Cauchy problem

initial value problem

- last time for classical fields propagating in flat spacetime ($\mathcal{C}1$ - $\mathcal{C}21$)
- today*) Cauchy problem in GR mostly for Einstein's vacuum field equations

mostly stated theorems
without proofs but with an „understanding"
some calculations indicated - often
quite long (for 3+1 splitting, see Text by O.S.)
pages enumerated by $\mathcal{C}_g$ X ...

*) and very probably next time

The generalization of the <u>Cauchy problem</u> for <u>generalized Klein-Gordon equation</u> which is linear but on general curved spacetime

$$\boxed{g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + A^\mu\nabla_\mu\phi + B\phi + C = 0} \quad (EI)$$

∇_μ – covariant derivative (but could be "∂_μ" since highest derivatives „matter")

$g^{\mu\nu}(x)$ – <u>given</u>, smooth Lorentzian metric,
i.e., there a possibility of $x^\mu \to X^\mu$ in which $\underbrace{g^{\mu\nu} \to \eta^{\mu\nu}}_{\text{„hyperbolicity"}}$
A^μ, B, C smooth functions of x (x^δ),
$(M, g_{\mu\nu})$ globally hyperbolic

<u>Thm</u>: Equation (EI) admits a well-posed initial value problem by giving, on Cauchy hs. Σ with normal n^α function $\phi|_\Sigma$ and $n^\alpha\nabla_\alpha\phi|_\Sigma$ as i.dat

(i.e. solution is unique on M and depends continuously on initial data)

Further generalization for n eqs. for n functions:

$$(EII) \quad \boxed{g^{\mu\nu}\nabla_\mu\nabla_\nu\phi_i + \sum_{j=1}^{n}(A_{ij})^\mu\nabla_\mu\phi_j + \sum_{j=1}^{n}B_{ij}\phi_j + C_i = 0} \quad i = 1,\dots n$$

Conditions on $g^{\mu\nu}$, A_{ij}, B_{ij}, C_i as „above"
initial data on Cauchy hypers. are ϕ_i, $n^\alpha\nabla_\alpha\phi_i$
again the problem is well-posed
(EII) is linear hyperbolic system of 2nd order

Our goal for Einstein's equations: bring them into the form of a quasilinear, hyperbolic system of the 2nd order:

$$(E\underline{\underline{III}}) \quad \boxed{g^{\alpha\beta}(x; \phi_j, \nabla_\gamma \phi_j)\, \nabla_\alpha \nabla_\beta \phi_i = F_i(x; \phi_j, \nabla_\gamma \phi_j) \qquad i = 1, \ldots n}$$

smooth Lorentzian metric (i.e. can be explicitly $-+++$) implies hyperbolic character

more general than previously:

- $g^{\alpha\beta}$ may depend on $\phi_j, \nabla_\gamma \phi_j$

$-(\ast)\nabla_0\nabla_0\phi_i + (\ast)\nabla_i\nabla_j\phi$ the highest derivatives linearly

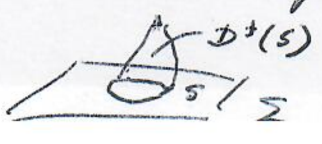
Theorem Leray (1952)

Let $(\phi_0)_1 \ldots (\phi_0)_n$ be solution of $(E\underline{\underline{III}})$ on M which is globally hyperbolic (i.e. contains a global Cauchy hypersurface), Σ smooth hypersurf. in $(M, (g_0)_{\alpha\beta})$, $(g_0)^{\alpha\beta} = g^{\alpha\beta}(x; (\phi_0)_j, \nabla_\gamma(\phi_0)_j)$

Then I.V.F for $(E\underline{\underline{III}})$ is well-posed in the following sense:

If we give initial data sufficiently close to data for $(\phi_0)_i$, then there exists a neighborhood $\mathcal{O}$ of hypersurface Σ (so "leave Σ upwards in time") such that $(E\underline{\underline{III}})$ has solution $\phi_1, \ldots \phi_n$ in neigh. $\mathcal{O}$ and $(\mathcal{O}, g_{\alpha\beta}(x; \phi_j, \nabla_\gamma \phi_j))$ is globally hyperbolic ("new piece of spacetime").

In addition, the solution is causal: if some other data $\tilde{\phi}_i$ agree with ϕ_i on a subset S of Σ then the solution $\tilde{\phi}_i$ agree with ϕ_i on $\mathcal{O} \cap D^+(S)$.



