Mathematical "sequel"

For some hypersurfaces one cannot formulate „reasonable" initial value problem depending on the form of equation.
They are called characteristics, the normals are „bicharacteristics."

General linear equation

$$\sum_{\substack{k_1,\dots k_n \\ k_1+k_2+\dots k_n = m}} A^{(m)}_{k_1,\dots k_n}(t, x_i) \frac{\partial^m z}{\partial t^{k_1}\partial x_2^{k_2}\dots\partial x_n^{k_n}} + \dots = 0$$

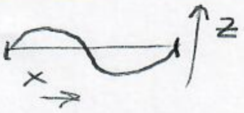
remaing terms

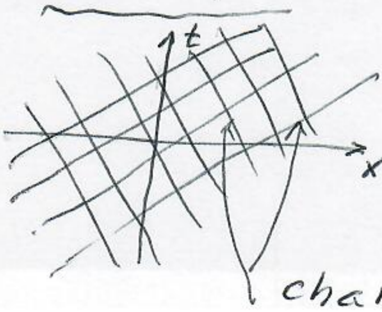
the highest derivatives
only these are now relevant

the "direction" cosines of characteristics given by $(\alpha_1, \alpha_2, \dots, \alpha_n)$ which are solutions of

$$\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = 1,$$

$$\boxed{\sum_{k_1+\dots k_n = m} A^{(m)}_{k_1\dots k_n} \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_n^{k_n} = 0}$$

Example: classical 1-dim „string"  or 1-d wave eq.



characteristics

$$\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} = 0$$

$$\alpha_1^2 + \alpha_2^2 = 1$$

$$\alpha_1^2 - \alpha_2^2 = 0$$

$$\Rightarrow \alpha_1 = \pm\alpha_2, \quad \alpha_1 = \pm\frac{\sqrt{2}}{2}$$

For elliptic eq. $\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} = 0$ no real characteristics exist

— For nonlinear PDEs the characteristics depend on a given solution

„ Cauchy problem" on a characteristic has either no solution or infinite number of solutions

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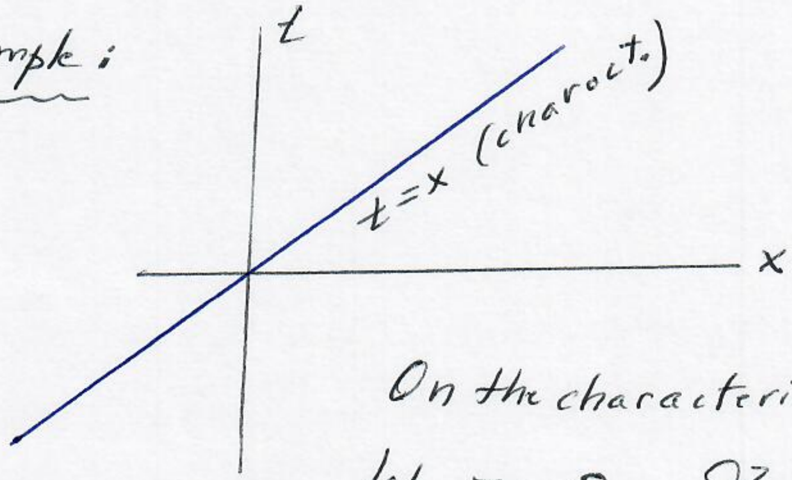
Example:



$$\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} = 0$$

On the characteristic $t = x$

let $z = 0$, $\left.\frac{\partial z}{\partial t}\right|_{t=x} = 0$

→ infinite number of solutions

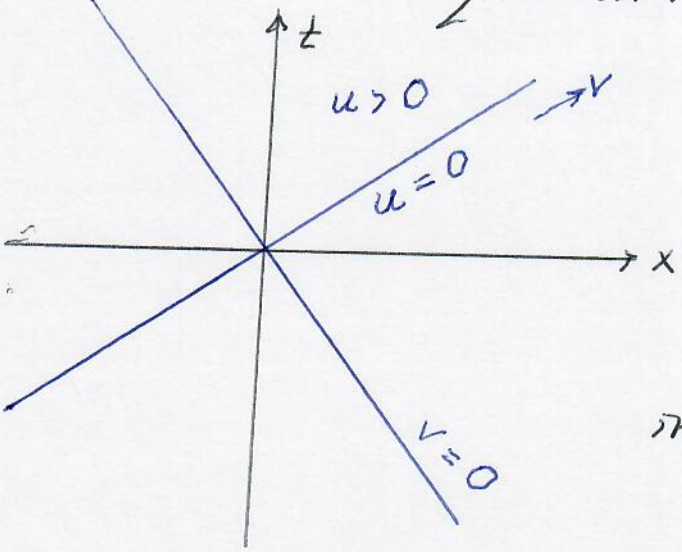
$$z = \alpha (t-x)^2 \Rightarrow z = 0 \text{ on } t = x$$

$$\frac{\partial z}{\partial t} = 0 \text{ on } t = x$$

α: arbitrary parameter

"Cauchy problem" on a characteristic – a more detailed view

Consider wave equation in two dimensions again



$$\left.\begin{array}{l} u = t - x \\ v = t + x \end{array}\right\} \text{null coordinates}$$

wave equation $\Psi_{,tt} - \Psi_{,xx} = 0$

in null coordinates

$$\Psi_{,uv} = 0, \qquad \Psi = \Psi(u,v)$$

Let us formulate "Cauchy" problem on the characteristic $u=0$:

$\Psi = \psi_0(v)$ on $u=0$, $\quad \Psi_{,u} = \psi_1(v)$ on $u=0$

(i) In order that $\Psi_{,uv} = 0$ is satisfied we must satisfy on $u=0$ condition $\partial_v \Psi_{,u} = \partial_v \psi_1(v) = 0$

$\Rightarrow$ $\psi_1(v) = $ const but standard init. value problem should admit general 'initial data', i.e, general $\psi_1(v)$ so Cauchy problem on a characteristic has no solution

(ii) in the special case when $\Psi_{,u} = \psi_1(v) = \text{const} = A$, $\Psi_{,uv} = 0$ is satisfied on $u=0$, and then integrating

$\Psi(u,v) = \psi_0(v) + Au + \chi(u)$ where we require

$$\chi(u=0) = 0, \quad \partial_u \chi(u)\Big|_{u=0} = 0 \quad (\bullet)$$

$\Rightarrow$ at $u=0$ indeed $\Psi = \psi_0(v)$ and $\Psi_{,u} = A = \psi_1(v)$

But then the solution contains arbitrary function $\chi(u)$ (satisfying $(\bullet)$); it is not unique.

Generalized Cauchy - Kowalevskaya for equations with higher derivatives

separate $x_1 \equiv t$

$$(\mathrm{I}) \quad \frac{\partial^m z}{\partial t^m} = f\left(t, x_2, \dots x_n; z, \dots \frac{\partial^k z}{\partial t^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}\right)$$

the highest time derivative is here explicitly „taken out"

$k_1 + \dots + k_n \leq m$

and $k_1 < m$

derivativ. w.r.t. t cannot be $\frac{\partial^m}{\partial t^m}$ this is „at the left"

Initial conditions for $z, \frac{\partial z}{\partial t} \dots \frac{\partial^{m-1} z}{\partial t^{m-1}}$:

$$\left.\begin{aligned} z(t = t_0, x_2 \dots x_n) &= f_0(x_2, \dots x_n) \\ \frac{\partial z}{\partial t}(t = t_0, x_2 \dots x_n) &= f_1(\text{—"—}) \\ &\vdots \\ \frac{\partial^{m-1} z}{\partial t^{m-1}}(t = t_0, x_2 \dots x_n) &= f_{m-1}(x_2, \dots x_n) \end{aligned}\right\} (*)$$

Note that by giving $(*)$ at $\{t_0, \underbrace{x_2, \dots x_n}_{\text{arbitrary}}\}$, i.e. everywhere on the initial hypersurface we can express all derivatives on the right-hand side of (I)

e.g. from $z(t_0, x_2, \dots x_n)$ can find $\left.\frac{\partial^k z}{\partial x_i^k}\right|_{t=t_0}$

Thm (Kowalewskaya)

Let f in (I) is analytic in all its variables in the neighborhood of a "point"

$$\left(t_0, x_2^0, \dots x_n^0, z_0, \dots f^0_{k_1, k_2 \dots k_n}\right)$$

$$= \frac{\partial z}{\partial t^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}\Bigg|_{\text{at } t_0, x_2^0, \dots x_n^0}$$

and let $f_0, f_1 \cdots f_{m-1}$ on the r.h.s. of ⊛ are analytic in the neighborhood of $(x_2^0, \dots x_n^0)$

Then there exists a neighborhood of $\left[t_0, x_2^0, \dots, x_n^0\right]$, i.e. around the initial hypersurface,

in which the solution of (I) is analytic and unique

Example of a string: $$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

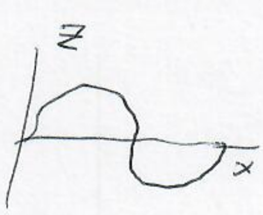
$a^2 = T/\rho_n$

tension density

speed of waves

$$z(0, x) = f_0(x)$$

$$\frac{\partial z}{\partial t}(0, x) = f_1(x)$$



clearly f as function of $\frac{\partial^2 z}{\partial x^2}$ is analytic

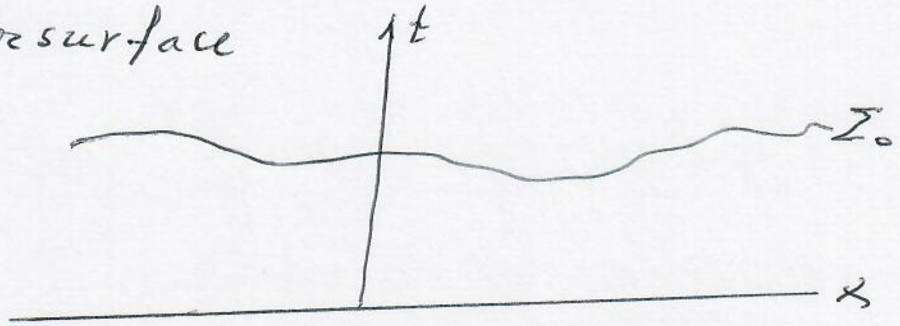
$\Rightarrow$ existence of analytic solution in neighborhood of (t_0, x_0)

if f_0 and f_1 are analytic in the neighborhood of (t_0, x_0)

Similarly for system of (linear) equations

Generalized Cauchy problem

initial data are not given on a $(n-1)$-dim. hyperplane but on a general $(n-1)$ dim hypersurface



suitable transf. to new coordinates (at least locally)

this brings us closer to the

Cauchy problem in General Relativity