Cauchy problem,

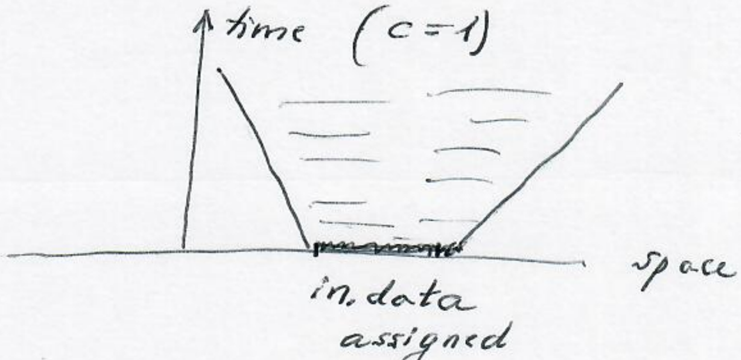
or, i.v.p. = initial value problem

for

- classical fields propagating in flat spacetime (SR)
- gravitational field (GR) "building curved ST"

I. Cauchy problem for particles & fields in flat spacetime

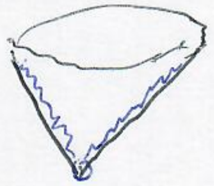
When a theory admits to give „suitable initial data" (possibly with constraints) in such a way that the evolution of the system is determined uniquely, then the theory admits <u>initial value formulation (i.v.f.)</u>; this is „well-posed" if the evolution is causal and stable under small changes of initial data.



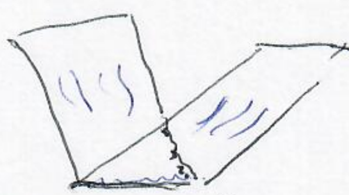
no signal propagates with $v > c$

Different <u>types</u> of i.v.f.:

„true" Cauchy p. — data on <u>spacelike</u> hypersurface



on a <u>null</u> cone



on 2 null hypersurfaces

the simplest case — n mass points interacting in classical mechanics, n degrees of freedom

$$\frac{d^2 q_i}{dt^2} = F_i\left(q_1, \dots q_n; \frac{dq_1}{dt}, \dots, \frac{dq_n}{dt}\right)$$

From theory of ordinary diff. eqs.: if we know $q_{10}, \dots q_{n0}$

$\left.\frac{dq_1}{dt}\right|_0, \dots, \left.\frac{dq_n}{dt}\right|_0 \Rightarrow$ a unique solution exists with these in. val. for at least final $\Delta t > 0$

Example of a <u>solution existing for only finite t</u> - at first sight not „pathological":

Consider eq. $\boxed{\dot{q} = q^{\alpha}}$, α number $\alpha > 1$, or $\alpha < 1$ $\alpha \neq 1$

separation var.

$$\Rightarrow \quad \frac{dq}{q^{\alpha}} = dt \quad \Rightarrow \quad \frac{-1}{\alpha - 1} q^{-(\alpha-1)} = t + C$$

↑ const.

$$\Rightarrow \quad \frac{1}{q^{\alpha-1}} = -(\alpha-1)t \underbrace{- C(\alpha-1)}_{\equiv +K \text{ const. } \neq 0}$$

solution is thus

$$q = \left[\frac{1}{K - (\alpha-1)t}\right]^{\frac{1}{\alpha-1}}$$

for $\alpha > 1$ the denominator

$$\frac{1}{K - (\alpha-1)t} \to \infty \quad \text{for } \overbrace{K = (\alpha-1)t}^{\text{finite}}, \quad \text{finite } t$$

so q stops to exist

for $\alpha < 1$ $\alpha - 1 < 0$

$$q = \left[\frac{1}{K + (1-\alpha)t}\right]^{-\frac{1}{1-\alpha}}$$

$$q = \left[K + (1-\alpha)t\right]^{\frac{1}{1-\alpha}}$$

is OK for any t

Maxwell equations (with sources in vacuum)

in CGS

$$(1)\quad \mathrm{rot}\,\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$(2)\quad \mathrm{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

