Hamiltonian – Canonical formalism (classical, STR, GTR)

PRELIMINARIES & (GENERALITIES / GENERALIZATIONS)

In standard classical mechanics, momentum defined as

$$p = \frac{\partial L}{\partial \dot{x}}$$

But in the Hamilton-Jacobi theory one can define

$$p = \frac{\partial S(x,t)}{\partial x}, \quad \text{and} \quad \overset{\text{energy}}{E} = -\frac{\partial S}{\partial t}$$

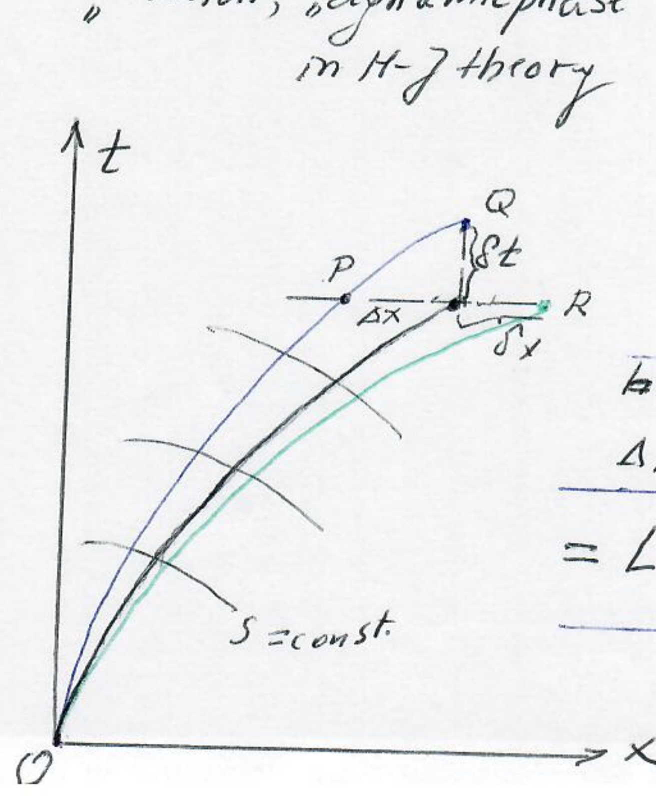
where $S(x,t) = I_{extremum}(x,t) = [\text{extremum of}] \int_{x',t'}^{x,t} L(x,\dot{x},t)\,dt$

changing action

fixed initial point

„action", „dynamic phase" in H-J theory

with respect to x and t



so trajectories are dynamical but with different end-points

$$\delta S = L\,\delta t + \int_{x',t'}^{x+\Delta x,t} \delta L\,dt =$$

here $\Delta x = \delta x$ for „green" trajectory QR

$\Delta x = -\dot{x}\,\delta t$ for „blue" trajectory QP (*)

$$= L\,\delta t + \int_{x',t'}^{x+\Delta x,t} \left(\frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x\right) dt \Rightarrow$$

(*) so general variation of the final point $\Rightarrow \Delta x = \delta x - \dot{x}\,\delta t \quad (\cdot)$

$$\rightarrow = L\,\delta t + \int_{x',t'}^{x+\Delta x,\,t} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x + \frac{\partial L}{\partial x}\delta x \right] dt$$

$$= L\,\delta t + \frac{\partial L}{\partial \dot{x}}\Delta x + \int_{x',t'} \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)}_{\substack{=0 \text{ because trajectories} \\ \text{are "real" extremizing action}}} \delta x\, dt$$

substituting for Δx the general variation of the final point — see (•) on preceeding page, $\Delta x = \delta x - \dot{x}\delta t$

we obtain

$$\boxed{\delta S = \underbrace{\frac{\partial L}{\partial \dot{x}}}\,\delta x - \underbrace{\left[\dot{x}\frac{\partial L}{\partial \dot{x}} - L\right]}\delta t}$$

$\Rightarrow$ rate of change of dynamic phase with position $=$ momentum $p = \frac{\partial L}{\partial \dot{x}}$

(free particle $p = m\dot{x}$)

−(rate of change of dynamic phase with time) = energy $E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L$

(viz free particle $L = \frac{1}{2}m\dot{x}^2 \Rightarrow E = \frac{1}{2}m\dot{x}^2$)

solving for $\dot{x}$ $\Rightarrow$

$$E = H(p, x, t)$$

and $$-\frac{\partial S}{\partial t} = H\left(\frac{\partial S}{\partial x}, x, t\right)$$

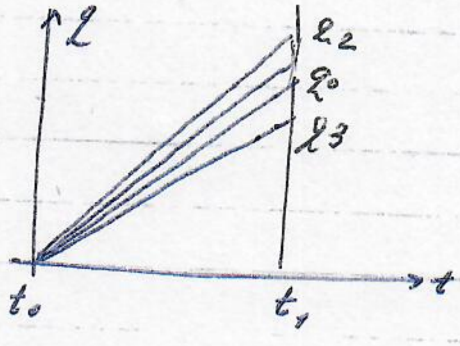
This is analogous to similar procedure in GR:

$\delta S = \int \pi^{ij}\delta g_{ij}\, d^3x\,,\quad \pi^{ij} = \frac{\delta S}{\delta g_{ij}}$ ... see later

