\left[\frac{1}{24}(D-2) + a\right]\right\}. \tag{12.152}$$

The right-hand side is a sum of terms, each of which contains the operator $(\alpha^I_{-m}\alpha^J_m - \alpha^J_{-m}\alpha^I_m)$ for a different value of m. Such terms cannot cancel each other, so the commutator above vanishes if and only if the coefficient in large braces vanishes for all positive integers m:

$$m\left[1 - \frac{1}{24}(D-2)\right] + \frac{1}{m}\left[\frac{1}{24}(D-2) + a\right] = 0, \quad \forall m \in \mathbb{Z}^+. \tag{12.153}$$

It suffices to examine this condition for $m = 1$ and $m = 2$ to conclude that each of the terms in brackets must simply vanish. We therefore have

$$1 - \frac{1}{24}(D-2) = 0 \quad \text{and} \quad \frac{1}{24}(D-2) + a = 0. \tag{12.154}$$

The first equation fixes the dimension of spacetime:

Summary:

The condition of Lorentz invariance of quantum string theory fixes the dimension of spacetime.

In the superstring theory which includes also Fermions a similar procedure fixes the dimensionality of spacetime to $D = 10$!

Does the string theory really predict D ?

?