} dynamics

rot ≡ curl

$$(3)\quad \mathrm{div}\,\vec{E} = 4\pi\rho$$

$$(4)\quad \mathrm{div}\,\vec{H} = 0$$

} constraints

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acting by "div" on (1) $\quad \underset{\text{„div rot"}}{0} = \frac{4\pi}{c}\mathrm{div}\,\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\mathrm{div}\,\vec{E}}_{=4\pi\rho}$

$$\Longrightarrow \quad \frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0$$

continuity eq. follows from Maxwell's eq.

In vacuum: $(2^*)\quad \mathrm{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$,

$(1^*)\quad \mathrm{rot}\,\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$, $\quad \mathrm{div}\,\vec{E} = 0$, $\mathrm{div}\,\vec{H} = 0$

Maxwell eqs. are of 1st order in $\vec{E}, \vec{H}$, so initial conditions are given by giving at initial hypers. $t = t_0 \equiv \Sigma$

$$\vec{E}(t = t_0, x^i), \quad \vec{H}(t = t_0, x^i)$$

but they must satisfy the constraints

$$\mathrm{div}\,\vec{E} = \frac{\partial E_1}{\partial x^1} + \frac{\partial E_2}{\partial x^2} + \frac{\partial E_3}{\partial x^3} = 0, \quad \mathrm{div}\,\vec{H} = \dots = 0$$

note that by knowing $\vec{E}(t = t_0, x^i), \vec{H}(\dots)$ we can calculate all derivatives $\frac{\partial E_j}{\partial x^i}, \frac{\partial H_j}{\partial x^i}$ on Σ

and so also $\mathrm{rot}\,\vec{E}$, $\mathrm{rot}\,\vec{H}$ at $t = t_0$

Now important assumption: analycity!

So we can Taylor expand in time, and write

$$(*) \quad \vec{E}(t_0+\delta t, x^i) = \vec{E}(t_0) + \frac{\partial \vec{E}}{\partial t}\Big|_{t_0}\delta t + \frac{1}{2!}\frac{\partial^2 \vec{E}}{\partial t^2}\Big|_{t_0} + \cdots$$

similarly
$(**)$ $\quad \vec{H}(t_0+\delta t, x^i) = \cdots$

since we know $\mathrm{rot}\,\vec{E}$ and $\mathrm{rot}\,\vec{H}$ at $t=t_0$

we can use dynamical Maxwell equations (1^*), (2^*) to calculate $\frac{\partial \vec{E}}{\partial t}\Big|_{t_0}$ and $\frac{\partial \vec{H}}{\partial t}\Big|_{t_0}$, so find $\vec{E}(t_0+\delta t)$, $\vec{H}(t_0+\delta t)$ to the first order in δt.

Further terms in Taylor expansions follow from taking time derivatives $\frac{\partial}{\partial t}$ of Maxwell eqs. (1^*), (2^*). In this way we determine

$\frac{\partial^2 \vec{E}}{\partial t^2}\Big|_{t_0}$, $\frac{\partial^2 \vec{H}}{\partial t^2}\Big|_{t_0}$ from (1^*), (2^*) since

we can calculate $\mathrm{rot}\,\frac{\partial \vec{E}}{\partial t}$, $\mathrm{rot}\,\frac{\partial \vec{H}}{\partial t}$ at t_0.

So obtain $\vec{E}, \vec{H}$ up to the second order $\sim (\delta t)^2$

in this way we can continue by making further $\partial/\partial t$ of (1^*), (2^*) and determine so full Taylor series $(*)$ and $(**)$.

