O. Semerák: Lecture Notes

H21

The stationarity of action is equivalent to the **Hamilton canonical equations**. Indeed,

$$\delta S := \delta \int_\Omega \mathcal{L}\sqrt{-g}\,\mathrm{d}^4x \xrightarrow{1+3} \delta \int_{t_1}^{t_2}\int_{\Sigma(t)} \mathcal{L}\sqrt{-g}\,\mathrm{d}^3x\,\mathrm{d}t = \delta \int_{t_1}^{t_2}\int_{\Sigma(t)} (\Pi\cdot\dot{q} - \mathcal{H})\,\mathrm{d}^3x\,\mathrm{d}t =$$

$$= \int_{t_1}^{t_2}\int_{\Sigma(t)} \left(\delta\Pi\cdot\dot{q} + \Pi\cdot\delta\dot{q} - \frac{\partial\mathcal{H}}{\partial q}\cdot\delta q - \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi - \frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q_{,i}\right)\mathrm{d}^3x\,\mathrm{d}t\,.$$

We covered this on p. H11 (with slightly different notation)

The second term we "per-partes", already dropping the boundary term as usual (better argument is that δq is assumed to vanish at the marginal times t_1 and t_2),

$$\int_{t_1}^{t_2}\int_{\Sigma(t)} \Pi\cdot\delta\dot{q}\,\mathrm{d}^3x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\Sigma(t)} \dot{\Pi}\cdot\delta q\,\mathrm{d}^3x\,\mathrm{d}t\,.$$

Similarly we process the last term (time integration is not important in it),

$$-\int_{\Sigma(t)} \frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q_{,i}\,\mathrm{d}^3x = -\int_{\Sigma(t)} \left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q\right)_{,i}\mathrm{d}^3x + \int_{\Sigma(t)} \left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}\cdot\delta q\,\mathrm{d}^3x = \int_{\Sigma(t)} \left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}\cdot\delta q\,\mathrm{d}^3x\,;$$

here the first term has dropped out, since by Gauss law it can be rewritten as an integral from $\frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q$ over the boundary $\partial\Sigma(t)$ (which may possibly lie at infinity), where we assume $\delta q = 0$.

To summarize,

$$\delta S = \int_{t_1}^{t_2}\int_{\Sigma(t)} \left[\delta\Pi\cdot\dot{q} - \dot{\Pi}\cdot\delta q - \frac{\partial\mathcal{H}}{\partial q}\cdot\delta q - \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi + \left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}\cdot\delta q\right]\mathrm{d}^3x\,\mathrm{d}t\,,$$

from where we see that

$$\delta S = 0 \quad\Longleftrightarrow\quad \dot{q} := \pounds_t q = \frac{\partial\mathcal{H}}{\partial\Pi}\,,\qquad \dot{\Pi} := \pounds_t\Pi = -\frac{\partial\mathcal{H}}{\partial q} + \left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}\,. \tag{26.3}$$

26.1 Klein-Gordon field and EM field: a warm up

Before embarking on the Einstein equations, let us illustrate the Hamiltonian approach on the Klein-Gordon scalar field and on the electromagnetic field. In the latter case, we will meet the important circumstance which later will also occur in the gravitation problem – thanks to a gauge freedom in the field variables, some of the field equations become constraints. recall lecture at the beginning of semester

Suppose, for simplicity, that we deal with a situation where $N^i = 0$, so, according to (25.7), $g^{\mu\nu} = \mathrm{diag}(-N^{-2}, h^{ik})$. The Lagrangian density* of the Klein-Gordon scalar field ($q \equiv \psi$) then reads

$$\mathcal{L} = -\frac{1}{2}\left(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} + m^2\psi^2\right) = -\frac{1}{2}\left(g^{tt}\dot{\psi}^2 + h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right).$$

From it, we have

$$\Pi := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\psi}} = -\sqrt{-g}\,g^{tt}\dot{\psi} \qquad\Longrightarrow\qquad \dot{\psi} = -\frac{\Pi}{\sqrt{-g}\,g^{tt}}$$

this is really geometrical density which we usually denoted as $\mathscr{L}$

*) here the density means the integrand in the action $\int \cdots \mathrm{d}^3x$

