$$\rightarrow = L\,\delta t + \int_{x',t'}^{x+\Delta x,t} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x + \frac{\partial L}{\partial x}\delta x \right] dt$$

$$= L\,\delta t + \frac{\partial L}{\partial \dot{x}}\Delta x + \int_{x',t'}^{x+\Delta x,t} \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)}_{=0 \text{ because trajectories are "real" extremizing action}} \delta x \, dt$$

substituting for Δx the general variation at the final point — see (•) on preceeding page, $\Delta x = \delta x - \dot{x}\,\delta t$

we obtain

$$\boxed{\delta S = \underbrace{\frac{\partial L}{\partial \dot{x}}}\,\delta x - \underbrace{\left[\dot{x}\frac{\partial L}{\partial \dot{x}} - L\right]}\,\delta t}$$

$\Rightarrow$ rate of change of dynamic phase with position $=$ momentum $p = \dfrac{\partial L}{\partial \dot{x}}$

(free particle $p = m\dot{x}$)

−(rate of change of dynamic phase with time) = energy $E = \dot{x}\dfrac{\partial L}{\partial \dot{x}} - L$

(viz free particle $L = \frac{1}{2}m\dot{x}^2 \Rightarrow E = \frac{1}{2}m\dot{x}^2$)

solving for $\dot{x} \Rightarrow$

$$E = H(p, x, t)$$

and $$-\frac{\partial S}{\partial t} = H\left(\frac{\partial S}{\partial x}, x, t\right)$$

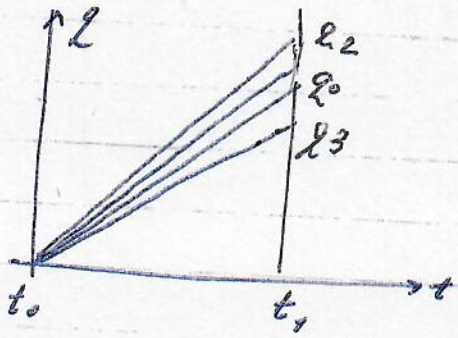
This is analogous to similar procedure in GR:

$$\delta S = \int \pi^{ij}\,\delta g_{ij}\,d^3x\,, \quad \pi^{ij} = \frac{\delta S}{\delta g_{ij}} \quad \ldots \text{ see later}$$

Simple examples

(I) Uniform motion

Change of final point



$$L = \frac{1}{2}\dot{q}^2, \quad q = q_0 + v(t-t_0), \quad v = \frac{q_1 - q_0}{t_1 - t_0}$$

$$S = \int_{t_0, q_0}^{t_1, q_1} \frac{1}{2} v^2 dt = \frac{1}{2} v^2 (t_1 - t_0) = \frac{1}{2}\left(\frac{q_1 - q_0}{t_1 - t_0}\right)^2 (t_1 - t_0)$$

for real motion $v = \text{const}$

$$\Longrightarrow \quad \boxed{S(q_0, t_0; q_1, t_1) = \frac{1}{2}\frac{(q_1 - q_0)^2}{(t_1 - t_0)}}$$

And, indeed,

$$\frac{\partial S}{\partial q_1} = \frac{q_1 - q_0}{t_1 - t_0} = v = p \checkmark$$

$$-\frac{\partial S}{\partial t_1} = + \frac{1}{2}\underbrace{\frac{(q_1 - q_0)^2}{(t_1 - t_0)^2}}_{= v^2} = E \checkmark$$