Simple examples

(I) Uniform motion

Change of final point



$$L = \frac{1}{2}\dot{q}^2, \quad q = q_0 + v(t-t_0), \quad v = \frac{q_1 - q_0}{t_1 - t_0}$$

$$S = \int_{t_0,q_0}^{t_1,q_1} \frac{1}{2} v^2 dt = \frac{1}{2} v^2 (t_1 - t_0) = \frac{1}{2}\left(\frac{q_1 - q_0}{t_1 - t_0}\right)^2 (t_1 - t_0)$$

for real motion $v = const$

$$\Longrightarrow \quad \boxed{S(q_0, t_0; q_1, t_1) = \frac{1}{2}\,\frac{(q_1 - q_0)^2}{(t_1 - t_0)}}$$

And, indeed,

$$\frac{\partial S}{\partial q_1} = \frac{q_1 - q_0}{t_1 - t_0} = v = p \checkmark$$

$$-\frac{\partial S}{\partial t_1} = +\frac{1}{2}\,\frac{(q_1 - q_0)^2}{(t_1 - t_0)} = E \checkmark$$

II. 1-dimensional free-fall in gravitational field

particle with $m=1$

$$T=\frac{1}{2}\dot{q}^2, \quad V=gq, \quad L=\frac{1}{2}\dot{q}^2-gq, \quad p=\frac{\partial L}{\partial \dot{q}}=\dot{q}$$

$$H=p\dot{q}-L=p^2-\frac{p^2}{2}+gq=\frac{p^2}{2}+gq$$

H.-J. equation $H\left(\frac{\partial S}{\partial q}, q\right)+\frac{\partial S}{\partial t}=0$

H is independent of t: $S=-Et+W(E,q)$, $\quad \frac{\partial S}{\partial q}=\frac{\partial W}{\partial q}$

$$\frac{1}{2}\left(\frac{dW}{dq}\right)^2+gq=E \Rightarrow \frac{dW}{dq}=-\sqrt{2(E-gq)}$$

choose $-$, so $W>0$

$$\Rightarrow W=\frac{1}{3g}\left[2(E-gq)\right]^{3/2}$$

$$\Rightarrow \boxed{S=-Et+\overbrace{\frac{1}{3g}\left[2(E-gq)\right]^{3/2}}^{W}} \quad (*)$$

motion:

$$\tau\,(=\text{konst.})=-\frac{\partial S}{\partial E}=t-\frac{\partial W}{\partial E}$$

$$\Rightarrow t-\tau=\frac{-1}{g}\sqrt{2(E-gq)}$$

$$\Rightarrow (t-\tau)^2=\frac{2E}{g^2}-\frac{2q}{g}$$

$$\Rightarrow \boxed{q=\frac{E}{g}-\frac{1}{2}g(t-\tau)^2}$$

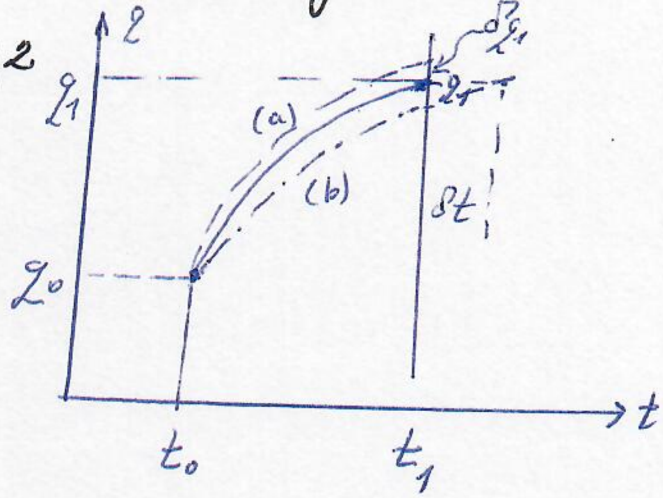
we shall use this expression

It is common to determine E, τ from initial conditions $q_0, \dot{q}_0$ at time t_0 see the following

Now consider 2 events, (q_0, t_0) and (q_1, t_1) connected by a real physical trajectory (worldline):

(1) $$q_0 = \frac{E}{g} - \frac{1}{2} g (t_0 - \tau)^2$$

(2) $$q_1 = \frac{E}{g} - \frac{1}{2} g (t_1 - \tau)^2$$



Clearly, to consider real motions, which all start at (q_0, t_0) but end generally at $(q_1 + \delta q_1, t_1 + \delta t_1)$ means to consider different energies E

(a) will come at given t_1 to different q : $q_1 + \delta q_1$

(b) will come to the same q_1 later, in time $t_1 + \delta t$

Given (q_0, t_0) and (q_1, t_1) determine uniquely the motion which connects them –