H22

and the Hamiltonian density this is now really geometrical density, Π is density

$$\mathcal{H} := \Pi\dot\psi - \sqrt{-g}\,\mathcal{L} = -\frac{\Pi^2}{\sqrt{-g}\,g^{tt}} + \frac{\sqrt{-g}}{2}\left(g^{tt}\frac{\Pi^2}{-g(g^{tt})^2} + h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right) =$$
$$= -\frac{\Pi^2}{2\sqrt{-g}\,g^{tt}} + \frac{\sqrt{-g}}{2}\left(h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right),$$

from which we finally find evolution equations

$$\dot\psi = \frac{\partial\mathcal{H}}{\partial\Pi} = -\frac{\Pi}{\sqrt{-g}\,g^{tt}} = \frac{\Pi N^2}{\sqrt{-g}},$$
$$\dot\Pi = -\frac{\partial\mathcal{H}}{\partial\psi} + \left(\frac{\partial\mathcal{H}}{\partial\psi_{,j}}\right)_{,j} = -\sqrt{-g}\,m^2\psi + \left(\sqrt{-g}\,h^{jk}\psi_{,k}\right)_{,j}.$$

This result really leads to the Klein-Gordon equation,

$$\Box\psi \equiv g^{\mu\nu}\psi_{;\mu\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,g^{\mu\nu}\psi_{,\mu}\right)_{,\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,g^{tt}\psi_{,t}\right)_{,t} + \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,h^{jk}\psi_{,j}\right)_{,k} =$$
$$= -\frac{\dot\Pi}{\sqrt{-g}} + \frac{1}{\sqrt{-g}}\left(\dot\Pi + \sqrt{-g}\,m^2\psi\right) = m^2\psi.$$

EM field

Second, let us test the Hamiltonian approach on a free EM field in the Minkowski space-time ($g_{\mu\nu}=\eta_{\mu\nu}$, $\sqrt{-g}=1$). Suppose the configuration variable is the four-potential in this case, $q\equiv A_\mu$. We split it to time and spatial components with respect to Σ_t, i.e. to the "scalar" and "vector" potentials

$$\phi := -A_\mu n^\mu, \qquad \vec{A} := A_\mu h^\mu_\alpha,$$ $h^\mu_\alpha = \delta^\mu_\alpha + n^\mu n_\alpha$ n^μ... normal to Σ_t

and write down the Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} = \frac{1}{8\pi}(E^2-B^2) = \frac{1}{8\pi}\underbrace{(\vec\nabla\phi+\dot{\vec{A}})}_{-\vec{E}}\cdot(\vec\nabla\phi+\dot{\vec{A}}) - \frac{1}{8\pi}\underbrace{(\vec\nabla\times\vec{A})}_{\vec{B}}\cdot(\vec\nabla\times\vec{A}),$$

where

$$E_\mu \equiv F_{\mu\nu}n^\nu, \quad B_\mu \equiv -{}^*F_{\mu\nu}n^\nu \qquad (\iff F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu + \epsilon_{\mu\nu\rho\sigma}n^\rho B^\sigma)$$

are the electric and magnetic fields defined with respect to Σ_t. For quantities "living on Σ_t" we have employed the three-vector notation, in particular

$$\vec{E} := -\vec\nabla\phi - \dot{\vec{A}}, \qquad \vec{B} := \vec\nabla\times\vec{A}.$$

The momenta conjugated to the scalar and vector potentials come out

$$\Pi_t := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot\phi} = 0, \qquad \vec\Pi := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\vec{A}}} = \frac{1}{4\pi}(\vec\nabla\phi+\dot{\vec{A}}) = -\frac{\vec{E}}{4\pi}.$$

Here comes *the issue*: the first of these relations cannot be inverted, namely, it is not possible to express from it $\dot\phi$, so it is also not possible to find the Hamiltonian density $\mathcal{H} = \Pi_t\dot\phi + \vec\Pi\cdot\dot{\vec{A}} - \mathcal{L}$. in its standard form
This "accident", related to the gauge freedom of the four-potential, is being remedied in a simple way: if Π_t vanishes identically, it is clearly not appropriate to consider ϕ a *dynamical* variable. If dropping

4/23

ϕ and only leaving $\vec{A}$ as configuration variables, we may continue: we express $\dot{\vec{A}} = 4\pi\vec{\Pi} - \vec{\nabla}\phi$ and submit it to the "restricted" Hamiltonian-density prescription,

$$\mathcal{H} = \vec{\Pi}\cdot\dot{\vec{A}} - \mathcal{L} = \vec{\Pi}\cdot(4\pi\vec{\Pi} - \vec{\nabla}\phi) - 2\pi\vec{\Pi}\cdot\vec{\Pi} + \frac{1}{8\pi}B^2 = 2\pi\Pi^2 - \vec{\Pi}\cdot\vec{\nabla}\phi + \frac{B^2}{8\pi}\,.$$

$$\mathcal{H} = 2\pi\,\vec{\Pi}^2 + \frac{\vec{B}^2}{8\pi} + \phi\,\vec{\nabla}\vec{\Pi} \underbrace{- \vec{\nabla}(\phi\vec{\Pi})}_{\text{contributes only to the surface term}}$$

