O. Semerák : Lecture Notes

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The stationarity of action is equivalent to the **Hamilton canonical equations**. Indeed,

$$\delta S := \delta \int_\Omega \mathcal{L}\sqrt{-g}\,\mathrm{d}^4x \xrightarrow{1+3} \delta \int_{t_1}^{t_2}\int_{\Sigma(t)} \mathcal{L}\sqrt{-g}\,\mathrm{d}^3x\,\mathrm{d}t = \delta \int_{t_1}^{t_2}\int_{\Sigma(t)} (\Pi\cdot\dot q - \mathcal{H})\,\mathrm{d}^3x\,\mathrm{d}t =$$
$$= \int_{t_1}^{t_2}\int_{\Sigma(t)} \left(\delta\Pi\cdot\dot q + \Pi\cdot\delta\dot q - \frac{\partial\mathcal{H}}{\partial q}\cdot\delta q - \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi - \frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q_{,i}\right)\mathrm{d}^3x\,\mathrm{d}t\,.$$

we covered this on p. H_{11} (with slightly different notation)

The second term we "per-partes", already dropping the boundary term as usual (better argument is that δq is assumed to vanish at the marginal times t_1 and t_2),

$$\int_{t_1}^{t_2}\int_{\Sigma(t)} \Pi\cdot\delta\dot q\,\mathrm{d}^3x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\Sigma(t)} \dot\Pi\cdot\delta q\,\mathrm{d}^3x\,\mathrm{d}t\,.$$

Similarly we process the last term (time integration is not important in it),

$$-\int_{\Sigma(t)}\frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q_{,i}\,\mathrm{d}^3x = -\int_{\Sigma(t)}\left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q\right)_{,i}\mathrm{d}^3x + \int_{\Sigma(t)}\left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}\cdot\delta q\,\mathrm{d}^3x = \int_{\Sigma(t)}\left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}\cdot\delta q\,\mathrm{d}^3x\,;$$

here the first term has dropped out, since by Gauss law it can be rewritten as an integral from $\frac{\partial\mathcal{H}}{\partial q_{,i}}\cdot\delta q$ over the boundary $\partial\Sigma(t)$ (which may possibly lie at infinity), where we assume $\delta q = 0$.

To summarize,

$$\delta S = \int_{t_1}^{t_2}\int_{\Sigma(t)}\left[\delta\Pi\cdot\dot q - \dot\Pi\cdot\delta q - \frac{\partial\mathcal{H}}{\partial q}\cdot\delta q - \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi + \left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}\cdot\delta q\right]\mathrm{d}^3x\,\mathrm{d}t\,,$$

from where we see that

$$\delta S = 0 \qquad \Longleftrightarrow \qquad \dot q := \pounds_t q = \frac{\partial\mathcal{H}}{\partial\Pi}\,, \qquad \dot\Pi := \pounds_t\Pi = -\frac{\partial\mathcal{H}}{\partial q} + \left(\frac{\partial\mathcal{H}}{\partial q_{,i}}\right)_{,i}. \tag{26.3}$$

26.1 Klein-Gordon field and EM field: a warm up

Before embarking on the Einstein equations, let us illustrate the Hamiltonian approach on the Klein-Gordon scalar field and on the electromagnetic field. In the latter case, we will meet the important circumstance which later will also occur in the gravitation problem – thanks to a gauge freedom in the field variables, some of the field equations become constraints. recall lecture at the beginning of semester

Suppose, for simplicity, that we deal with a situation where $N^i = 0$, so, according to (25.7), $g^{\mu\nu} = \mathrm{diag}(-N^{-2}, h^{ik})$. The Lagrangian density* of the Klein-Gordon scalar field ($q \equiv \psi$) then reads

$$\mathcal{L} = -\frac{1}{2}\left(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} + m^2\psi^2\right) = -\frac{1}{2}\left(g^{tt}\dot\psi^2 + h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right).$$

From it, we have

$$\Pi := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot\psi} = -\sqrt{-g}\,g^{tt}\dot\psi \qquad\Longrightarrow\qquad \dot\psi = -\frac{\Pi}{\sqrt{-g}\,g^{tt}}$$

this is really geometrical density which we usually denoted as $\mathscr{L}$

*) here the density means the integrand in the action $\int \cdots \mathrm{d}^3x$

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and the Hamiltonian density *this is now really geometrical density)*

$$\mathcal{H} := \Pi\dot{\psi} - \sqrt{-g}\,\mathcal{L} = -\frac{\Pi^2}{\sqrt{-g}\,g^{tt}} + \frac{\sqrt{-g}}{2}\left(g^{tt}\frac{\Pi^2}{-g(g^{tt})^2} + h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right) =$$
$$= -\frac{\Pi^2}{2\sqrt{-g}\,g^{tt}} + \frac{\sqrt{-g}}{2}\left(h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right),$$

Π is density

from which we finally find evolution equations

$$\dot{\psi} = \frac{\partial\mathcal{H}}{\partial\Pi} = -\frac{\Pi}{\sqrt{-g}\,g^{tt}} = \frac{\Pi N^2}{\sqrt{-g}},$$
$$\dot{\Pi} = -\frac{\partial\mathcal{H}}{\partial\psi} + \left(\frac{\partial\mathcal{H}}{\partial\psi_{,j}}\right)_{,j} = -\sqrt{-g}\,m^2\psi + \left(\sqrt{-g}\,h^{jk}\psi_{,k}\right)_{,j}.$$

This result really leads to the Klein-Gordon equation,

$$\Box\psi \equiv g^{\mu\nu}\psi_{;\mu\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,g^{\mu\nu}\psi_{,\mu}\right)_{,\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,g^{tt}\psi_{,t}\right)_{,t} + \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,h^{jk}\psi_{,j}\right)_{,k} =$$
$$= -\frac{\dot{\Pi}}{\sqrt{-g}} + \frac{1}{\sqrt{-g}}\left(\dot{\Pi} + \sqrt{-g}\,m^2\psi\right) = m^2\psi.$$

EM field

Second, let us test the Hamiltonian approach on a free EM field in the Minkowski space-time ($g_{\mu\nu} = \eta_{\mu\nu}$, $\sqrt{-g} = 1$). Suppose the configuration variable is the four-potential in this case, $q \equiv A_\mu$. We split it to time and spatial components with respect to Σ_t, i.e. to the "scalar" and "vector" potentials

$$\phi := -A_\mu n^\mu, \qquad \vec{A} := A_\mu h^\mu_\alpha,$$

$h^\mu_\alpha = \delta^\mu_\alpha + n^\mu n_\alpha$ $\quad n^\mu$... normal to Σ_t

and write down the Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} = \frac{1}{8\pi}(E^2 - B^2) = \frac{1}{8\pi}\underbrace{(\vec{\nabla}\phi + \dot{\vec{A}})}_{-\vec{E}}\cdot(\vec{\nabla}\phi + \dot{\vec{A}}) - \frac{1}{8\pi}\underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{B}}\cdot(\vec{\nabla}\times\vec{A}),$$

where

$$E_\mu \equiv F_{\mu\nu}n^\nu, \quad B_\mu \equiv -{}^*F_{\mu\nu}n^\nu \qquad (\iff \quad F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu + \epsilon_{\mu\nu\rho\sigma}n^\rho B^\sigma)$$

are the electric and magnetic fields defined with respect to Σ_t. For quantities "living on Σ_t" we have employed the three-vector notation, in particular

$$\vec{E} := -\vec{\nabla}\phi - \dot{\vec{A}}, \qquad \vec{B} := \vec{\nabla}\times\vec{A}.$$