– „thick sandwich theorem" in mechanics

From (1) and (2) above we can express E, τ in terms (q_0, t_0), (q_1, t_1)

$$2g(q_1 - q_0) = -g^2 (t_1 - \tau)^2 + g^2 (t_0 - \tau)^2; \quad \frac{2}{g}(q_1 - q_0) =$$

$$= t_0^2 - t_1^2 + 2\tau(t_1 - t_0)$$

$$\Rightarrow \boxed{\tau = \frac{q_1 - q_0}{g(t_1 - t_0)} + \frac{t_1 + t_0}{2}}$$

$$\frac{1}{2} g (q_1 + q_0) = E - \frac{1}{4} g^2 \left[(t_1 - \tau)^2 + (t_0 - \tau)^2 \right]$$

$$\Rightarrow E = \frac{1}{2} g (q_1 + q_0) + \frac{1}{4} g^2 \underbrace{\left[\left\{ \underset{A}{\frac{t_1 - t_0}{2}} - \underset{B}{\frac{q_1 - q_0}{g(t_1 - t_0)}} \right\}^2 + \left\{ \underset{-A}{\frac{t_0 - t_1}{2}} - \underset{B}{\frac{q_1 - q_0}{g(t_1 - t_0)}} \right\}^2 \right]}$$

$$(A - B)^2 + (-A - B)^2 = A^2 - 2AB + B^2 + A^2 + 2AB + B^2 = 2A^2 + 2B^2$$

$$E = \frac{1}{2}g(q_1+q_0) + \frac{1}{4}g^2\left\{2\left(\frac{t_1-t_0}{2}\right)^2 + 2\left(\frac{q_1-q_0}{g(t_1-t_0)}\right)^2\right\}$$

$$= \frac{1}{2}g(q_1+q_0) + \frac{1}{2}g^2\left\{\left(\frac{t_1-t_0}{2}\right)^2 + \left(\frac{q_1-q_0}{g(t_1-t_0)}\right)^2\right\} \Rightarrow$$

$$(*) \quad \boxed{E = \frac{1}{2}g(q_1+q_0) + \frac{1}{2}\left[\frac{g(t_1-t_0)}{2}\right]^2 + \underbrace{\frac{1}{2}\left[\frac{q_1-q_0}{t_1-t_0}\right]^2}}$$

corresponds to a free, uniformly moving particle (via put $g \to 0$ in E)

| Calculating action as function of the final points |: $\int_{t_0,q_0}^{t_1,q_1} L\,dt = \int\left(\frac{1}{2}\dot{q}^2 - gq\right)dt =$

= we do it for real motion, i.e. $q = \frac{E}{g} - \frac{1}{2}g(t-\tau)^2$

$\dot{q} = -g(t-\tau)$

$$= \int_{t_0,q_0}^{t_1,q_1}\left[\frac{1}{2}g^2(t-\tau)^2 - E + \frac{1}{2}g(t-\tau)^2\right]dt =$$

$$= \int_{t_0,q_0}^{t_1,q_1}\left[g^2(t-\tau)^2 - E\right]dt = \left[\frac{1}{3}g^2(t-\tau)^3 - Et\right]_{t_0\ (q_0)}^{t_1\ (q_1)}$$

$$= \frac{1}{3}g^2\left[(t_1-\tau)^3 - (t_0-\tau)^3\right] - E(t_1-t_0),$$ where for τ, E it is necessary to substitute results in terms of $(q_0, t_0)(q_1, t_1)$

$$(A-B)^3 - (-A-B)^3 = (A-B)^3 + (A+B)^3 =$$
$$= A^3 + 3AB^2 - \cancel{3A^2B} - \cancel{B^3} + A^3 + 3AB^2 + \cancel{3A^2B} + \cancel{B^3}$$
$$= 2A^3 + 6AB^2$$

$\boxed{H7}$

So that

$$I_{extr.}(q_0, t_0; q_1, t_1) =$$

$$= \frac{1}{3} g^2 \left[\Big\{ \underbrace{\frac{t_1 - t_0}{2}}_{A} - \underbrace{\frac{q_1 - q_0}{g(t_1 - t_0)}}_{B} \Big\}^3 - \Big\{ \underbrace{\frac{t_0 - t_1}{2}}_{-A} - \underbrace{\frac{q_1 - q_0}{g(t_1 - t_0)}}_{B} \Big\}^3 \right]$$

$$- \left[\frac{1}{2} g (q_1 + q_0) + \frac{1}{2} \Big\{ g \frac{(t_1 - t_0)}{2} \Big\}^2 + \frac{1}{2} \Big\{ \frac{q_1 - q_0}{t_1 - t_0} \Big\}^2 \right] (t_1 - t_0)$$

$$\Rightarrow I_{extr.} = \frac{1}{3} g^2 \left[2 \left(\frac{t_1 - t_0}{2} \right)^3 + 6 \frac{(t_1 - t_0)}{2} \left(\frac{q_1 - q_0}{g(t_1 - t_0)} \right)^2 \right]$$

$$- \left[\frac{1}{2} g (q_1 + q_0) + \frac{1}{2} \Big\{ \frac{g(t_1 - t_0)}{2} \Big\}^2 + \frac{1}{2} \Big\{ \frac{q_1 - q_0}{t_1 - t_0} \Big\}^2 \right] (t_1 - t_0)$$

$$= \frac{g^2}{12} (t_1 - t_0)^3 + \frac{(q_1 - q_0)^2}{(t_1 - t_0)} - \frac{1}{2} g (q_1 + q_0)(t_1 - t_0)$$

$$- \frac{g^2}{8} (t_1 - t_0)^3 - \frac{1}{2} \frac{(q_1 - q_0)^2}{(t_1 - t_0)}$$