Also, ϕ_i depend continuously on initial data

<u>Indication of the proof</u> - only "rough" but "intuitive"

Assume we know $(\phi_0)_i$. Substitute this known solution to $g^{\alpha\beta}(x, \phi_i, \nabla_\gamma \phi_i)$ on the left-hand side of (E III) and into the right-hand side $F_i(\dots)$ of (E III).

Get the linear system

$$\underbrace{g^{\alpha\beta}(\dots)}_{\text{known}} \nabla_\alpha \nabla_\beta \phi_i = \underbrace{F_i(\dots)}_{\text{known}}$$

Solve this system for new initial conditions by using solution of (E II) (p.).
Substitute the solution found again into $g^{\alpha\beta}(\dots)$ and $F_i(\dots)$ in (E III). Find then solutions $\underset{(2)}{\phi_i}$ using again (E II).

$\Rightarrow$ sequence of solutions $\underset{(k)}{\phi_i}$.

One can then show that "reasonably" close data these solutions converge to the solution of the "new" nonlinear system (E III).

JEAN LERAY

7 November 1906 — 10 November 1998

Elected ForMemRS 1983

BY MARTIN ANDLER

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Jean Leray was one of the major mathematicians of the twentieth century. His primary focus in mathematics came from applications; indeed, a majority of his contributions were in the theory of partial differential equations arising in physics, notably his 1934 paper on the Navier–Stokes equation. World War II, during which he was a prisoner-of-war in Austria for five years, induced him to turn to pure mathematics to avoid helping the German war effort. There he worked in topology, developing two radically new ideas: sheaf theory and spectral sequences. After 1950 he came back to partial differential equations and became interested in complex analysis, writing a remarkable series of papers on the Cauchy problem. Leray remained mathematically active until the end of his life; in the course of his career he wrote 133 papers. His influence on present mathematics is tremendous. On the one hand, sheaf theory and spectral sequences became essential tools in contemporary pure mathematics, reaching far beyond their initial scope in topology. On the other hand, Leray can rightly be considered the intellectual guide of the distinguished French school of applied mathematics.

Jean Leray, who died on 10 November 1998—three days after his 92nd birthday, was a major figure of twentieth-century mathematics. Between 1931, when his first research announcement (in fluid mechanics) was published, and 1999, when his last (posthumous) paper appeared, his complete list of publications contains 133 entries. Leray's collected works (in fact a *Selecta*) (22)*, published shortly before his death by Springer-Verlag and the Société mathématique de France, consists of a three-volume set of 1600 pages, containing 53 papers, mostly in French, written over a period of more than 60 years. Until the mid-1990s he had

* Numbers in this form refer to the bibliography at the end of the text.