The momenta conjugated to the scalar and vector potentials come out

$$\Pi_t := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\phi}} = 0, \qquad \vec{\Pi} := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\vec{A}}} = \frac{1}{4\pi}(\vec{\nabla}\phi + \dot{\vec{A}}) = -\frac{\vec{E}}{4\pi}.$$

Here comes *the issue*: the first of these relations cannot be inverted, namely, it is not possible to express from it $\dot{\phi}$, so it is also not possible to find the Hamiltonian density $\mathcal{H} = \Pi_t\dot{\phi} + \vec{\Pi}\cdot\dot{\vec{A}} - \mathcal{L}$. *in its standard form* This "accident", related to the gauge freedom of the four-potential, is being remedied in a simple way: if Π_t vanishes identically, it is clearly not appropriate to consider ϕ a *dynamical* variable. If dropping

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ϕ and only leaving $\vec{A}$ as configuration variables, we may continue: we express $\dot{\vec{A}} = 4\pi\vec{\Pi} - \vec{\nabla}\phi$ and submit it to the "restricted" Hamiltonian-density prescription,

$$\mathcal{H} = \vec{\Pi} \cdot \dot{\vec{A}} - \mathcal{L} = \vec{\Pi} \cdot (4\pi\vec{\Pi} - \vec{\nabla}\phi) - 2\pi\vec{\Pi} \cdot \vec{\Pi} + \frac{1}{8\pi}B^2 = 2\pi\Pi^2 - \vec{\Pi} \cdot \vec{\nabla}\phi + \frac{B^2}{8\pi}.$$

The Hamilton equations yield

$$\mathcal{H} = 2\pi \vec{\Pi}^2 + \frac{\vec{B}^2}{8\pi} + \phi \vec{\nabla}\vec{\Pi} \underbrace{- \vec{\nabla}(\phi\vec{\Pi})}_{\text{contributes only to the surface term}}$$

$$\dot{\vec{A}} = \frac{\partial \mathcal{H}}{\partial \vec{\Pi}} = 4\pi\vec{\Pi} - \vec{\nabla}\phi = -\vec{E} - \vec{\nabla}\phi,$$

$$\dot{\vec{\Pi}} \left(= -\frac{\dot{\vec{E}}}{4\pi}\right) = -\cancel{\frac{\partial \mathcal{H}}{\partial \vec{A}}} + \left(\frac{\partial \mathcal{H}}{\partial \vec{A}_{,j}}\right)_{,j} = \frac{1}{8\pi}\left(\frac{\partial B^2}{\partial \vec{A}_{,j}}\right)_{,j} = -\frac{\vec{\nabla} \times \vec{B}}{4\pi}.$$

The last equality of the latter equation can best be computed "in components":

$$B^2 \equiv B_k B^k = \epsilon_{klm} A^{m,l} \epsilon^{kno} A_{o,n} = (\delta^n_l \delta^o_m - \delta^n_m \delta^o_l) A^{m,l} A_{o,n} = A^{m,l}(A_{m,l} - A_{l,m}),$$

so

$$\frac{\partial B^2}{\partial A_{i,j}} = \frac{\partial}{\partial A_{i,j}}\left[A^{m,l}(A_{m,l} - A_{l,m})\right] = \delta^{mi}\delta^{lj}(A_{m,l} - A_{l,m}) + A^{m,l}(\delta^i_m \delta^j_l - \delta^i_l \delta^j_m) =$$
$$= 2(A^{i,j} - A^{j,i}) \equiv 2F^{ji}$$
$$\Longrightarrow \quad \left(\frac{\partial B^2}{\partial A_{i,j}}\right)_{,j} = 2F^{ji}{}_{,j} = 2\,\epsilon^{jik} B_{k,j} \equiv -2(\vec{\nabla} \times \vec{B})^i.$$

ϕ is the the Lagrange multiplier

$$\frac{\delta H}{\delta \varphi} = 0 \Rightarrow \vec{\nabla}\vec{E} = 0$$

Constraint

The first equation thus reproduces the expression of $\vec{E}$ in terms of the potentials, thanks to which (plus $\vec{B} = \vec{\nabla} \times \vec{A}$) holds the second set of Maxwell equations, $\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \dot{\vec{A}} = -\dot{\vec{B}}$ and $\vec{\nabla} \cdot \vec{B} = 0$. The second Hamilton equation yields the Maxwell equation $\vec{\nabla} \times \vec{B} = \dot{\vec{E}}$. Finally, for consistence, it is necessary to add the equation for the derivative by non-dynamical variable ϕ,

$$0 = \dot{\Pi}_t = -\cancel{\frac{\partial \mathcal{H}}{\partial \phi}} + \left(\frac{\partial \mathcal{H}}{\partial \phi_{,j}}\right)_{,j} = -\left[\frac{\partial(\vec{\Pi} \cdot \vec{\nabla}\phi)}{\partial \phi_{,j}}\right]_{,j} = -\left(\Pi^j\right)_{,j} = -\vec{\nabla} \cdot \vec{\Pi} = \frac{1}{4\pi}\vec{\nabla} \cdot \vec{E}.$$

This Maxwell equation represents a **constraint**. (Sure, it cannot be an evolution equation, because it does not contain the time derivative. The same also applies to the similar equation $\vec{\nabla} \cdot \vec{B} = 0$.)

The EM field thus exemplifies the **Hamiltonian system with a constraint**. This more delicate type of problem occurs when the configuration variables possess a gauge freedom. As a consequence, some of them are *not dynamical*, effectively playing the role of Lagrangian multipliers which enforce the fulfilment of certain **constraints**.

26.2 Gravitational field

We will start from the Lagrangian density $\mathcal{L}_g = R - 2\Lambda$, rewriting it to the "3+1" form. For simplicity, we will not take into account "surface" terms – those given by divergence of some vector field (here we have covariant divergence in mind), because such can be expressed, thanks to the Gauss law, as integrals from the flows of the respective vector fields over the boundary of the integration region. Let us emphasize, however, that we are speaking now about the *Lagrangian itself* rather than about

Hamiltonian formalism for gravitational field (in full General Relativity)

In the standard Hamiltonian (canonical) formalism dynamical variables are the canonical coordinates (q) and momenta (p) which can be initially chosen arbitrarily and the Hamilton equations determine their evolution in time.