Action:

$$\Rightarrow \; = S(q_0, t_0; q_1, t_1)$$

$$\boxed{I_{extr.}(q_0, t_0; q_1, t_1) = \frac{1}{2} \frac{(q_1 - q_0)^2}{(t_1 - t_0)} - \frac{g^2}{24} (t_1 - t_0)^3 - \frac{1}{2} g (q_1 + q_0)(t_1 - t_0)}$$

the term corresponding to the freely moving particle remaining after $g \to 0$

$$\frac{\partial S}{\partial q_1} = \frac{(q_1 - q_0)}{(t_1 - t_0)} - \frac{1}{2} g (t_1 - t_0)$$

$$p_1 = \dot{q}_1 = -g(t_1 - \tau) = -g \left[\frac{t_1 - t_0}{2} - \frac{q_1 - q_0}{g(t_1 - t_0)} \right] =$$

$$= \frac{q_1 - q_0}{t_1 - t_0} - \frac{1}{2} g (t_1 - t_0)$$

So, indeed: $\boxed{p = \frac{\partial S}{\partial q},}$

when we think of q_0, t_0 as fixed and consider action calculated over real trajectories as the function of the final point (q_1, t_1)

Similarly,

$$\frac{\partial S}{\partial t_1} = -\frac{1}{2}\frac{(q_1-q_0)^2}{(t_1-t_0)^2} - \frac{g^2}{8}(t_1-t_0)^2 - \frac{1}{2}g(q_1+q_0)$$

this follows from (*), p. H_6

$$-E = -\frac{1}{2}g(q_1+q_0) - \frac{1}{2}\frac{g^2(t_1-t_0)^2}{4} - \frac{1}{2}\left[\frac{q_1-q_0}{t_1-t_0}\right]$$

So, it is also true that

$$\boxed{E = -\frac{\partial S}{\partial t}}$$

Next: on p. H_4, Eq. (*) we have

$$S = -Et + \frac{1}{3g}(2E-2gq)^{3/2}$$

on p. H_6 we have (the 3rd line from bottom)

$$\int_{t_0}^{t_1} L\,dt = \left[-Et + \frac{1}{3}g^2(t-\tau)^3\right]_{t_0}^{t_1} \qquad (\cdot)$$

but also $q = \frac{E}{g} - \frac{1}{2}g(t-\tau)^2 \Rightarrow 2E - 2gq = g^2(t-\tau)^2$

$\Rightarrow (t-\tau) = \frac{1}{g}(2E-2gq)^{\frac{1}{2}}$

after subst. into $(\cdot)$: $\int_{t_0}^{t_1} \cancel{L\,dt =} \left[-Et + \frac{1}{3}g^2\frac{1}{g^3}(2E-2gq)^{3/2}\right]_{t_0}^{t_1}$

$$\int^{t} L\,dt = \left[-Et + \frac{1}{3g}(2E-2gq)^{3/2}\right] + \text{const (inside is } t_0)$$

(indeed $\int_{t_0}^{t} L\,dt = S(t)$)

General strategy

1) Introduce the configuration space V_q

q ... instantenous values of some in general tensorial field ψ given on a spacelike hypersurface Σ_t

∞-dimensional — „tangent bundle"

2) space of momenta – dual to the conf-space

V_q^* „cotangent bundle"

intuitively if tangent vectors in V_q are tensors (k,l) δq^k_l then π_k^l maps δq^k_l into $\mathbb{R}$ by means of

$$\delta q \to \int_{\Sigma_t} \pi \, \delta q \, d^3x = \int_{\Sigma_t} \pi^l_k \, \delta q^k_l \, d^3x$$

3) Hamiltonian $H[q,\pi]$ ... functional on Σ_t

$$H = \int_{\Sigma_t} \mathcal{H} \, d^3x$$

$\mathcal{H}$ ... Hamiltonian density local function of q, π and their spatial derivatives

Hamilton equations

$$\boxed{\begin{aligned} &(*) \quad \dot{q} \overset{df}{\equiv} \pounds_t q = \frac{\delta H}{\delta \pi} \\ &(**) \quad \dot{\pi} \overset{df}{\equiv} \pounds_t \pi = -\frac{\delta H}{\delta q} \end{aligned}}$$

$$\frac{\delta H}{\delta q} = \frac{\partial \mathcal{H}}{\partial q} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{H}}{\partial q_{,i}}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} \longrightarrow \dot{q} = \dot{q}(q, \pi), \quad \boxed{\mathcal{H}(q,\pi) = \pi \dot{q} - \mathcal{L}} \quad (\mathcal{H})$$

Thm. If the Hamiltonian is so chosen, the Hamilton equations are equivalent to eqs. of motion from action and Lagrange eqs.