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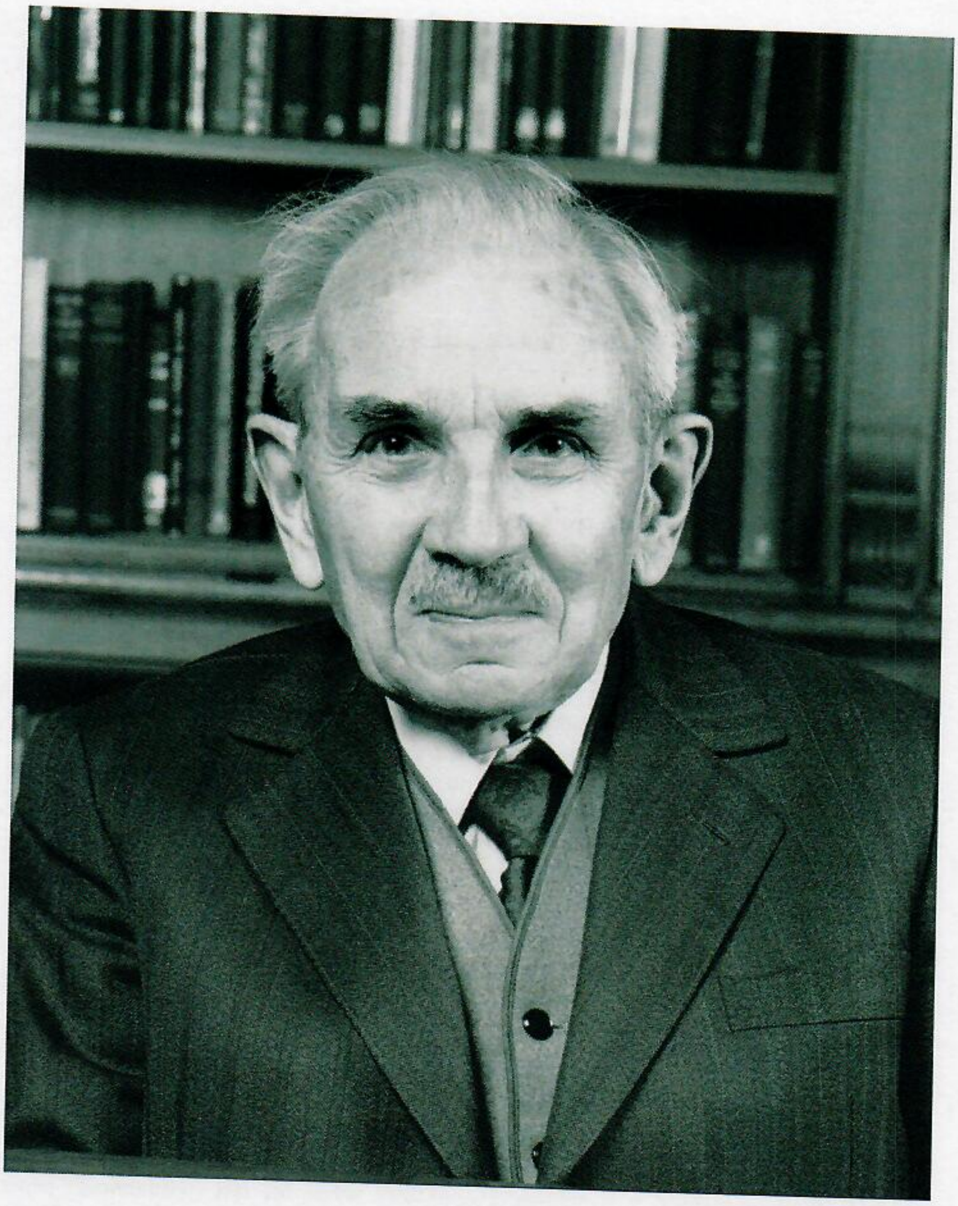

J. Leray

refused to yield to the pressure of publishing his collected works, arguing that he was still working.

Leray worked in three different fields: topology, partial differential equations, and complex analysis. He considered himself an applied mathematician, and it is no easily-solved paradox that a large part of his work should have been in pure mathematics.

When Leray was a student at the École normale supérieure in Paris, between 1926 and 1929, the mathematics faculty at the University of Paris was not at its best: long gone were the prewar days when French analysts were inventing modern integration, when Henri Poincaré was, with Hilbert, the foremost mathematician in Europe. Poincaré died prematurely in 1912; many others, such as Emile Borel, Paul Painlevé and Emile Picard, turned away from mathematics and became involved in politics or administration. In the 1920s, only two senior Paris mathematicians stood out: Jacques Hadamard and Élie Cartan. With very few exceptions, the intermediate generation had been wiped out by the war (see Andler 1994, 2006).

Leray and his fellow students at École normale supérieure were a brilliant group, including such men as André Weil, Henri Cartan, Jean Dieudonné, Claude Chevalley, Charles Ehresmann and Jacques Herbrand. But whereas the latter, who were true heirs at least in that respect to the strongly biased attitudes against applied mathematics prevalent in France—and not only in France, of course; in England G. H. Hardy was no less prejudiced—were interested in pure mathematics and found a major source of inspiration in Hilbert's formalism and the German school, Jean Leray, who had a strong interest in physics, leaned towards the applied aspects and felt more in tune with the intellectual resources available in Paris. All of them, however, shared an endurable admiration for Élie Cartan, Henri's father, as is evident in this quotation from a letter to Pierre Lamandé in October 1990, kindly shown to us by Jean Leray's daughter Françoise Pecker: 'Cartan's work proved to be fundamental and his teaching, constantly renewed, was shining with all its brilliance'. The philosophical difference between André Weil and his friends (who would create the influential Bourbaki group in the 1930s), and Jean Leray would prove to be long lasting.

In 1933, Leray started working in fluid mechanics and wrote his thesis on that topic under Henri Villat, a 'mechanician', Professor of Fluid Mechanics at the University of Paris, who had been elected to the French Académie des sciences in 1932. In that powerful position Villat provided, again and again, a considerable support for Leray at the early stages of his career. They remained very close until Villat's death in 1972.