However, in GR ${}^{(4)}g_{\mu\nu}$ and ${}^{(4)}\dot{g}_{\mu\nu}$ cannot be specified freely because of constraints. In GR the dynamical variables and kinematical variables - space & time are interconnected. We already saw this in the formulation of the Cauchy problem - the metric $h_{\mu\nu}$ and lapse N and shift N_α.

It is instructive to reformulate classical dynamics so that it becomes „closer" to the Hamiltonian formulation of general relativity...

Parametrized formalism for a non-relativistic particle in classical mechanics

First consider just 1-dimensional problem: particle in the potential $V(x)$

X ... cartesian coordinate, T ... Newtonian time

Action
$$S = \int L\, dT = \int dT \left[\frac{1}{2} m \left(\frac{dX}{dT}\right)^2 - V(X)\right]$$

We now do not seek $X = X(T)$ but express T as a function of arbitrary „coordinate" time t
$$T = T(t) \quad \longleftrightarrow \quad t = t(T)$$
and search $\quad X = X(t), \quad T = T(t)$

When we find this, we can return back to the „deparametrized" form $X = X(T)$.

In the parametrized form X and T appear in a more „equivalent" way

Let $\quad \dot{} \equiv \frac{d}{dt}$

Then action becomes
$$S = \int dt\, \dot{T} \left[\frac{1}{2} m \frac{\dot{X}^2}{\dot{T}^2} - V(x)\right] =$$
$$= \int dt \underbrace{\left[\frac{1}{2} m \frac{\dot{X}^2}{\dot{T}} - V(x)\, \dot{T}\right]}_{\text{(new) Lagrangian } \mathcal{L}} \qquad \mathcal{L} = \mathcal{L}(X, T, \dot{X}, \dot{T}, \cancel{t})$$

$\mathcal{L}$ indep. of t

Define momentum

$$\Pi_x = \frac{\partial \mathcal{L}}{\partial \dot{X}} = m \frac{\dot{X}}{\dot{T}}$$

Hamilton form of action:

$$S = \int dt \underbrace{\left[\Pi_x \dot{X} - \underbrace{\left(\frac{1}{2m} \Pi_x^2 + V(x)\right) \dot{T}}_{\mathcal{H}} \right]}_{\mathcal{L}} \qquad (*)$$

We can freely vary X and Π_x; $T(t)$ is prescribed
But we can in fact vary also T (it is function of indp. variable t). Will get

$$\delta S = \int \dots - \underbrace{\left(\frac{1}{2m} \Pi_x^2 + V(x)\right) \frac{d}{dt} \delta T}$$

by parts $\rightsquigarrow \left\{ \frac{d}{dt} \left(\frac{1}{2m} \Pi_x^2 + V(x)\right) \right\} \delta T$ (arbitrary)

$$- \left[\frac{d}{dt} \left(\frac{1}{2m} \Pi_x^2 + V\right) \delta T \right] = 0$$

$\delta T = 0$ on time boundary

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2m} \Pi_x^2 + V(x) \right) =$$

$$= \left[\frac{1}{2m} \Pi_x^2 + V(X) \right]^{\bullet} = 0$$

Integrand in $(*)$ is linear in $\dot{T}$ and $\dot{X}$

$\Rightarrow$ we introduce Π_T, $\Pi_T = \frac{\partial \mathcal{L}}{\partial \dot{T}} = -\left(\frac{1}{2m} \Pi_x^2 + V(x)\right)$ (see $(*)$)

is energy ... momentum (canonical) conjugated with T

Hence the action now reads

$$S = \int dt \left[\Pi_x \dot{x} + \Pi_T \dot{T} \right]$$

it has homogeneous form.

However, momentum Π_T cannot be freely varied. It has to satisfy the constraint

$$\Pi_T = -\left(\frac{1}{2m} \Pi_x^2 + V(x) \right)$$

We rewrite it into the form

$$\mathcal{H} \equiv \Pi_T + \frac{1}{2m} \Pi_x^2 + V(x) = 0$$

And add it to the action multiplied by Lagrange multiplier $(-N)$:

$$\boxed{S = \int dt \left[\Pi_x \dot{x} + \Pi_T \dot{T} - N \mathcal{H} \right]}$$

Here X, Π_x and T, Π_T and N can now be varied freely and all correct equations of motion follow (varying N, the constraint $\mathcal{H} = 0$ follows).

This is fully analogical to the gravitational action as we shall see!

$N \leftrightarrow$ analogue of the lapse function

$\mathcal{H} \leftrightarrow$ analogue of superhamiltonian

for solutions („on-shell") $\mathcal{H} = 0$

Analogue of the shift vector only if consider field theory in Minkowski and introduce general coordinates t, ξ^i instead of Minkowski T, X^i (hope in good „seminar")

For a single free relativistic particle

$$L = -m\sqrt{1-\delta_{ik}\frac{dX^i}{dT}\frac{dX^k}{dT}}$$

similarly, after parameterizing with t, we got the super-Hamiltonian constraint

$$\tilde{\mathcal{H}} = \Pi_T + \sqrt{\delta^{ik}\Pi_i\Pi_k + m^2} = 0$$

In the more convenient squared form,

$$\mathcal{H} = -\Pi_T^2 + \delta^{ik}\Pi_i\Pi_k + m^2 \Rightarrow$$

$$\mathcal{H} = \eta^{\iota\kappa}\Pi_\iota\Pi_\kappa + m^2 = 0$$

$$\eta_{\iota\kappa} = (-+++)$$

Writing down the action as

$$S = \int dt\left(\Pi_\iota \dot{X}^\iota - N\left(-\Pi_T^2 + \delta^{ik}\Pi_i\Pi_k + m^2\right)\right)$$

Varying with respect to Π_T and requiring $\delta S = 0$,

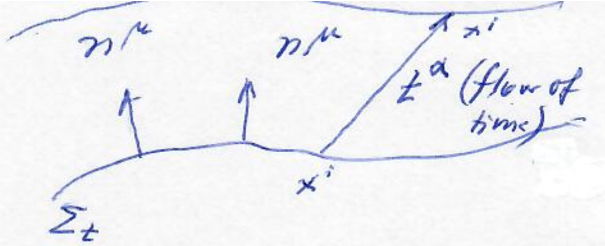
$$\delta S = \int dt\left[\delta\Pi_T\dot{T} + 2\Pi_T N\delta\Pi_T\right] = 0$$

for arbitrary $\delta\Pi_T$

$$\Rightarrow \quad N = \frac{1}{2}\frac{\dot{T}}{\sqrt{m^2 + \delta^{ik}\Pi_i\Pi_k}}$$

Can also parametrize e.g. scalar field propagating in flat spacetime with Minkowski $X^\iota = (T, X^i)$ parametrize by curvilinear coordinates $x^\iota = (t, x^i)$

$\Rightarrow$ analogue of lapse & shift – see seminar



$K_{\mu\nu} \equiv n_{\alpha;\beta}\, h^\alpha_\mu h^\beta_\nu$ extrinsic curvature

$= n_{\mu;\nu} + a_\mu n_\nu = \frac{1}{2} \pounds_n h_{\mu\nu}$ (24.12)

$a_\mu := n_{\mu;\beta} n^\beta$, "acceleration" of n^μ

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its *variation*, so we are *not* claiming that the "surface" terms vanish as a consequence of vanishing of the variations on the boundary – actually, we saw in Section 23.2.6 that the "surface" parts of the Lagrangian typically *aren't* zero and depend on the geometric properties of the boundary.