Note:

Another notation (in fact more useful for the 2nd variation (3rd.., etc)

usually one writes variation of q as

$q \to \tilde{q} = q + \delta q$ and one wants the action to be stationary under this change, $\delta S = 0$

But we can consider q to be parameterized by a small parameter e, $q(e;x)$

and denote $\overset{(\text{unvaried})}{q}$ as $q(0;x)$, $\tilde{q} \Leftrightarrow q(e;x)$

$$q(e;x) = q(0;x) + \underbrace{\frac{dq}{de}\Big|_{e=0} e}_{\Leftrightarrow\ \delta q}$$

$$\delta S = S(0) + \underbrace{\frac{dS}{de}\Big|_{e=0}}_{= S'(0)} e - S(0)$$

"$'$" here always $= \frac{d}{de}$

i.e. stationarity of action:

$S'(0) = 0$ for arbitrary $q'(0)$ which vanish on the boundary $\partial\Omega$

$$S = \int_\Omega \mathcal{L}\, d^4x$$

Back to the Thm. that Hamilton formulation $\Leftrightarrow$ Lagrange formulation

Introduce $J = \int_{t_1}^{t_2} H\,dt = \int_{t_1}^{t_2} dt \int_{\Sigma_t} \mathcal{H}\, d^3x =$

(I)
$$= \int_{t_1}^{t_2} dt \int_{\Sigma_t} \left(\pi \dot{q} - \mathcal{L}\right) d^3x =$$

$$= \underbrace{- \int_{t_1}^{t_2} dt \int_{\Sigma_t} \mathcal{L}\, d^3x}_{=\text{ action } S} + \int_{t_1}^{t_2} dt \int_{\Sigma_t} \pi \dot{q}\, d^3x = - S + \int_{t_1}^{t_2} dt \int_{\Sigma_t} \pi \dot{q}\, d^3x$$

variation

(II)
$$\left.\frac{dJ}{de}\right|_{e=0} = - \left.\frac{dS}{de}\right|_{e=0} + \int_{t_1}^{t_2} dt \int_{\Sigma_t} \Big(\pi \underbrace{\delta \dot{q}}_{=\left.\frac{d\dot{q}}{de}\right|_{e=0}} + \underbrace{\delta \pi}_{\left.\frac{d\pi}{de}\right|_{e=0}} \dot{q}\Big) d^3x$$

$$= - \left.\frac{dS}{de}\right|_{e=0} + \int_{t_1}^{t_2} dt \int_{\Sigma_t} \left[(\pi\, \delta q)^{\boldsymbol{\cdot}} - \dot{\pi}\, \delta q + \dot{q}\, \delta \pi \right] d^3x$$

but from (I) we also have

(III)
$$\left.\frac{dJ}{de}\right|_{e=0} = \int_{t_1}^{t_2} dt \int_{\Sigma_t} \left(\frac{\delta H}{\delta q} \delta q + \frac{\delta H}{\delta \pi} \delta \pi \right) d^3x$$

Comparing (II) and (III):

$$- \left.\frac{dS}{de}\right|_{e=0} + \int_{t_1}^{t_2} dt \int_{\Sigma_t} \Big[\underbrace{(\pi\, \delta q)^{\boldsymbol{\cdot}}}_{\to 0} - \dot{\pi}\, \delta q + \dot{q}\, \delta \pi \Big] d^3x =$$

$$= \int_{t_1}^{t_2} dt \int_{\Sigma_t} \left[\frac{\delta H}{\delta q} \delta q + \frac{\delta H}{\delta \pi} \delta \pi \right] d^3x$$

Hence, when

$$\underbrace{\left.\frac{dS}{de}\right|_{e=0} = 0}_{\text{Lagrange formulation}} \iff \boxed{\frac{\delta H}{\delta q} = - \dot{\pi}, \quad \frac{\delta H}{\delta \pi} = \dot{q}}$$

Hamilton equations

Variational principle, Lagrangian & Hamiltonian formalism for perturbations („2nd variations")

Let $S = \int_\Omega d^4x \, \mathcal{L}(\varphi^A ; \varphi^A{}_{,\mu})$

derivation of $\dfrac{\partial \mathcal{L}}{\partial \varphi^A} - \dfrac{\partial}{\partial x^\mu}\left(\dfrac{\partial \mathcal{L}}{\partial \varphi^A{}_{,\mu}}\right) = 0$

Let field variables are functions of some small parameter e, $\varphi^A = \varphi^A(x^\mu ; e)$ (e.g. $e \equiv$ small mass or charge of a particle)

then action is also function of e:

$$S(e) = \int_\Omega d^4x \, \mathcal{L}\left(\varphi^A(x;e) ; \varphi^A{}_{,\mu}(x;e)\right)$$

Require

$$S'(0) \equiv \left.\frac{dS}{de}\right|_{e=0} = 0 \quad \text{for arbit. } \underbrace{\varphi'^A(0)}_{\equiv d\varphi^A/de|_{e=0}}$$

vanishing on $\partial\Omega$

We have

$$S'(e) = \int \Big[\frac{\partial \mathcal{L}}{\partial \varphi^A} \varphi'^A + \frac{\partial \mathcal{L}}{\partial \varphi^A{}_{,\mu}} \underbrace{(\varphi^A{}_{,\mu})'}_{= \frac{\partial}{\partial e}\frac{\partial \varphi^A}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}(\varphi'^A)}\Big] d^4x$$

by parts "p.p"

$$\Rightarrow S'(e) = \underbrace{\int \Big[\frac{\partial \mathcal{L}}{\partial \varphi^A} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \varphi^A{}_{,\mu}}\right)\Big]}_{\equiv F_A(e)} \varphi'^A d^4x + \underbrace{\int \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \varphi^A{}_{,\mu}} \varphi'^A\right) d^4x}_{\to \int_{\partial\Omega} \to 0}$$