The Hamilton equations yield

$$\dot{\vec{A}} = \frac{\partial\mathcal{H}}{\partial\vec{\Pi}} = 4\pi\vec{\Pi} - \vec{\nabla}\phi = -\vec{E} - \vec{\nabla}\phi\,,$$

$$\dot{\vec{\Pi}}\left(= -\frac{\dot{\vec{E}}}{4\pi}\right) = -\cancel{\frac{\partial\mathcal{H}}{\partial\vec{A}}} + \left(\frac{\partial\mathcal{H}}{\partial\vec{A}_{,j}}\right)_{,j} = \frac{1}{8\pi}\left(\frac{\partial B^2}{\partial\vec{A}_{,j}}\right)_{,j} = -\frac{\vec{\nabla}\times\vec{B}}{4\pi}\,.$$

The last equality of the latter equation can best be computed "in components":

$$B^2 \equiv B_kB^k = \epsilon_{klm}A^{m,l}\epsilon^{kno}A_{o,n} = (\delta^n_l\delta^o_m - \delta^n_m\delta^o_l)A^{m,l}A_{o,n} = A^{m,l}(A_{m,l} - A_{l,m})\,,$$

so

$$\frac{\partial B^2}{\partial A_{i,j}} = \frac{\partial}{\partial A_{i,j}}\left[A^{m,l}(A_{m,l} - A_{l,m})\right] = \delta^{mi}\delta^{lj}(A_{m,l} - A_{l,m}) + A^{m,l}(\delta^i_m\delta^j_l - \delta^i_l\delta^j_m) =$$
$$= 2(A^{i,j} - A^{j,i}) \equiv 2F^{ji}$$
$$\Longrightarrow\quad \left(\frac{\partial B^2}{\partial A_{i,j}}\right)_{,j} = 2F^{ji}{}_{,j} = 2\,\epsilon^{jik}B_{k,j} \equiv -2(\vec{\nabla}\times\vec{B})^i\,.$$

ϕ is the the Lagrange multiplier

$$\frac{\delta H}{\delta\varphi} = 0 \Rightarrow \vec{\nabla}\vec{E} = 0$$

Constraint

The first equation thus reproduces the expression of $\vec{E}$ in terms of the potentials, thanks to which (plus $\vec{B} = \vec{\nabla}\times\vec{A}$) holds the second set of Maxwell equations, $\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\dot{\vec{A}} = -\dot{\vec{B}}$ and $\vec{\nabla}\cdot\vec{B} = 0$. The second Hamilton equation yields the Maxwell equation $\vec{\nabla}\times\vec{B} = \dot{\vec{E}}$. Finally, for consistence, it is necessary to add the equation for the derivative by non-dynamical variable ϕ,

$$0 = \dot{\Pi}_t = -\cancel{\frac{\partial\mathcal{H}}{\partial\phi}} + \left(\frac{\partial\mathcal{H}}{\partial\phi_{,j}}\right)_{,j} = -\left[\frac{\partial(\vec{\Pi}\cdot\vec{\nabla}\phi)}{\partial\phi_{,j}}\right]_{,j} = -\left(\Pi^j\right)_{,j} = -\vec{\nabla}\cdot\vec{\Pi} = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{E}\,.$$

This Maxwell equation represents a **constraint**. (Sure, it cannot be an evolution equation, because it does not contain the time derivative. The same also applies to the similar equation $\vec{\nabla}\cdot\vec{B} = 0$.)

The EM field thus exemplifies the **Hamiltonian system with a constraint**. This more delicate type of problem occurs when the configuration variables possess a gauge freedom. As a consequence, some of them are *not dynamical*, effectively playing the role of Lagrangian multipliers which enforce the fulfilment of certain **constraints**.

26.2 Gravitational field

We will start from the Lagrangian density $\mathcal{L}_g = R - 2\Lambda$, rewriting it to the "3+1" form. For simplicity, we will not take into account "surface" terms – those given by divergence of some vector field (here we have covariant divergence in mind), because such can be expressed, thanks to the Gauss law, as integrals from the flows of the respective vector fields over the boundary of the integration region. Let us emphasize, however, that we are speaking now about the *Lagrangian itself* rather than about