Leray's thesis and first papers on partial differential equations

The mathematical difficulties of fluid mechanics are formidable. Although the equations for an incompressible fluid, the so-called Navier–Stokes equations, have been known since the nineteenth century, very little progress had been made on solving them before Leray. The Navier–Stokes equations are partial differential equations; that is, equations involving the function and some of its partial derivatives. Many other problems in physics have a mathematical formulation involving partial differential equations (the heat equation, the wave equation, Maxwell's equations, Einstein's equations…). For such equations, several questions immediately arise:

(i) Can one prove mathematically the existence and possibly uniqueness of solutions (under appropriate conditions on the data)?

We wish to convert Einst. eqs. into the form (E III)

By direct calculations one finds (in vacuum)

all terms linear in $\partial\partial g_{..}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R =$$

$$= -\frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \Big[-2 \partial_\beta \partial_{(\nu} g_{\mu)\alpha} + \partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu \partial_\nu g_{\alpha\beta} \Big] \qquad (\sim)$$

$$+ \frac{1}{2} \sum_{\alpha,\beta,\rho,\sigma} g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} \Big[-\partial_\beta \partial_\rho g_{\sigma\alpha} + \partial_\alpha \partial_\beta g_{\rho\sigma} \Big]$$

$$+ \tilde{F}_{\mu\nu}(g, \partial g) = 0$$

$\tilde{F}_{\mu\nu}$ are nonlinear functions of $g_{\alpha\beta}$ and $\partial g_{..}$ but independent of "$\partial^2 g_{\alpha\beta}$"; (of course with $\partial_\beta \partial_\nu g_{\mu\alpha}$ above there is $g^{\alpha\beta}$ which is nonlinear in $g_{\alpha\beta}$)

But the system $(\sim)$ above is not in the form (E III), p. $\mathcal{E}_g 3$ because it involves (contains) also 4 <u>Constraints</u>

$$\sum_\nu G_{\mu\nu} n^\nu = 0, \quad \text{or } G_{\mu 0} = 0 \quad (\text{with } n^\nu = 1,0,0,0)$$

where n^ν is normal to Σ (given in special coordinates by $t = \text{const}$); so in these coordinates $\sum G_{\mu\nu} n^\nu = 0$ follows from $(\sim)$. These constraints do <u>not</u> contain $g_{\alpha\beta,tt}$ — whereas (E III) does contain $g^{00}(\ldots) \nabla_0 \nabla_0 g_{\alpha\beta}$

Lemma

It is sufficient to satisfy the constraints only on the initial hypersurface $t = t_0$, i.e. $G_{0\beta} = 0$ at $t = t_0$, provided that the "dynamical spatial" equations $G_{ab} = 0$, $(a, b = 1,2,3)$, are satisfied. The "satisfaction" of the constraints at any $t > t_0$ is the consequence of the Bianchi identities.

Proof: Bianchi $\boxed{\nabla^\alpha G_{\alpha\beta} = 0}$ (Bi)

1) Take $\beta = b$ $(1, 2, 3)$

$$\Rightarrow \quad 0 = \nabla^a G_{ab} + \nabla^0 G_{0b} = \underbrace{g^{a\sigma}\nabla_\sigma}_{=\nabla^a} G_{ab} + g^{0\sigma}\nabla_\sigma G_{0b} =$$

$$= g^{a\sigma}\left(\partial_\sigma G_{ab} - \Gamma^\rho_{\sigma a} G_{\rho b} - \Gamma^\rho_{\sigma b} G_{a\rho}\right) +$$

$$+ g^{0\sigma}\left(\partial_\sigma G_{0b} - \Gamma^\rho_{\sigma 0} G_{\rho b} - \Gamma^\rho_{\sigma b} G_{0\rho}\right) \qquad (*)$$

assume $\underbrace{G_{\rho 0} = 0 \text{ at } t = t_0}$, $G_{ab} = 0$, $\partial_t G_{ab} = 0$ at $t \geq t_0$

$\Rightarrow \partial_i G_{\alpha 0} = 0$ (stay at Σ)

at $t = t_0$ then $(*)$ implies $g^{00}\partial_t G_{0b} = 0 \Rightarrow \partial_t G_{0b} = 0$