From here it is seen that if we demand $S'(0) = 0$ for arbitrary $\varphi'^A(0)$ (which $\to 0$ on $\partial\Omega$)

$$\Rightarrow \quad F_A(0) = \left[\frac{\partial \mathcal{L}}{\partial \varphi^A} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \varphi^A}\right)\right]_{e=0} = 0$$

$\Rightarrow$ standard Euler-Lagrange equations

Assume now we have solution $\varphi^A(0)$ satisfying $F_A(0) = 0$

Let $\varphi^A(e)$ is a solution "very near" to $\varphi^A(0)$, and let it solves equations $F_A(e) = 0$

Then the difference $\varphi^A(e) - \varphi^A(0) \cong \left.\frac{d\varphi^A}{de}\right|_{e=0} e$ $= \varphi'^A(0) e$, i.e. the perturbation(s) satisfy equation

$$\boxed{F'_A(0) \equiv \left.\frac{dF_A}{de}\right|_{e=0} = 0}$$

$$F_A(e) = \frac{\partial \mathcal{L}}{\partial \varphi^A} - \left(\frac{\partial \mathcal{L}}{\partial \varphi^A_{,\mu}}\right)_{,\mu}$$

basic solution satisfies field eqs., i.e.

$$F_A\left[\varphi^A(x;0);\ \varphi^A_{,\mu}(x;0);\ \varphi^A_{,\mu\nu}(x;0)\right] = 0$$

that perturbed solution satisfies field eqs. means

$$F_A\left[\varphi^A(x;e);\ \varphi^A_{,\mu}(x;e);\ \varphi^A_{,\mu\nu}(x;e)\right] = 0$$

„both" F_A the same functions. $\quad F'_A(0) = 0$

From the last equation

$$F_A\left[\varphi^A(x;0)+\varphi'^A e;\ \varphi^A_{,\mu}(x;0)+\varphi'^A_{,\mu}e;\ \varphi^A_{,\mu\nu}\dots\right]=0$$

taken at $e=0$

expand to 1st order

$$\Rightarrow \overbrace{F_A\left[\varphi^A(x;0);\ \varphi^A_{,\mu}(x;0);\ \varphi^A_{,\mu\nu}(x;0)\right]}^{\otimes}+$$

$$+\frac{\partial F_A}{\partial\varphi^A}\varphi'^A e+\frac{\partial F_A}{\partial\varphi^A_{,\mu}}\varphi'^A_{,\mu}e+\frac{\partial F_A}{\partial\varphi^A_{,\mu\nu}}\varphi'^A_{,\mu\nu}e=0$$

$\otimes = 0$, since this term are just unperturbed field eqs. for unpert. φ's („background")

$\Rightarrow$ Field eqs. for perturbations are (after $\times\ 1/e$)

$$\text{(P)}\qquad \frac{\partial F_A}{\partial\varphi^A}\varphi'^A(0)+\frac{\partial F_A}{\partial\varphi^A_{,\mu}}\varphi'^A_{,\mu}(0)+\frac{\partial F_A}{\partial\varphi^A_{,\mu\nu}}\varphi'^A_{,\mu\nu}(0)=0$$

i.e. $\left.\dfrac{dF_A}{de}\right|_{e=0}=0$ here $\dfrac{d}{de}$ is the total derivative

the coefficients here depend on the unperturbed solution

<u>(P) Equations for perturbations</u> (linearized in $\varphi'^A;\varphi'^A_{,\mu\dots}$)

One can show that (P) can be rewritten explicitly in terms of the Lagrangian as follows:

$$\left.\frac{dF_A}{de}\right|_{e=0}=\left[\frac{\partial^2\mathcal{L}}{\partial\varphi^A\partial\varphi^B}\varphi'^B+\frac{\partial^2\mathcal{L}}{\partial\varphi^A\partial\varphi^B_{,\mu}}\varphi'^B_{,\mu}\right.$$

$$\left.-\left(\frac{\partial^2\mathcal{L}}{\partial\varphi^A_{,\mu}\partial\varphi^B}\varphi'^B+\frac{\partial^2\mathcal{L}}{\partial\varphi^A_{,\mu}\partial\varphi^B_{,\nu}}\varphi'^B_{,\nu}\right)_{,\mu}\right]_{e=0}=0$$

For the proof start from

$$F_A = \frac{\partial \mathcal{L}}{\partial \varphi^A} - \left(\frac{\partial \mathcal{L}}{\partial \varphi^A_{,\mu}}\right)_{,\mu}$$

so that

$$\frac{dF_A}{de} = \frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B} \varphi'^B + \frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B_{,\mu}} \varphi'^B_{,\mu} -$$

$$- \frac{\partial}{\partial e}\left[\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \varphi^A_{,\mu}}\right)\right] = \dots$$

How these equations can be derived from a variational principle:

At the bottom of H_{12} we derived the result

$$S'(e) = \int F_A(e)\, \varphi'^A(e)\, d^4x + \int \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \varphi^A_{,\mu}} \varphi'^A\right) d^4x$$