Klein-Gordon equation in Minkowski

$$\partial_\alpha \partial^\alpha \phi - m^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - m^2 \phi = 0 \qquad (KG)$$

Note:

for $m = 0$ and on the left $a^2 \frac{\partial^2 \phi}{\partial t^2}$, and $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0$

$\Rightarrow$ equation for the vibrating string

(KG) tells us how to calculate $\partial^2 \phi / \partial t^2$ if we know ϕ and $\partial \phi / \partial t$ at $t = t_0$ (on Σ_0).

Note: In fact, (KG) determines $\partial^2 \phi / \partial t^2$ at t_0 without need of knowing $\partial \phi / \partial t$ at t_0 but for more general form of Σ_0, say given by $S(t, x^i) = 0$ or in curved background, one needs to know $\partial \phi / \partial t |_\Sigma$

1) Analytical initial values

Assume ϕ and $\partial \phi / \partial t$ are analytical functions on Σ_0

$\Rightarrow$ can calculate $\partial^2 \phi / \partial x^{i\,2}$ on Σ_0 – analytical f.

(KG) $\Rightarrow$ $\frac{\partial^2 \phi}{\partial t^2}$ on Σ_0 and from this can calculate all $\frac{\partial}{\partial x^i} \frac{\partial^2 \phi}{\partial t^2}$ on Σ_0, etc./make $\frac{\partial}{\partial t}$ of (KG)

$\Rightarrow$ get $\frac{\partial^3 \phi}{\partial t^3}$ etc $\Rightarrow$ can write down Taylor series formula for

$$\phi(x^i; t_0 + \delta t) = \phi(x^i; t_0) + \frac{\partial \phi}{\partial t}\Big|_{t_0} \delta t + \frac{1}{2!} \frac{\partial^2 \phi}{\partial t^2}\Big|_{t_0} (\delta t)^2 + \ldots$$

Existence and uniqueness of the Taylor Series soltns with finite radius of convergence (see Courant & Hilbert etc)

Theorem (Cauchy-Kowalewski)

Let $\underbrace{t, x^1, \dots x^{m-1}}_{\equiv x^a}$ are coordinates in R^m

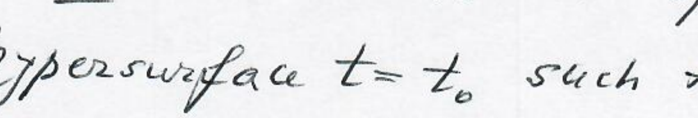
Let a system of n partial differential eqs. (PDE) for n unknown functions ϕ_i, $i = 1, \dots n$ in R^m has the form

$$(+) \quad \boxed{\frac{\partial^2 \phi_i}{\partial t^2} = F_i \left(t, x^1, \dots x^{m-1}; \phi_j, \frac{\partial \phi_j}{\partial t}, \frac{\partial \phi_j}{\partial x^a}, \frac{\partial^2 \phi_j}{\partial t \partial x^a}, \frac{\partial^2 \phi_j}{\partial x^a \partial x^b} \right)}$$

where all functions F_i are analytic in all variables. Let $f_i(x^a)$, $g_i(x^a)$ are analytic.

Then there exists an open neighborhood $\mathcal{O}$ of hypersurface $t = t_0$ such that there exists unique analytic solution of the system (+) such that

$$\phi_i(t_0, x^a) = f_i(x^a), \qquad \frac{\partial \phi_i}{\partial t}(t_0, x^a) = g_i(x^a)$$



$t = t_0$

Bure, five years his junior. The de Bure family were printers and booksellers, and published most of Cauchy's works.[6] Aloïse and Augustin were married on April 4, 1818, with great Roman Catholic pomp and ceremony, in the Church of Saint-Sulpice. In 1819 the couple's first daughter, Marie Françoise Alicia, was born, and in 1823 the second and last daughter, Marie Mathilde.[7]

The conservative political climate that lasted until 1830 suited Cauchy perfectly. In 1824 Louis XVIII died, and was succeeded by his even more reactionary brother Charles X. During these years Cauchy was highly productive, and published one important mathematical treatise after another. He received cross-appointments at the Collège de France, and the Faculté des sciences de Paris [fr].