2) Take $\beta = 0$ in (Bi)

$$\Rightarrow \quad 0 = g^{\alpha\rho}\nabla_\rho G_{\alpha 0} = 0 \Rightarrow 0 = g^{\alpha\rho}\left(\partial_\rho G_{\alpha 0} - \Gamma^\sigma_{\rho\alpha} G_{\sigma 0} - \Gamma^\sigma_{\rho 0} G_{\alpha\sigma}\right)$$

assuming at $t = t_0$ as in 1) and taking into account that from 1) we know already that $\partial_t G_{0b} = 0$, it is easy to see that the last relation implies

$\partial_t G_{00} = 0$ so the constraint $G_{0\beta} = 0$ propagates in time $\|$ NUM. REL.! $\|$

Note: Analogous situation in electrodynamics

assume vacuum, at $t = t_0$ constraint and evolution equations satisfied:

$$\operatorname{div} \vec{E} = 0 = \operatorname{div} \vec{H}, \quad \frac{\partial \vec{E}}{\partial t} = \operatorname{rot} \vec{H}, \; \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot} \vec{E}$$

$$\Rightarrow \frac{\partial}{\partial t}(\operatorname{div} \vec{E}) = \operatorname{div} \operatorname{rot} \vec{H} = 0, \quad \frac{\partial}{\partial t}(\operatorname{div} \vec{H}) = 0$$

so assuming constraints $\operatorname{div} \vec{E} = \operatorname{div} \vec{H} = 0$ they „propagate" in time due to dynamical Maxw. eqs.

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back to EFEs ('Einstein Field Equations')

They represent an <u>'underdetermined' system</u>: only 6 evolution equations (remaining 4 are constraints) for 10 variables $g_{\alpha\beta}$! We have 4 arbitrary functions in choosing coordinates x^μ; choosing "the gauge", i.e., suitable x^μ can only guarantee the possibility of proving the uniqueness of the solution.

Commonly is used <u>harmonic gauge (harmonic coordinates)</u> – „harmonic" functions in math – those satisfying Laplace's or wave equation.

Here, harmonic coordinates x^μ satisfy

$$\boxed{H^\mu \equiv \nabla_\alpha \nabla^\alpha x^\mu = 0}$$

x^μ – set of 4 scalars

Hence, $H^{\mu} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta x^{\mu} = \frac{1}{\sqrt{-g}} \Big(\sqrt{-g}\, g^{\alpha\beta} \underbrace{\frac{\partial x^{\mu}}{\partial x^{\beta}}}_{=\delta^{\mu}_{\beta}} \Big)_{,\alpha}$

$$\Rightarrow H^{\mu} = 0 \quad \to \boxed{\left(\sqrt{-g}\, g^{\alpha\mu}\right)_{,\alpha} = 0} \qquad (H)$$

usual form of conditions for harmonic gauge

In the <u>linear</u> theory the harmonic gauge conditions imply „de Donder/Lorenz gauge conditions":

$$\gamma^{\alpha\mu}{}_{,\mu} = 0, \quad \text{where } \gamma_{\alpha\mu} = h_{\alpha\mu} - \tfrac{1}{2}\eta_{\alpha\mu} h$$

Indeed, $g_{\alpha\mu} = \eta_{\alpha\mu} + h_{\alpha\mu}$, $g^{\alpha\mu} = \eta^{\alpha\mu} - h^{\alpha\mu}$

the determinant $-g = 1 + h$, $h = h^{\sigma}_{\sigma}$ (cp. „skripta")

$$\sqrt{-g}\, g^{\alpha\mu} = \sqrt{1+h}\,(\eta^{\alpha\mu} - h^{\alpha\mu}) = (1 + \tfrac{1}{2}h)\eta^{\alpha\mu} - h^{\alpha\mu}$$
$$= \eta^{\alpha\mu} - (h^{\alpha\mu} - \tfrac{1}{2} h \eta^{\alpha\mu}) = \eta^{\alpha\mu} - \gamma^{\alpha\mu}$$

Writing explicitly (H) in terms of "sums":

$$0 = H^{\mu} = \sum_{\alpha} \Big(\partial_\alpha g^{\alpha\mu} + \tfrac{1}{2} g^{\alpha\mu} \sum_{\rho,\sigma} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} \Big) \qquad (\tilde{H})$$

Notice that H^{μ} contains only 1st. derivatives of $g_{\rho\sigma}$ but $\partial_\alpha H^{\mu}$ contains $\partial^2 g_{\rho\sigma}$ – will be used later – now!