So from here we can calculate

$S''(e)$. The result reads:

$$S''(e) = \int F_A \varphi''^A d^4x + \int \left(\frac{\partial \mathcal{L}}{\varphi^A_{,\mu}} \varphi''^A\right)_{,\mu} d^4x$$

$$+ \int \left[\frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B} \varphi'^A \varphi'^B + 2\frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B_{,\mu}} + \right.$$

$$\left. + \frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B_{,\nu}} \varphi'^A_{,\mu} \varphi'^B_{,\nu}\right] d^4x$$

The derivation - my 'older' calculations - are given on the following page

$$S''(e) = \int F_A \varphi''^A d^4x + \int F_A' \varphi'^A d^4x +$$

$$+ \int \left[\frac{\partial \mathcal{L}}{\partial \varphi^A_{,\mu}} \varphi''^A\right]_{,\mu} d^4x + \int \left[\frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B} \varphi'^A \varphi'^B\right]_{,\mu} d^4x$$

$$+ \int \left[\frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B_{,\nu}} \varphi'^A \varphi'^B_{,\nu}\right]_{,\mu} d^4x$$

$\Rightarrow$

$$S''(e) = \int F_A \varphi''^A d^4x + \int \left\{ \frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B} \varphi'^B \varphi'^A + \frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B_{,\mu}} \varphi'^B_{,\mu} \varphi'^A \right.$$

$$- \left(\frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B} \varphi'^B\right)_{,\mu} \varphi'^A - \left(\frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B_{,\nu}} \varphi'^B_{,\nu}\right)_{,\mu} \varphi'^A$$

$$+ \left[\frac{\partial \mathcal{L}}{\partial \varphi^A_{,\mu}} \varphi''^A\right]_{,\mu} + \left[\frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B} \varphi'^A \varphi'^B\right]_{,\mu} + \left[\frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B_{,\nu}} \varphi'^A \varphi'^B_{,\nu}\right]_{,\mu}$$

this term separately

$$= \int F_A \varphi''^A d^4x + \int \left[\frac{\partial \mathcal{L}}{\partial \varphi^A_{,\mu}} \varphi''^A\right]_{,\mu} d^4x$$

these add together

$$+ \int \left\{ \frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B} \varphi'^B \varphi'^A + \frac{\partial^2 \mathcal{L}}{\partial \varphi^A \partial \varphi^B_{,\mu}} \varphi'^B_{,\mu} \varphi'^A + \frac{\partial^2 \mathcal{L}}{\partial \varphi^{A}_{,\mu} \partial \varphi^B} \varphi'^B \varphi'^A_{,\mu} \right.$$

$$\left. + \frac{\partial^2 \mathcal{L}}{\partial \varphi^A_{,\mu} \partial \varphi^B_{,\nu}} \varphi'^A_{,\mu} \varphi'^B_{,\nu} \right\} d^4x$$

~~Recall~~

Now assuming that $\varphi^A(x;0)$ is the solution of $F_A(0) = 0$, i.e. this is ~~the~~ the unperturbed solution and assuming $\varphi''(0) = 0$ on $\partial\Omega$ (so that the divergence term can be omitted) the resulting expression for $S''(0)$ becomes

$$S''(0) = \int \left(\frac{\partial^2 \mathcal{L}}{\partial\varphi^A \partial\varphi^B} \varphi'^A \varphi'^B + \frac{2\,\partial^2 \mathcal{L}}{\partial\varphi^A \partial\varphi^B_{,\mu}} \varphi'^A \varphi'^B_{,\mu} + \frac{\partial^2 \mathcal{L}}{\partial\varphi^A_{,\mu} \partial\varphi^B_{,\nu}} \varphi'^A_{,\mu} \varphi'^B_{,\nu} \right) d^4x$$

Consider now φ'^A (not unperturbed φ^A) as function of x and of some parameter f (not „e")

Then $S''(0)$ above is function of f through $\varphi'^A, \varphi'^B_{,\mu}$

Denote $\mathcal{T}(f) = S''(0)$

and look at Lagrange-Euler equations implied by the condition

$$\left(\frac{d\mathcal{T}}{df}\right)_{f=0} = 0$$

so extremizing $\mathcal{T}$

this is the

„Second variation" ! of φ

one finds

$$\delta\mathcal{T} = 2\int F'_A \,\delta\varphi'^A d^4x$$

$$\left.\begin{aligned} &+ 2\int \left(\frac{\partial^2 \mathcal{L}}{\partial\varphi^A \partial\varphi^B_{,\mu}} \varphi'^A \delta\varphi'^B\right)_{,\mu} d^4x \\ &+ 2\int \left(\frac{\partial^2 \mathcal{L}}{\partial\varphi^A_{,\mu} \partial\varphi^B_{,\nu}} \varphi'^A_{,\mu} \delta\varphi^B\right)_{,\nu} d^4x \end{aligned}\right\} \text{divergences}$$

If variations φ'^A vanish on $\partial\Omega$,

the functional $S''(0)$ becomes extremal,

i.e. $\delta\mathcal{T}$ or $\left.\frac{d\mathcal{T}}{df}\right|_{f=0} = 0$ for those φ'^A which satisfy

$$F_A'(0) = 0 \quad \text{equations for perturbations}$$

Summary

Solutions $\varphi^A(x;0)$ of Lagrange equations $F_A(\varphi) = 0$ are those for which $S'(0) = 0$, where

$$S(e) = \int_{\Omega} d^4x\, \mathcal{L}\left(\varphi^A(x;e), \varphi^A{}_{,\mu}(x;e)\right),$$

for such that φ'^A (corresponding to the first variation $\delta\varphi$) which vanish on $\partial\Omega$.