Will be suitable to slightly modify the 3+1 form of the scalar curvature (25.35). If writing (remember that $K = n^\alpha{}_{;\alpha}$)

$$K_{;\gamma} n^\gamma = (K n^\gamma)_{;\gamma} - K n^\gamma{}_{;\gamma} = (K n^\gamma)_{;\gamma} - K^2 ,$$

the Ricci scalar can be cast into the form (24.35) in M. Mezera talk (O. S. notes)

4d

$$R = {}^{(3)}R + K^2 + K_{\nu\gamma}K^{\nu\gamma} - 2a^\delta{}_{;\delta} + 2K_{,\gamma}n^\gamma =$$
$$= {}^{(3)}R - K^2 + K_{\nu\gamma}K^{\nu\gamma} - 2a^\delta{}_{;\delta} + 2(Kn^\gamma)_{;\gamma} . \tag{26.4}$$

3d

Therefore, if omitting the surface terms (given by divergences), the gravitational Lagrangian density reads *)

$$\boxed{\mathcal{L}_{\rm g} = {}^{(3)}R - K^2 + K_{\nu\gamma}K^{\nu\gamma} - 2\Lambda} = {}^{(3)}R + K_{\kappa\lambda}K_{\rho\sigma}(h^{\kappa\rho}h^{\lambda\sigma} - h^{\kappa\lambda}h^{\rho\sigma}) - 2\Lambda . \tag{26.5}$$

As configuration variables on a given Cauchy hypersurface Σ_t, we choose the latter's metric $h_{\mu\nu}$, and the lapse and shift functions N and N_α. Wishing to define the respective canonical-momentum densities as derivatives of the Lagrangian density by "velocities"

$$\dot h_{\mu\nu} := h^\alpha_\mu h^\beta_\nu \pounds_{\mathbf t} h_{\alpha\beta} , \qquad \dot N := \pounds_{\mathbf t} N = N_{,\sigma} t^\sigma , \qquad \dot N_\alpha := h^\beta_\alpha \pounds_{\mathbf t} N_\beta ,$$

we must express the $\mathcal{L}_{\rm g}$ in terms of the latter. The time derivative of $h_{\mu\nu}$ occurs in the extrinsic curvature, as we know from equation (25.14), i.e. from the formula $K_{\mu\nu} = \frac{1}{2N} h^\alpha_\mu h^\beta_\nu (\pounds_{\mathbf t} h_{\alpha\beta} - \pounds_{\mathbf N} h_{\alpha\beta})$. Rewriting its second term as

$$\pounds_{\mathbf N} h_{\alpha\beta} = h_{\alpha\beta;\sigma}N^\sigma + N^\sigma{}_{;\alpha}h_{\sigma\beta} + N^\sigma{}_{;\beta}h_{\alpha\sigma} = h_{\alpha\beta;\sigma}h^\sigma_\rho t^\rho + N_{\sigma;\alpha}h^\sigma_\beta + N_{\sigma;\beta}h^\sigma_\alpha$$
$$\Longrightarrow \quad h^\alpha_\mu h^\beta_\nu \pounds_{\mathbf N} h_{\alpha\beta} = \cancel{h_{\mu\nu|\rho}}\, t^\rho + N_{\nu|\mu} + N_{\mu|\nu} , \tag{26.6}$$

the extrinsic curvature assumes the form

$$\boxed{K_{\mu\nu} = \frac{1}{2N} h^\alpha_\mu h^\beta_\nu (\pounds_{\mathbf t} h_{\alpha\beta} - \pounds_{\mathbf N} h_{\alpha\beta}) = \frac{1}{2N}\left(\dot h_{\mu\nu} - N_{\nu|\mu} - N_{\mu|\nu}\right).} \tag{26.7}$$

Now it is possible to define the "momenta" canonically conjugated to $h_{\mu\nu}$,

$$\boxed{\Pi^{\mu\nu} := \frac{\partial(\sqrt{-g}\mathcal{L}_{\rm g})}{\partial \dot h_{\mu\nu}}} = \frac{\partial(\sqrt{-g}\mathcal{L}_{\rm g})}{\partial K_{\alpha\beta}} \frac{\partial K_{\alpha\beta}}{\partial \dot h_{\mu\nu}} .$$

Substituting from (26.5)

$$\frac{\partial(\sqrt{-g}\mathcal{L}_{\rm g})}{\partial K_{\alpha\beta}} = \sqrt{-g}\left(\delta^\alpha_\kappa \delta^\beta_\lambda K_{\rho\sigma} + K_{\kappa\lambda}\delta^\alpha_\rho\delta^\beta_\sigma\right)\left(h^{\kappa\rho}h^{\lambda\sigma} - h^{\kappa\lambda}h^{\rho\sigma}\right) = 2\sqrt{-g}\,(K^{\alpha\beta} - Kh^{\alpha\beta})$$

and from (26.7) $\frac{\partial K_{\alpha\beta}}{\partial \dot h_{\mu\nu}} = \frac{1}{2N}\delta^\mu_\alpha \delta^\nu_\beta$, we finally arrive at

$$\boxed{\Pi^{\mu\nu} = \frac{\sqrt{-g}}{N}(K^{\mu\nu} - Kh^{\mu\nu}) = \sqrt{h}\,(K^{\mu\nu} - Kh^{\mu\nu}) .} \tag{26.8}$$

$\Rightarrow K^{\mu\nu} = \frac{1}{\sqrt{h}}\left(\Pi^{\mu\nu} - \frac{\Pi}{2} h^{\mu\nu}\right)$, $\Pi = -2\sqrt{h}K$

*) again, the "density" here is used as "integrand" in total L_g
"real" (geometrical) density is $\sqrt{-g}R = \sqrt{h}N\left[{}^{(3)}R + K_{\alpha\beta}K^{\alpha\beta} - K^2\right]$

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The momenta conjugated with N and N_α identically vanish, because Lagrangian does not contain $\dot{N}$ and $\dot{N}_\alpha$. In analogy with the situation occurring in electrodynamics, we interpret it in such a way that N, N_α in fact *aren't dynamical variables*. Similarly as in the EM case, it is related to gauge freedom: N and N_α do not describe any *intrinsic* properties of space-time, they are *elective* components of the time vector t^μ. Specifically, N tells "how far" it is from Σ_{t_1} to Σ_{t_2}, so it scales the time coordinate, and N_α specify the coordinates that cover the Σ_t hypersurfaces. In other words, the choice of N and N_α basically corresponds to the choice of coordinates and it is rather arbitrary. Hence, we will not take these quantities into account in designing the Lagrangian, we will rather understand them as Lagrange multipliers which will then enforce certain constraints within the Hamilton equations.