In exile[edit]

A. Cauchy

In July 1830, the July Revolution occurred in France. Charles X fled the country, and was succeeded by the non-Bourbon king Louis-Philippe (of the House of Orléans). Riots, in which uniformed students of the École Polytechnique took an active part, raged close to Cauchy's home in Paris.

These events marked a turning point in Cauchy's life, and a break in his mathematical productivity. Cauchy, shaken by the fall of the government, and moved by a deep hatred of the liberals who were taking power, left Paris to go abroad, leaving his family behind.[8] He spent a short time at Fribourg in Switzerland, where he had to decide whether he would swear a required oath of allegiance to the new regime. He refused to do this, and consequently lost all his positions in Paris, except his membership of the Academy, for which an oath was not required. In 1831 Cauchy went to the Italian city of Turin, and after some time there, he accepted an offer from the King of Sardinia (who ruled Turin and the surrounding Piedmont region) for a chair of theoretical physics, which was created especially for him. He taught in Turin during 1832–1833. In 1831, he was elected a foreign member of the Royal Swedish Academy of Sciences, and the following year a Foreign Honorary Member of the American Academy of Arts and Sciences.[9]

In August 1833 Cauchy left Turin for Prague, to become the science tutor of the thirteen-year-old Duke of Bordeaux Henri d'Artois (1820–1883), the exiled Crown Prince and grandson of Charles X.[10] As a professor of the École Polytechnique, Cauchy had been a notoriously bad lecturer, assuming levels of understanding that only a few of his best students could reach, and cramming his allotted time with too much material. The young Duke had neither taste nor talent for either mathematics or science, so student and teacher were a perfect mismatch. Although Cauchy took his mission very seriously, he did this with great clumsiness, and with surprising lack of authority over the Duke.

During his civil engineering days, Cauchy once had been briefly in charge of repairing a few of the Parisian sewers, and he made the mistake of mentioning this to his pupil; with great malice, the young Duke went about saying Mister Cauchy started his career in the sewers of Paris. His role as tutor lasted until the Duke became eighteen years old, in September 1838.[8] Cauchy did hardly any research during those five years, while the Duke acquired a lifelong dislike of mathematics. The only good that came out of this episode was Cauchy's promotion to baron, a title by which Cauchy set great store. In 1834, his wife and two daughters moved to Prague, and Cauchy was finally reunited with his family after four years in exile.

$\Rightarrow$ KG theory (eq.) admits in. value formulation for analytical data

there exist as many analytic soltns. of KG equation as there are pairs of arbitrary analytic functions of 3 space variables x^a (f_1, f_2 above)

However, Cauchy-Kowalewskaya does not show that i.v.p. is „well-posed":

1) no stability demonstration in the sense that the relation of the solution for $t > t_0$ to the i.values on $t = t_0$ is continuous, i.e., if the init. data are changed „a little", also the solution...

one has to define norms („distance") of functions f_1, f_2 on Σ_0 by, for example,

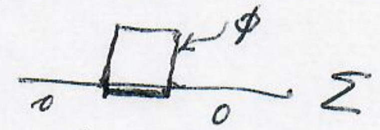
$$\| f_1 - f_2 \| = \underset{x \in \Sigma_0}{\text{"least upper bound"}} \left| f_1(x) - f_2(x) \right|$$

$$(N) \qquad + \sum_{k_1, k_2, k_3} \underset{x \in \Sigma_0}{\text{"l.u.b."}} \left| \frac{\partial^{k_1+k_2+k_3}(f_1 - f_2)}{\partial x^{k_1} \partial y^{k_2} \partial z^{k_3}} \right|$$

$$k_1 + k_2 + k_3 \leq k$$

open balls in this norm - basis of introducing topology on the set of functions f

2) Cauchy–Kowalewski give no information about the causal propagation of the field
– analytic function is determined by its value at a given point and by all derivatives at this point – by Taylor series; of course, it is determined also by these values in arbitrary small neighborhood of the point.