Solutions φ'^A of equations $F_A'(\varphi;\varphi') = 0$ where φ^A satisfy Lagrange equations and are the coefficients in the linear dif. equations $F_A'(\varphi;\varphi') = 0$ are such that $S''(0)$ becomes extremal.

The equations for perturbations (all terms at $e = 0$)

$$\frac{\partial^2\mathcal{L}}{\partial\varphi^A\partial\varphi^B}\varphi'^B + \frac{\partial^2\mathcal{L}}{\partial\varphi^A\partial\varphi^B{}_{,\mu}}\varphi'^B{}_{,\mu} - \left[\frac{\partial^2\mathcal{L}}{\partial\varphi^A{}_{,\mu}\partial\varphi^B}\varphi'^B + \frac{\partial^2\mathcal{L}}{\partial\varphi^A{}_{,\mu}\partial\varphi^B{}_{,\nu}}\right]_{,\mu} = 0$$

are obtained by extremizing the functional (quadratic in φ'^A)

$$S''(0) = \int\left[\frac{\partial^2\mathcal{L}}{\partial\varphi^A\partial\varphi^B}\varphi'^A\varphi'^B + \frac{2\,\partial^2\mathcal{L}}{\partial\varphi^A\partial\varphi^B{}_{,\mu}}\varphi'^A\varphi'^B{}_{,\mu} + \frac{\partial^2\mathcal{L}}{\partial\varphi^A{}_{,\mu}\partial\varphi^B{}_{,\nu}}\varphi'^A{}_{,\mu}\varphi'^B{}_{,\nu}\right]d^4x$$

with respect to changes ("variations of the variations" φ'^A)

$$\varphi'^A(x;f) = \varphi'(x;0) + \left.\frac{d\varphi'^A}{df}\right|_{f=0}\cdot f \quad \text{"}\delta\varphi'^A\text{" 2nd}$$

Hamiltonian formalism

Assume $S = \int_\Omega \mathcal{L}(\varphi^A; \varphi^A{}_{,\mu})\, d^4x$,

rewrite it as

$$S = \int_\Omega d^4x \left[\pi_A \varphi^A{}_{,0} - \mathcal{H}(\pi_A, \pi_{A,i}; \varphi^A; \varphi^A{}_{,i})\right]$$

$i = 1,2,3$

where momenta

$$\pi_A = \frac{\partial \mathcal{L}}{\partial \varphi^A{}_{,0}}$$

The requirement that S is stationary w.r.t. arbitrary independent variations of π_A, φ^A which $\to 0$ on $\partial\Omega$ leads to the Hamilton equations

(HE)

$$\frac{\partial \varphi^A}{\partial t} = \frac{\partial \mathcal{H}}{\partial \pi_A} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}}{\partial \pi_{A,i}}\right) = \frac{\delta H}{\delta \pi_A}$$

$$\frac{\partial \pi_A}{\partial t} = -\frac{\partial \mathcal{H}}{\partial \varphi^A} + \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}}{\partial \varphi^A_{,i}}\right) = -\frac{\delta H}{\delta \varphi^A}$$

Hamilt. eqs. $\Leftrightarrow$ Lagrange-Euler eqs. as we saw already

Further we can proceed in a full analogy with the Lagrange formalism.

We express S on 1-parameter family of functions $\pi_A(x^\mu; e)$, $\varphi^A(x^\mu; e)$, i.e., we obtain $S(e)$. Calculate $S'(0)$ and $S''(0)$.

The condition $S'(0) = 0$ for all $\pi'_A(0)$, $\varphi'^A(0)$ which vanish on $\partial\Omega$ leads to the Hamilton equations (HE) on the preding page for $\pi_A(0)$ and $\varphi^A(0)$. Expression $S''(0)$ is the variational integral for equations for small perturbations (we vary π'_A, φ'^A in $S''(0)$)

which read

$$(HE)' \quad \frac{\partial \varphi'^A(0)}{\partial t} = \frac{d}{de}\left[\frac{\partial \mathcal{H}}{\partial \pi_A} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}}{\partial \pi_{A,i}}\right)\right]\Bigg/_{e=0},$$

$$\frac{\partial \pi'_A(0)}{\partial t} = -\frac{d}{de}\left[\frac{\partial \mathcal{H}}{\partial \varphi^A} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}}{\partial \varphi^A_{,i}}\right)\right]\Bigg/_{e=0}.$$

$(HE)'$ arose just by taking $\frac{d}{de}$ (HE)

Analogous to the fact that Lagrange equations for perturbations are $\frac{d}{de} F_A\Big/_{e=0} = 0$, i.e. the derivative of the original Lagrange equations.

A. H. Taub: Stability of General Relativistic Gaseous Masses and Variational Principles, Commun. math. Phys. 15, 235–254 (1969)

J. Bičák: On the theories of the interacting perturbations of the Reissner-Nordström black hole Czechoslov. Journal of Phys. B29, 945–980 (1979)

and citations there to V. Moncrief's papers