<u>Intermezzo</u>:

Vladimir Alexandrovich Fo(c)k 1898 – 1974 in St Peterburgh

fundamental work in Quantum Mechanics (Hartree – Fock eq.), Klein – Gordon – Fock eq., Quantum-field theory – Fock space … founder of the school of theor. physics in Leningrad – the book: „The Theory of Space, Time and Gravitation" Defender of GR in USSR against Marxists „Propagandist" for using harmonic coordinat

Reduced Einstein's Eqs (using ∂H^μ) vacuum

Eq 9

$$R^{harm.}_{\mu\nu} \overset{def}{\equiv} R_{\mu\nu} + \sum_\alpha \underline{g_{\alpha(\mu}\partial_{\nu)} H^\alpha}$$

"full", standard $R_{\mu\nu}$ containing various $\partial^2 g_{\alpha\beta}$

Contains $\partial^2 g_{\alpha\beta}$ which combine with those in $R_{\mu\nu}$

indeed, these two "parts" lead to

$$\boxed{R^{harm}_{\mu\nu} = -\frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \hat{F}_{\mu\nu}(g, \partial g)}$$

this expression for Ricci tensor valid only in harmonic coordinates

$\Rightarrow \quad R_{\mu\nu} = 0 \iff R^{harm}_{\mu\nu} = 0$ & harmonic conditions $(H), (\tilde{H})$

Nice result is that $R^{harm}_{\mu\nu} = 0$, i.e.,

$$\underline{\underline{-\frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \hat{F}_{\mu\nu}(g, \partial g) = 0}}$$

is in the form for which Leray theorem can be applied!

So given initial conditions $g_{ab} = h_{ab}$, $a,b = 1,2,3$, on init. $t = t_0$ (Σ), and $\frac{\partial h_{ab}}{\partial t}$ the local existence of solutions is guaranteed, $\frac{\partial g_{0a}}{\partial t}$ can be given freely so that $H^\mu = 0$ on Σ

In harmonic coordinates we saw that

$$0 = R^{harm.}_{\mu\nu} - R_{\mu\nu}$$

In addition one can show that for the Ricci scalar (scalar curvature) we obtain

$$R^{harm.} = g^{\mu\nu} R^{harm}_{\mu\nu} = R + \frac{1}{2} g^{\mu\nu} g_{\alpha\mu} \partial_\nu H^\alpha + \frac{1}{2} g^{\mu\nu} g_{\alpha\nu} \partial_\mu H^\alpha$$

$$= R + \frac{1}{2}\partial_\alpha H^\alpha + \frac{1}{2} \partial_\alpha H^\alpha$$

$$= R + \partial_\alpha H^\alpha \qquad (+)$$

How does harmonic gauge condition $H^\mu = 0$ „propagates" in time

Choose Σ, $t = t_0$, assume constraints $G_{\mu\nu} n^\nu = 0$ satisfied on Σ, and assume $H^\mu = 0$ on Σ

we have

$$0 = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu} = R^{harm}_{\mu\nu} - \frac{1}{2} R^{harm} g_{\mu\nu} - \sum_\alpha \Big[g_{\alpha(\mu} \partial_{\nu)} H^\alpha - \frac{1}{2} g_{\mu\nu} \partial_\alpha H^\alpha \Big]$$

multiply the last equation by n^ν,

$\sum R_{\mu\nu} n^\nu - \frac{1}{2} R n_\mu = 0$ and assuming $R^{harm}_{\mu\nu} = 0$ on Σ and using $(+)$ above, we get

$$- \sum_{\nu, \alpha} g_{\alpha(\mu} \partial_{\nu)} H^\alpha n^\nu + \frac{1}{2} n_\mu \sum_{\rho, \sigma} g^{\rho\sigma} \sum_\alpha g_{\alpha(\rho} \partial_{\sigma)} H^\alpha = 0$$