Hence, changing a little initial conditions around the point, in analytical case, we have to change them on whole Σ_0. To judge causality, need to choose non-analytical initial data, say 

or at least C^∞, or δ-function type, etc.

<u>New approach of demonstrating „well posedness" in M^4</u> (Wald p. 247–248) <u>using conserv. of energy-mom.</u>

(note: „energy estimates" are used in recent proofs of stability of Minkowski, Kerr, etc – in GR)

Example – scalar field in Minkowski

(EM) $$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}\eta_{\alpha\beta}\left(\phi_{,\gamma}\phi^{,\gamma} + m^2\phi^2\right)$$

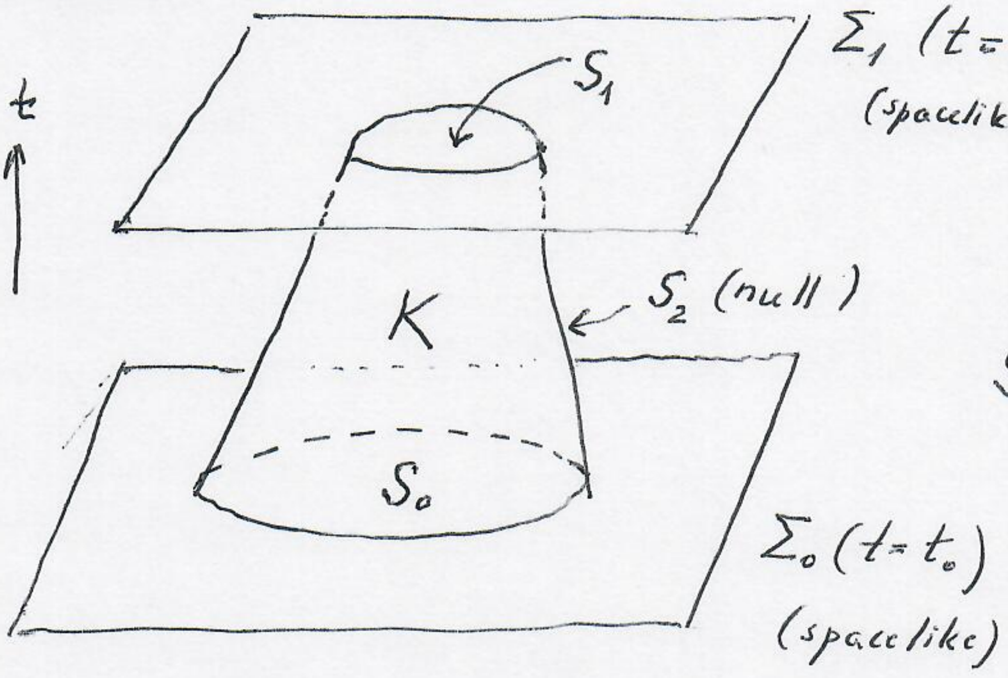
Let $\xi^\alpha = \frac{\partial}{\partial t}$ is Killing vector, perpendicular to Σ_0

$$\therefore \boxed{\partial^\alpha\left(T_{\alpha\beta}\xi^\beta\right) = 0} \Leftrightarrow \nabla_\alpha\left(T^\alpha_{\ \beta}\xi^\beta\right) = 0$$

More generally,

$$\nabla_\alpha\left(T^\alpha_{\ \beta}\xi^\beta\right) = \underbrace{\nabla_\alpha T^\alpha_{\ \beta}}\,\xi^\beta + T^{\alpha\beta}\underbrace{\nabla_\alpha\xi_\beta} = 0$$

Consider:



ℰ 10

K „interior" (4 dim)