Now we have everything to find the gravitational part of the Hamiltonian density, more accurately its "bulk" part (surface terms are not included, as already stressed above),

$$\begin{aligned}
&\mathcal{H}_{\mathrm{g}} = \Pi^{\mu\nu}\dot{h}_{\mu\nu} - \sqrt{-g}\,\mathcal{L}_{\mathrm{g}} = \\
&= \sqrt{h}\,(K^{\mu\nu} - Kh^{\mu\nu})(2NK_{\mu\nu} + N_{\nu|\mu} + N_{\mu|\nu}) - N\sqrt{h}\left({}^{(3)}\!R - K^2 + K_{\mu\nu}K^{\mu\nu} - 2\Lambda\right) = \\
&= 2\sqrt{h}\,(K^{\mu\nu} - Kh^{\mu\nu})\,N_{\mu|\nu} + N\sqrt{h}\left(K_{\mu\nu}K^{\mu\nu} - K^2 - {}^{(3)}\!R + 2\Lambda\right).
\end{aligned}$$

Employing the formula (26.8) for momenta $\Pi^{\mu\nu}$, one easily finds the relations

$$\begin{aligned}
&\Pi_{\mu\nu}\Pi^{\mu\nu} = h\left(K_{\mu\nu}K^{\mu\nu} + K^2\right), \qquad \Pi^2 := (\Pi^\mu_\mu)^2 = (-2\sqrt{h}\,K)^2 = 4hK^2 \\
&\Longrightarrow \quad 2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2 = 2h\left(K_{\mu\nu}K^{\mu\nu} - K^2\right),
\end{aligned}$$

thanks to which one finally arrives to

$$\boxed{\mathcal{H}_{\mathrm{g}} = 2\Pi^{\mu\nu}N_{\mu|\nu} + \frac{N}{2\sqrt{h}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) - N\sqrt{h}\left({}^{(3)}\!R - 2\Lambda\right)}\,. \tag{26.9}$$

Note that only the symmetric part of $N_{\mu|\nu}$ matters in the first term.

26.2.1 Hamilton equations: constraints

Let us first look at how *constraints* arise, given by vanishing of the derivatives of $\mathcal{H}_{\mathrm{g}}$ with respect to the gauge-fixing variables N and N_α:

$$\begin{aligned}
&0\left(= \dot{\Pi}_N\right) = -\frac{\partial\mathcal{H}_{\mathrm{g}}}{\partial N} + \cancel{\left(\frac{\partial\mathcal{H}_{\mathrm{g}}}{\partial N_{|\iota}}\right)_{|\iota}} = -\frac{1}{2\sqrt{h}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) + \sqrt{h}\left({}^{(3)}\!R - 2\Lambda\right) \\
&\Longleftrightarrow \quad 2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2 = 2h\left({}^{(3)}\!R - 2\Lambda\right), \qquad (26.10) \\
&0\left(= \dot{\Pi}_{N_\alpha}\right) = -\frac{\partial\mathcal{H}_{\mathrm{g}}}{\partial N_\alpha} + \left(\frac{\partial\mathcal{H}_{\mathrm{g}}}{\partial N_{\alpha|\iota}}\right)_{|\iota} = 2\Pi^{\alpha\nu}{}_{|\nu} \\
&\Longleftrightarrow \quad \Pi^{\alpha\nu}{}_{|\nu} = 0\,. \qquad (26.11)
\end{aligned}$$

The shortest way how to derive the second equation is to rewrite the first term of (26.9) as

$$\Pi^{\mu\nu}N_{\mu|\nu} = (\Pi^{\mu\nu}N_\mu)_{|\nu} - \Pi^{\mu\nu}{}_{|\nu}N_\mu$$

and discard the first, "surface" part given by divergence.

Plugging to the above equations $\Pi^{\mu\nu}$ reveals that they represent the **Hamiltonian and momentum constraints**, (25.40) and (25.41). We beg to point out that the constraints of course *must not* be plugged back into the Hamiltonian – the Hamiltonian has to remain "off shell" (i.e. not evaluated along the actual evolution), in order that one may compute its derivatives "in arbitrary direction".

ReWriting the Hamiltonian density (26.9) in a slightly modified form:

$$\boxed{\mathcal{H}_g = \sqrt{h}\left\{N\left(-{}^{(3)}R + 2\Lambda + \frac{1}{h}\Pi_{\mu\nu}\Pi^{\mu\nu} - \frac{1}{2h}\Pi^2\right) - 2N_\mu\left[D_\nu\left(\frac{1}{\sqrt{h}}\Pi^{\nu\mu}\right)\right]\right\}} \quad (*)$$

First three terms are the same as corresponding terms in (26.9), the last term arises from the first term in (26.9): (See also O.S. below (26.11))

$$2\Pi^{\mu\nu}D_\mu N_\nu =$$

$$= \sqrt{h}\left\{-2N_\mu\left[D_\nu\left(\frac{1}{\sqrt{h}}\Pi^{\mu\nu}\right)\right] + \underbrace{2D_\mu\left(\frac{1}{\sqrt{h}}N_\nu\Pi^{\mu\nu}\right)}_{\text{omit- divergence}}\right\}$$

$\Rightarrow \mathcal{H}_g$ above $(*)$

$$= 2N_\nu D_\mu\left(\frac{1}{\sqrt{h}}\Pi^{\mu\nu}\right) + \frac{2}{\sqrt{h}}\Pi^{\mu\nu}D_\mu N_\nu$$

Total Hamiltonian $H_g = \int_{\Sigma_t} \mathcal{H}_g\, d^3x$

From $(*)$ it is clearly seen how lapse and shift can be considered as Lagrange multipliers and in el/mag. case $\mathcal{H}_{el.} = 2\pi\,\Pi^2 + \frac{\vec{B}^2}{8\pi} + \phi\,\vec{\nabla}\vec{\pi}$, $\frac{\delta H}{\delta\phi} = 0$
$\Rightarrow$ constraint $\vec{\nabla}\vec{E} = 0$

Here in gravity

$$\frac{\delta H_g}{\delta N} = 0 \Rightarrow -{}^{(3)}R + 2\Lambda + \frac{1}{h}\Pi_{\mu\nu}\Pi^{\mu\nu} - \frac{1}{2h}\Pi^2 = 0 \qquad \text{(HC)}$$

Hamiltonian constraint

$$\frac{\delta H_g}{\delta N_\mu} = 0 \quad \Rightarrow \quad D_\nu \underbrace{\left(\frac{1}{\sqrt{h}} \Pi^{\nu\mu} \right)}_{} = 0 \quad (MC)$$

momentum constraint   tensor, $\Pi^{\nu\mu}$ density

Substituting the constraints onto the Hamiltonian density (*) one finds

$$\mathcal{H}_g = 0$$

As noticed at the bottom of p. 412 in Q.S. notes, when using $\mathcal{H}_g$ to derive all Hamilton equations which in the end are equivalent to Einstein's eqs, one does <u>not</u> substitute some specific solution, so constraints are not put equal to zero.