From here it follows:

if $H^\alpha = 0$ on Σ, so also $\partial_i H^\alpha = 0$

then also $\partial_t H^\alpha = 0$

Now turn to Bianchi identities:

$$0 = \nabla^\mu G_{\mu\nu} = \nabla^\mu \left(R^{harm}_{\mu\nu} - \frac{1}{2} R^{harm} g_{\mu\nu} \right)$$
$$- \nabla^\mu \sum_\alpha \left[g_{\alpha(\mu} \partial_{\nu)} H^\alpha - \frac{1}{2} g_{\mu\nu} \partial_\alpha H^\alpha \right]$$

assume $R^{harm}_{\mu\nu} - \frac{1}{2} R^{harm} g_{\mu\nu} = 0$ is satisfied

$\Rightarrow$

$$0 = - \sum_{\rho,\mu,\alpha} \frac{1}{2} g_{\alpha\nu} g^{\rho\mu} \partial_\rho \partial_\mu H^\alpha + (\text{terms without } \partial^2 H^\alpha)$$

$/ \times g^{\lambda\nu}$

$$\Rightarrow \boxed{0 = - \sum_{\rho,\mu} g^{\rho\mu} \partial_\rho \partial_\mu H^\lambda + (\dots)}$$

This is in <u>the form of the Leray system</u>

With initial conditions $H^\lambda = 0$, $\partial_t H^\lambda = 0$

at $t = t_0$ (on Σ)

Leray $\Rightarrow$ $H^\lambda = 0$ at $t \geq t_0$

In harmonic coordinates the uniqueness of solutions

Yvonne Choquet-Bruhat (97) well-posedness of EFEs

using harmonic coordinates - from 1956 - (first paper 1948)

1979 French Academy (from 1666) - first woman

tradition exchange the same field - from 1987 more

flexible - Lichnerowicz, Darmois, <u>Leray</u>

short history arxiv 1410.3490, 13.10.2014

„new" books, visit of Prague

Using 3+1 splitting of the metric (lecture notes by OS), (Eg 1:
or "Looking" at the constraints "geometrically"

Gauss-Codazzi equations can be used to write down constraints in terms of quantities „living" just on 3-dim. hypersurface Σ and in the form invariant under coordinate transformations:

$$\boxed{0} = h_\alpha^{\ \beta} G_{\beta\gamma} n^\gamma = h_\alpha^{\ \beta}\left(R_{\beta\gamma} - \tfrac{1}{2} g_{\beta\gamma} R\right) n^\gamma$$

(C1)
$$= h_\alpha^{\ \beta} R_{\beta\gamma} n^\gamma - \tfrac{1}{2} R \underbrace{h_\alpha^{\ \beta} n_\beta}_{=0} =$$

$$= \underline{K_{\alpha|\beta}^{\ \beta} - K_{\beta|\alpha}^{\ \beta}}$$

cp. Eq. (24.38) in (S)

where | is covariant derivative intrinsic to Σ
if there is a source, $\varkappa h_\alpha^{\ \beta} T_{\beta\gamma} n^\gamma$ stands on l.h.s.

In addition

$${}^{(3)}R_{\alpha\beta\gamma}^{\ \ \ \ \delta} = h_\alpha^{\ \varkappa} h_\beta^{\ \lambda} h_\gamma^{\ \mu} h_\nu^{\ \delta} R_{\varkappa\lambda\mu}^{\ \ \ \ \nu} - K_{\alpha\gamma} K_\beta^{\ \delta}$$
$$+ K_{\beta\gamma} K_\alpha^{\ \delta},$$

contracting in $(\alpha\gamma)$ and $(\beta\delta)$:

$$\Rightarrow \quad {}^{(3)}R = \underbrace{h^{\varkappa\mu} h_\nu^{\ \lambda} R_{\varkappa\lambda\mu}^{\ \ \ \ \nu}} - (K_\alpha^{\ \alpha})^2 + K_{\alpha\beta} K^{\alpha\beta}$$

$$= R_{\varkappa\lambda\mu\nu}\left(g^{\varkappa\mu} + n^\varkappa n^\mu\right)\left(g^{\nu\lambda} + n^\nu n^\lambda\right)$$

$$= R + 2 R_{\alpha\gamma} n^\alpha n^\gamma = 2 G_{\alpha\gamma} n^\alpha n^\gamma$$

so (C2) $$\boxed{0 = G_{\alpha\beta} n^\alpha n^\beta = \tfrac{1}{2}\left[{}^{(3)}R + (K_\alpha^{\ \alpha})^2 - K_{\alpha\beta} K^{\alpha\beta}\right]}$$

if not vacuum, source is $\varkappa T_{\alpha\beta} n^\alpha n^\beta$ = „$\varkappa T_{00}$"

cp. Eq. (24.39) in (S)