S_0 3-dim region („ball") on Σ_0

S_1 — " — in Σ_1

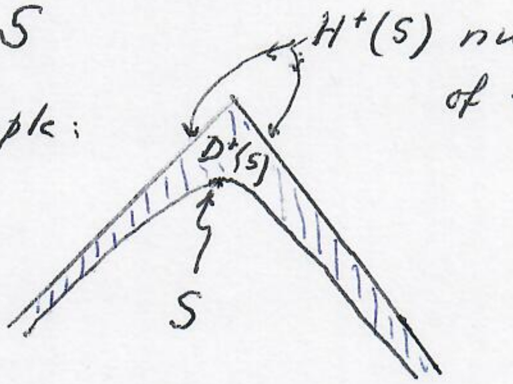
Precisely:
$$K = D^+(S_0) \cap J^-(\Sigma_1)$$
$$S_1 = D^+(S_0) \cap \Sigma_1$$

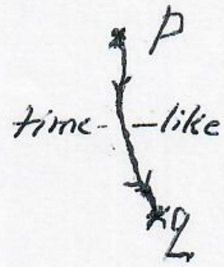
Intermezzo (more details in a seminar after Easter based on the book P.S. Joshi, Global Aspects in Gravitation and Cosmology 1996)

$D^+(S)$ .. future domain of dependence of S

set of points p, for which every past causal curve (timelike or null) intersects S

Example:





the causal past of point p:

$$J^-(p) = \left\{ \begin{array}{l} \text{the set of points } q \text{ in } M \text{ such that there exists} \\ \text{a timelike curve going to the past connecting} \\ q \text{ with } p \end{array} \right\}$$

the causal past of set S: $J^-(S) = \bigcup_{p \in S} J^-(p)$

Note: if we would take just $D^+(S_0)$

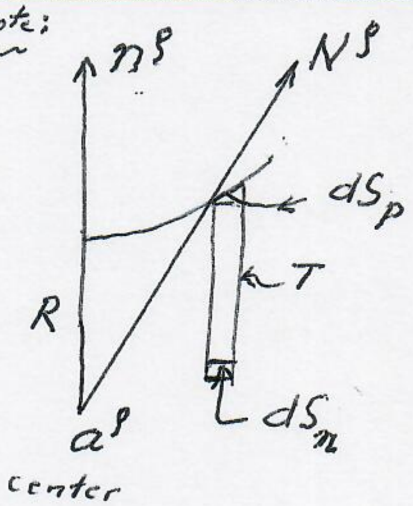
Integrate $\partial^\alpha (T_{\alpha\beta}\,\xi^\beta) = 0$ over 4-dim region K and use Gauss theorem, converting the $\int$ over K to 3 integrals over S_0, S_1, S_2 (use that ξ^α is perpendicular to $t=t_0$ and $t=t_1$

$$\text{(T2)}\quad \boxed{\int_{S_1} T_{\alpha\beta}\,\xi^\alpha\xi^\beta dV + \int_{S_2} T_{\alpha\beta}\, l^\alpha \xi^\beta d\omega = \int_{S_0} T_{\alpha\beta}\,\xi^\alpha\xi^\beta dV}$$

l^α is future-directed normal to S_2 which is null, $d\omega$ is the Lorentz inv. 2-content of a 3-dim element on a null cone S_2 (see below)

S_2 l^α

Note:



Pseudo-sphere with center at a^ρ, radius R:

$$\eta_{\rho\sigma}(x^\rho - a^\rho)(x^\sigma - a^\sigma) = -R^2 \qquad (\sim)$$

let n^ρ by any timelike unit vector pointing to future

T is a thin tube parallel to n^ρ,

dS_p ... 3 volume cut from the pseudosphere by T

N^ρ ... unit normal to the pseudosphere $N^\rho = \dfrac{x^\rho - a^\rho}{R}$ from $(\sim)$

Projection formula $dS_n = dS_p\,|N^\rho n_\rho|$
(„cosinus" theor.)