<u>Note</u> Very similar situation, also on the level of Lagrangian formalism arises in the Dirac theory, for example.

Standard form of the Dirac equation is

$$\boxed{(i\hbar \hat{\nabla} - mc)\psi = 0,} \qquad (D)$$

$$\hat{\nabla} = \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^0 \frac{\partial}{\partial x^0} + \vec{\gamma}\cdot\vec{\nabla}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_4 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\psi^+ = (\psi_1^*, \cdots \psi_4^*)$$

probability density $\rho = \psi^+ \psi = \sum_{\sigma=1}^{4} \psi_\sigma^* \psi_\sigma$

Define $\bar{\Psi}(x) = \psi^{+}\gamma_0$ conjugated spinor

Lagrangian density from which Dirac eq. (D) follows is — with $\hbar = 1, c = 1$

$$\mathcal{L}(x) = \bar{\Psi}(x)\left(i\hat{\nabla} - m\right)\psi(x).$$

So $\mathcal{L}(x) = 0$ for $\psi(x)$ satisfying Dirac eq. (D)

Still, it is "correct" Lagrangian leading to (D) and various conservation laws.

In fact, notice that for GR the scalar curvature R is the Lagrangian density which is 0 in vacuum.

$$\dot{h}_{\alpha\beta} = \frac{\delta H_G}{\delta \Pi^{\alpha\beta}} = \frac{\partial \mathcal{H}_G}{\partial \Pi^{\alpha\beta}}\,,\quad \dot{\Pi}^{\alpha\beta} = -\frac{\delta H_G}{\delta h_{\alpha\beta}}$$

H33

26.2.2 Hamilton equations: evolutions

The main work is still to be done. The **evolution equations** are given by Hamilton equations

$$\dot{h}_{\alpha\beta} = \frac{\partial \mathcal{H}_g}{\partial \Pi^{\alpha\beta}}\,, \qquad \dot{\Pi}^{\alpha\beta} := h^\alpha_\mu h^\beta_\nu \pounds_t \Pi^{\mu\nu} = -\frac{\partial \mathcal{H}_g}{\partial h_{\alpha\beta}} + \left(\frac{\partial \mathcal{H}_g}{\partial h_{\alpha\beta,\iota}}\right)_{,\iota} + \left(\text{terms given by } \frac{\partial \mathcal{H}_g}{\partial h_{\alpha\beta,\iota\kappa}}\right).$$

Computing

$$\frac{\partial \Pi^2}{\partial \Pi^{\alpha\beta}} = \frac{\partial}{\partial \Pi^{\alpha\beta}}(\Pi^{\kappa\lambda}\Pi^{\rho\sigma}h_{\kappa\lambda}h_{\rho\sigma}) = 2\delta^\kappa_\alpha\delta^\lambda_\beta\Pi^{\rho\sigma}h_{\kappa\lambda}h_{\rho\sigma} = 2\Pi h_{\alpha\beta}\,,$$

we have, from (26.9),

$$\boxed{\dot{h}_{\alpha\beta} = \frac{\partial \mathcal{H}_g}{\partial \Pi^{\alpha\beta}} = 2N_{(\alpha|\beta)} + \frac{N}{\sqrt{h}}(2\Pi_{\alpha\beta} - \Pi h_{\alpha\beta})\,,} \quad \checkmark \tag{26.12}$$

which exactly "repeats" the relations (26.7) and (26.8).

The second equation is more labourious; let us treat each of the terms of (26.9) separately:

- In the last term, $-N\sqrt{h}\left({}^{(3)}R - 2\Lambda\right)$, we use the knowledge from the variational derivation of Einstein equations. We learned there – see equation (23.11) – that if dropping the surface terms (given by behaviour of the metric *derivatives* on the integration-region boundary), then

$$\delta\left[\sqrt{-g}\,(R-2\Lambda)\right] = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}$$
$$\Longrightarrow \quad \frac{\partial}{\partial g^{\mu\nu}}\left[\sqrt{-g}\,(R-2\Lambda)\right] = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right).$$

However, we would prefer to know the derivative with respect to *covariant* metric, which reverses the sign, as we know from Section 23.2.1: specifically, we obtained there $\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}$, so

$$\delta\left[\sqrt{-g}\,(R-2\Lambda)\right] = -\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta}\right)\delta g_{\alpha\beta}\,,$$
$$\Longrightarrow \quad \frac{\partial}{\partial g_{\alpha\beta}}\left[\sqrt{-g}\,(R-2\Lambda)\right] = -\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta}\right).$$

In the 3D analogy to this result, we can claim that

$$\frac{\partial}{\partial h_{\alpha\beta}}\left[\sqrt{h}\left({}^{(3)}R - 2\Lambda\right)\right] = -\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right). \tag{26.13}$$

- In the middle term of (26.9), $\frac{N}{2\sqrt{h}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right)$, we rewrite

$$2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2 = \Pi^{\kappa\lambda}\Pi^{\mu\nu}\left(2h_{\kappa\mu}h_{\lambda\nu} - h_{\kappa\lambda}h_{\mu\nu}\right)$$

in order to differentiate it by $h_{\mu\nu}$,

$$\frac{\partial}{\partial h_{\alpha\beta}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) = \Pi^{\kappa\lambda}\Pi^{\mu\nu}\left(2\delta^\alpha_\kappa\delta^\beta_\mu h_{\lambda\nu} + 2h_{\kappa\mu}\delta^\alpha_\lambda\delta^\beta_\nu - \delta^\alpha_\kappa\delta^\beta_\lambda h_{\mu\nu} - h_{\kappa\lambda}\delta^\alpha_\mu\delta^\beta_\nu\right) =$$
$$= 4\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - 2\Pi\Pi^{\alpha\beta}\,.$$

H_{34}

Recalling Section 23.2.1 once again, specifically equation (23.6) for the derivative of the metric determinant, $\frac{\partial(-g)}{\partial g_{\mu\nu}} = (-g)g^{\mu\nu}$, we analogously take $\frac{\partial h}{\partial h_{\alpha\beta}} = hh^{\alpha\beta}$ here on Σ_t, so

$$\frac{\partial}{\partial h_{\alpha\beta}}\left(\frac{1}{\sqrt{h}}\right) = -\frac{1}{2h^{3/2}}\frac{\partial h}{\partial h_{\alpha\beta}} = -\frac{1}{2h^{3/2}}\,hh^{\alpha\beta} = -\frac{1}{2\sqrt{h}}\,h^{\alpha\beta}\,.$$

Hence, in total,

$$\begin{aligned}&\frac{\partial}{\partial h_{\alpha\beta}}\left(\frac{2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2}{\sqrt{h}}\right) = \\ &= \frac{2}{\sqrt{h}}\left(2\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - \Pi\Pi^{\alpha\beta}\right) - \frac{1}{2\sqrt{h}}\,h^{\alpha\beta}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right).\end{aligned} \tag{26.14}$$

- Finally, in the first term of (26.9), $2\Pi^{\mu\nu}N_{\mu|\nu}$, the momenta $\Pi^{\mu\nu}$ are taken as independent of $h_{\alpha\beta}$, and we rewrite

$$N_{\mu|\nu} = h_{\mu\kappa}N^{\kappa}{}_{|\nu} = h_{\mu\kappa}N^{\kappa}{}_{,\nu} + \frac{1}{2}\left(h_{\mu\nu,\lambda} + h_{\lambda\mu,\nu} - h_{\nu\lambda,\mu}\right)N^{\lambda}\,,$$

so

$$\begin{aligned}\frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta}} &= 2\Pi^{\alpha\nu}N^{\beta}{}_{,\nu} = \Pi^{\alpha\nu}N^{\beta}{}_{,\nu} + \Pi^{\beta\nu}N^{\alpha}{}_{,\nu}\,,\\ \frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta,\iota}} &= \Pi^{\mu\nu}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\iota}_{\lambda} + \delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\iota}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\lambda}\delta^{\iota}_{\mu}\right)N^{\lambda} = \Pi^{\alpha\beta}N^{\iota} + \cancel{\Pi^{\beta\iota}N^{\alpha} - \Pi^{\iota\alpha}N^{\beta}}\end{aligned}$$

(the last forms have been claimed on the basis of the necessary symmetry of both the expressions in α and β). Putting the two terms together, we thus have

$$\begin{aligned}&-\frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta}} + \left[\frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta,\iota}}\right]_{,\iota} = \\ &= \Pi^{\alpha\beta}{}_{,\iota}N^{\iota} + \Pi^{\alpha\beta}N^{\iota}{}_{,\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{,\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{,\nu} = \\ &= \left(\Pi^{\alpha\beta}{}_{|\iota} - \cancel{\Gamma^{\alpha}_{\iota\sigma}\Pi^{\sigma\beta}} - \cancel{\Gamma^{\beta}_{\iota\sigma}\Pi^{\alpha\sigma}}\right)N^{\iota} + \Pi^{\alpha\beta}N^{\iota}{}_{,\iota} - \\ &\quad - \Pi^{\alpha\nu}\left(N^{\beta}{}_{|\nu} - \cancel{\Gamma^{\beta}_{\nu\sigma}N^{\sigma}}\right) - \Pi^{\beta\nu}\left(N^{\alpha}{}_{|\nu} - \cancel{\Gamma^{\alpha}_{\nu\sigma}N^{\sigma}}\right) = \\ &= \Pi^{\alpha\beta}{}_{|\iota}N^{\iota} + \Pi^{\alpha\beta}N^{\iota}{}_{,\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu} = \\ &= \Pi^{\alpha\beta}{}_{|\iota}N^{\iota} + \Pi^{\alpha\beta}\sqrt{h}\left(\frac{N^{\iota}}{\sqrt{h}}\right)_{|\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu} = \\ &= \sqrt{h}\,(h^{-1/2}\Pi^{\alpha\beta}N^{\iota})_{|\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu}\,.\end{aligned}$$

Therefore, collecting the above results for the three Hamiltonian terms, we conclude that

$$\boxed{\begin{aligned}\dot{\Pi}^{\alpha\beta} &:= h^{\alpha}_{\mu}h^{\beta}_{\nu}\pounds_{t}\Pi^{\mu\nu} = -\frac{\partial\mathcal{H}_{g}}{\partial h_{\alpha\beta}} + \left(\frac{\partial\mathcal{H}_{g}}{\partial h_{\alpha\beta,\iota}}\right)_{,\iota} = \\ &= -N\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right) - \\ &\quad - \frac{N}{\sqrt{h}}\left(2\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - \Pi\Pi^{\alpha\beta}\right) + \frac{N}{4\sqrt{h}}\,h^{\alpha\beta}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) + \\ &\quad + \sqrt{h}\,(h^{-1/2}\Pi^{\alpha\beta}N^{\iota})_{|\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu}\,.\end{aligned}} \tag{26.15}$$

Summary: Constraints (HC), p. , (MC), p. and evolution equations (26.12) and (26.15) $\Longleftrightarrow R_{\mu\nu} = 0$!
„constrained Hamiltonian form of GR"

Additional „observations" on Hamiltonian formalism

A) The problem of configuration and phase spaces in the systems with constraints

I) Electromagnetism

Since there are constraints, our phase is "too big" - some of phase variables can be exluded from constraints

the constraint $\vec{\nabla}\vec{E} = 0 \Leftrightarrow \vec{\nabla}\vec{\pi} = 0$ (recall $\vec{\pi} = -\vec{E}$)

Phase variables are $\vec{A}$ and $\vec{\pi}$

But $\vec{A}$ is not unique $\vec{A} - \vec{\nabla}\Lambda$, $\Lambda = \Lambda(x^\mu)$

gives the same physical electromagnetic field

$\Rightarrow$ configuration space should be the classes of equivalence of vector potentials

denote a class $\vec{\tilde{A}}$ ... it is represented by some $\vec{A}$ and all $\vec{A} - \vec{\nabla}\Lambda$.