$\Longrightarrow \quad dS_n = (dS_p/R)\,|n_\rho (x^\rho - a^\rho)|$

define 2 content $d\omega$ of the pseudosphere with 3-volume dS_p as $R \to 0$, $dS_p \to 0$ (approaching null cone)

pseudosphere is

So 3volume $dS_p \to 0$ as $R \to 0$ but

the 3-element possesses an absolute 2content

$$\boxed{d\omega = \lim_{R \to 0} \frac{dS_p}{R} = \underbrace{\frac{dS_n}{|n_\varsigma (x^\varsigma - a^\varsigma)|}}_{\text{the fraction is indep. of the choice of } n^\varsigma}}$$

inv. 2-content on the null cone (null hypers.)

Going back to (T2) (previous page).

We assume – a physically natural – "dominant energy condition":

$T_{\alpha\beta}$ satisfies the dominant energy condition, if for 1) any future-directed timelike vector V^α (i.e. "an observer") the 4-current $-T^\alpha_{\ \beta} V^\beta$ is future-directed timelike or null, so mass-energy cannot flow faster than light, 2) $T_{\alpha\beta} V^\alpha V^\beta \geq 0$, matter-energy observed by observer with V^α must be positive.

Simple example: dust $T^\alpha_{\ \beta} = \mu U^\alpha U_\beta$

then $-T^\alpha_{\ \beta} V^\beta = -\mu U^\alpha \underbrace{U_\beta V^\beta}_{<0}$ -- future directed

<0 since both U^α and V^α are timelike and future directed

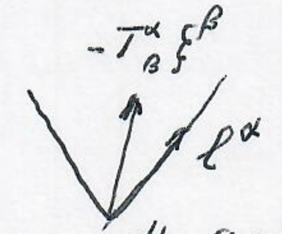
$\Rightarrow \mu U^\alpha$ future-directed, time like

ad 2) $T_{\alpha\beta} V^\alpha V^\beta = \mu \overbrace{U_\alpha \underbrace{U_\beta V^\alpha V^\beta}_{\leq 0}}^{\leq 0} \geq 0$

if the observer moves with the dust $T_{00} = \mu > 0$

Returning back to (T2) and assuming dominant energy condition, and taking ξ^α as the future-directed timelike vector (i.e. as „V^α" in energy cond)

$$\Rightarrow \quad \underbrace{\ell_\alpha}_{\text{future directed null}} \underbrace{(-T^\alpha_{\ \beta}\,\xi^\beta)}_{\text{future dir. timelike or null}} \leqq 0$$



the scalar product must be ≤ 0

cp. t / x in special LT

$$\Rightarrow \quad \ell^\alpha \xi^\beta T_{\alpha\beta} \geqslant 0$$

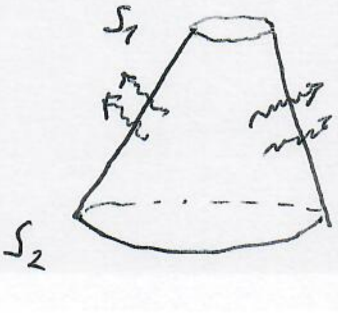
($T_{\alpha\beta}\,\xi^\alpha \xi^\beta$ is clearly ≥ 0 – condition 2), $T_{\alpha\beta}V^\alpha V^\beta \geq 0$)

Therefore Gauss theorem coming from $\partial^\alpha(T_{\alpha\beta}\,\xi^\beta) = 0$ implies

$$\int_{S_1} T_{\alpha\beta}\,\xi^\alpha \xi^\beta dV \leq \int_{S_0} T_{\alpha\beta}\,\xi^\alpha \xi^\beta dV,$$

or, after substituting for $T_{\alpha\beta}$ in terms of ϕ from (EM), page C9, we get an inequality (putting $\xi = \frac{\partial}{\partial t}$,

$$\text{(B)} \quad \underbrace{\int_{S_1}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + |\vec\nabla\phi|^2 + m^2\phi^2\right] dV}_{\text{energy in } S_1} \leq \underbrace{\int_{S_0}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + |\vec\nabla\phi|^2 + m^2\phi^2\right] dV}_{\text{energy in } S_0}$$