To get an associated cotangent space of momenta we form the space of linear variations of $\tilde{A}$ which depends only on the equivalence classes, i.e., $\vec{\pi}$

Form space of linear variations of $\vec{A}$ which depends only on equivalence class(es):

(EH) $$\int \vec{\pi}\,[\delta(\vec{A} - \vec{\nabla}\Lambda)]\,d^3x = \int \vec{\pi}\,\delta\vec{A}\,d^3x$$

i.e. $$\int \vec{\pi}\,\vec{\nabla}\delta\Lambda = \underbrace{\int \vec{\nabla}(\delta\Lambda\,\pi)}_{\text{boundary}} - \int \delta\Lambda\,\vec{\nabla}\vec{\pi} = 0 \quad (\text{omitting } "d^3x")$$

$$\Rightarrow \vec{\nabla}\vec{\pi} = 0 \checkmark \quad \text{constraint}$$

So new configuration space is formed from

$\tilde{\vec{A}}$ and $\vec{\pi}$ satisfying $\vec{\nabla}\vec{\pi}=0$

The Hamiltonian then becomes simply

$$\tilde{\mathcal{H}}_{ELM} = \frac{1}{2}\left(\vec{\pi}\vec{\pi} + \vec{B}\vec{B}\right) \qquad \cancel{+\varphi\,\underbrace{\vec{\nabla}\vec{\pi}}_{=0}}$$

where $\vec{B} = \vec{\nabla}\times\vec{A}$ which does not change when $\vec{A}\to\vec{A}-\vec{\nabla}\Lambda$ so $\vec{B}$ is function of the class $\tilde{\vec{A}}$ in fact —

Hamilton equations following from $\tilde{\mathcal{H}}_{ELM}$ then imply

$$(1) \qquad \dot{\tilde{\vec{A}}} = \frac{\delta\tilde{H}_{ELM}}{\delta\vec{\pi}} = \tilde{\vec{\pi}} = \vec{\pi}\ (\text{must be } \vec{\nabla}\vec{\pi}=0)$$

$$(2) \qquad \dot{\vec{\pi}} = -\frac{\delta\tilde{H}_{ELM}}{\delta\tilde{\vec{A}}} = -\vec{\nabla}\times\vec{B}$$

In (1) there are equivalence classes but these can be eliminated by applying $\vec{\nabla}\times$ ("rot, curl") on both sides

$$\vec{\nabla}\times\dot{\tilde{\vec{A}}} = \vec{\nabla}\times(\dot{\vec{A}}-\vec{\nabla}\dot{\Lambda}) = \vec{\nabla}\times\dot{\vec{A}}$$

Concerning $\vec{\pi}$, recall that every vector can be decomposed in two vectors, $\vec{\pi}=\vec{\pi}_0+\vec{\pi}_1$, one "divergence-free", the second "curl-free", i.e. $\vec{\nabla}\cdot\vec{\pi}_0=0$, $\vec{\nabla}\times\vec{\pi}_1=0$

Substituting into (1), one gets after taking $\vec{\nabla}\times$ on r.h.s.

$$\vec{\nabla}\times\tilde{\vec{\pi}} = \vec{\nabla}\times\vec{\pi}_0 \quad \text{for which} \quad \vec{\nabla}\cdot\vec{\pi}_0 = 0$$

Then (1) and (2) with $\vec{E}=-\vec{\pi}$ is equiv. to Maxwell eqs.

II) Gravitation (GR)

Similar to electromagnetism – gauge choice for $h_{\alpha\beta}$ – induced by gauge transformation $x'^{\alpha} = x^{\alpha} + w^{\alpha}(x)$

$h_{\alpha\beta}$ and $h'_{\alpha\beta} = h_{\alpha\beta} + D_{(\alpha} w_{\beta)}$ diffeom ψ on Σ_t

So we should consider equivalence classes $\psi^* h_{\alpha\beta}$

$\tilde{h}_{\alpha\beta}$ Riemannian metrics on Σ_t are equivalent if one can be transformed to another by a diffeomorphism

Configuration space formed by such equivalence classes is called usually <u>superspace</u> (J.A. Wheeler – Mark Twain – choose „correct word" – difference between „lightening" × „lightening bug")

Superspace: all 3-geometries – any point in superspace describes a complete 3-geometry, with all curvatures ...

„Geometrodynamics" (also Wheeler) analyzes how the spatial (3-dim) geometry of a hypersurface Σ_t changes if we push it through a spacetime. As one evolves the 3-geometry, it moves along a path in superspace.

Formally, if Riem(M) is the set of all spatial metrics defined over a 3-dim manifold, superspace

$$\mathcal{S}(M) = \mathrm{Riem}(M) / \underbrace{\mathrm{Diff}(M)}_{\text{diffeomorphism on } M}$$

In analogy with (EM) on p. H_{35}, i.e.

$$\int \vec{\pi}\left[\delta(\vec{A}-\vec{\nabla}\Lambda)\right] d^3x = \int \vec{\pi}\,\delta\vec{A}\, d^3x$$

now the gravitational canonical momentum $\pi^{\alpha\beta}$ must satisfy

$$\int \pi^{\alpha\beta}\left(\delta h_{\alpha\beta} + D_{(\alpha} w_{\beta)}\right) d^3x = \int \pi^{\alpha\beta}\,\delta h_{\alpha\beta}\, d^3x$$

$$\Longrightarrow \int \pi^{\alpha\beta} D_{(\alpha} w_{\beta)}\, d^3x = 0, \text{ or}$$

$$0 = \tfrac{1}{2}\left[\int \pi^{\alpha\beta} D_\alpha w_\beta\, d^3x + \int \pi^{\alpha\beta} D_{\underset{\downarrow}{\beta}}\, w_{\underset{\downarrow}{\alpha}}\, d^3x\right]$$

$$\qquad\qquad\qquad\qquad \alpha \quad \beta$$

recall $\pi^{\alpha\beta} = \underbrace{\sqrt{h}}_{\sqrt{\det h_{\mu\nu}}}\, K^{\alpha\beta}$ ← tensor ⟶ so $\dfrac{\pi^{\alpha\beta}}{\sqrt{h}}$ is tensor

↑ density on Σ

$$0 = \int \pi^{\alpha\beta} D_\alpha w_\beta \frac{\sqrt{h}}{\sqrt{h}}\, d^3x = \int \underbrace{D_\alpha\left(\frac{1}{\sqrt{h}}\pi^{\alpha\beta} w_\beta\right)}_{\text{divergence } \to 0} \sqrt{h}\, d^3x$$

$$- \int w_\beta \left(D_\alpha\left(\frac{\pi^{\alpha\beta}}{\sqrt{h}}\right)\right)\sqrt{h}\, d^3x$$

w_β arbitrary

$$\Rightarrow \boxed{D_\alpha\left(\frac{\pi^{\alpha\beta}}{\sqrt{h}}\right) = 0}$$

this is the momentum constraint

it is thus, automatically satisfied

The momentum constraint is eliminated by the choice of superspace as the configuration space („ classes of 3-geometries ")

But the Hamilton constraint remains!

It is analogical to the constraint which arises when we parameterized classical theory for the motion of a particle on a fixed background by introducing into the Lagrangian a new time function t (in addition to the original T see p. H_{25}) and consider it to be a dynamical variable. But in "ordinary" theories we can the theory „deparametrize back" by solving the constraint for the momentum conjugated with the time function.

However, in GR the Hamiltonian constraint is quadratic in momenta and such „deparameterization does not seems to be possible.

It appears impossible to find a configuration space for which in the associated phase space are contained only „real dynamical degrees of freedom"

It appears impossible to get rid of the Hamiltonian constraint.

Obstacle in the canonical quantization of General Relativity ?