$\Rightarrow$ <u>Thm: The solution is uniquely determined by initial data on S_0:</u>

Proof by 'contradiction'

Assume ϕ_1, ϕ_2 are two solutions –

(– of course, they must be (at least) C^2 so that 2nd. derivatives in wave eqs. are existing, but <u>no</u> analycity assumed)

both coming from the same initial data on S_0 (or Σ_0)

Form their difference $\psi = \phi_2 - \phi_1$. Eq. for ϕ is linear, so ψ is also the solution of KG equation.

But $\psi = 0$ on Σ_0 (S_0) because we assume the same init. data for both ϕ_1 and ϕ_2. Hence, the integral $\int_{S_0 \Sigma_0} \left[\left(\frac{\partial \psi}{\partial t}\right)^2 + |\vec{\nabla} \psi|^2 + m^2 \psi^2 \right] dV = 0$

The last relation on previous page (C13) implies that the same integral over S_1 must be also vanishing but the sum of squares, so

$\psi = 0$ also on S_1 ! $\Rightarrow$ Solution is uniquely determined by initial data (this is valid for both $m \neq 0$ and $m = 0$)

Moreover, the other requirement for well-posedness is satisfied – i.e. <u>causality</u>

any change of initial data outside the region S_0 cannot influence the solution in $D^+(S_0)$.

Equation (B) on p. (C13) can also be shown (C15)
to prove that solutions ϕ depend continuously on the initial data. For proof, involving „more" mathematical considerations, see Wald, p. 248 – 250 and problem 1, p. 267.

Just for a „curiosity" – the starting points:
Realize that (B), i.e., the inequality

$$\int_{S_1}\left[\left(\frac{\partial\phi}{\partial t}\right)^2+|\vec\nabla\phi|^2+m^2\phi^2\right] \leqq \int_{S_0}\left[\left(\frac{\partial\phi}{\partial t}\right)^2+|\vec\nabla\phi|^2+m^2\phi^2\right] \quad (B)$$

holds also for all partial derivatives of ϕ, $\frac{\partial\phi}{\partial x^k}$
because for these derivatives the original KG eq.

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - m^2\phi \quad \text{also holds.}$$

just make $\partial/\partial x^k$ of this equation. Can integrate and obtain inequalities

$$\|\phi\|_{S_1,k} \leq C_{1,k} \||\phi\||_{S_0,k} + C_{2,k} \left\|\left|\frac{\partial\phi}{\partial t}\right|\right\|_{S_0,k-1}$$

where the norms are defined by

$$\left.\begin{aligned} \|\phi\|^2_{S_1,k} &= \int_{S_1}\{|\phi|^2+\ldots\sum_i|\partial^{k_i}\phi|^2\} \\ \||\phi\||^2_{S_0,k} &= \int_{S_0}\{|\phi|^2+\ldots+\sum_i|D^{k_i}\phi|^2\} \end{aligned}\right\} \text{Sobolev norms}$$

here $\partial^{k_i}\phi = \frac{\partial^k\phi}{\partial x^k}$, or $\frac{\partial^k\phi}{\partial t^k}$, D^{k_i} only deriv. w.r.t. space coordin.

$$\Rightarrow \ldots \Rightarrow \underbrace{\text{l.u.b.}}_{x\in D^+(S_0)} |\phi| \leq \text{Sobolev norms on } S_0, \underset{\uparrow}{=} 3 \ldots$$

! in spacetime 3rd deriv.

similarly for l.u.b. $|\partial^m\phi| \leq$ — — characterizes