On the boundary terms and ADM energy

(Arnowitt-Deser-Misner)

ReWriting the Hamiltonian density (26.9) in a slightly modified form:

$$\boxed{\mathcal{H}_g = \sqrt{h}\left\{N\left(-{}^{(3)}R \overset{+2\Lambda}{+} \frac{1}{h}\Pi_{\mu\nu}\Pi^{\mu\nu} - \frac{1}{2h}\Pi^2\right) - 2N_\mu\left[D_\nu\left(\frac{1}{\sqrt{h}}\Pi^{\nu\mu}\right)\right]\right\}} \quad (*)$$

First three terms are the same as corresponding terms in (26.9), the last term arises from the first term in (26.9): (See also O.S. below (26.11))

$$2\Pi^{\mu\nu}D_\mu N_\nu =$$

$$= \sqrt{h}\left\{-2N_\mu\left[D_\nu\left(\frac{1}{\sqrt{h}}\Pi^{\mu\nu}\right)\right] + \underbrace{2D_\mu\left(\frac{1}{\sqrt{h}}N_\nu\Pi^{\mu\nu}\right)}_{\text{omit - divergence}}\right\}$$

$$= 2N_\nu D_\mu\left(\frac{1}{\sqrt{h}}\Pi^{\mu\nu}\right) + \frac{2}{\sqrt{h}}\Pi^{\mu\nu}D_\mu N_\nu$$

$\Rightarrow \mathcal{H}_g$ above $(*)$

Total Hamiltonian $H_g = \int_{\Sigma_t} \mathcal{H}_g \, d^3x$

From $(*)$ it is clearly seen how lapse and shift can be considered as Lagrange multipliers and in el/mag. case $\mathcal{H}_{em} = 2\pi\Pi^2 + \frac{\vec{B}^2}{8\pi} + \phi\vec{\nabla}\vec{\pi}$, $\frac{\delta H}{\delta\phi} = 0$

$\Rightarrow$ constraint $\vec{\nabla}\vec{E} = 0$

Here in gravity

$$\frac{\delta H_g}{\delta N} = 0 \Rightarrow -{}^{(3)}R \overset{+2\Lambda}{+} \frac{1}{h}\Pi_{\mu\nu}\Pi^{\mu\nu} - \frac{1}{2h}\Pi^2 = 0 \quad \text{(HC)}$$

Hamiltonian constraint

$$\boxed{\frac{\delta H_g}{\delta N_\mu} \simeq D_\nu\left(\frac{1}{\sqrt{\Pi}}\Pi^{\nu\mu}\right)} \quad \text{Momentum constraint}$$

$$\boxed{H_{30}^{*}}$$

$$\underbrace{\frac{\delta H_g}{\delta N_\mu}}_{\text{shiftvector}} = -2\left[D_\nu\left(\frac{1}{\sqrt{h}}\pi^{\nu\mu}\right)\right]$$

$\Rightarrow$ Momentum constraint

$$\boxed{D_\nu\left[\frac{1}{\sqrt{h}}\pi^{\nu\mu}\right] = 0}$$

$$\dot{h}_{\alpha\beta} = \frac{\delta H_G}{\delta \Pi^{\alpha\beta}} = \frac{\partial \mathcal{H}_G}{\partial \Pi^{\alpha\beta}}\,, \quad \dot{\Pi}^{\alpha\beta} = -\frac{\delta H_G}{\delta h_{\alpha\beta}}$$

H_{33}

26.2.2 Hamilton equations: evolutions

The main work is still to be done. The **evolution equations** are given by Hamilton equations

$$\dot{h}_{\alpha\beta} = \frac{\partial \mathcal{H}_g}{\partial \Pi^{\alpha\beta}}\,, \qquad \dot{\Pi}^{\alpha\beta} := h^\alpha_\mu h^\beta_\nu \pounds_t \Pi^{\mu\nu} = -\frac{\partial \mathcal{H}_g}{\partial h_{\alpha\beta}} + \left(\frac{\partial \mathcal{H}_g}{\partial h_{\alpha\beta,\iota}}\right)_{,\iota} + \left(\text{terms given by } \frac{\partial \mathcal{H}_g}{\partial h_{\alpha\beta,\iota\kappa}}\right).$$

Computing

$$\frac{\partial \Pi^2}{\partial \Pi^{\alpha\beta}} = \frac{\partial}{\partial \Pi^{\alpha\beta}} (\Pi^{\kappa\lambda}\Pi^{\rho\sigma} h_{\kappa\lambda} h_{\rho\sigma}) = 2\delta^\kappa_\alpha \delta^\lambda_\beta \Pi^{\rho\sigma} h_{\kappa\lambda} h_{\rho\sigma} = 2\Pi h_{\alpha\beta}\,,$$

we have, from (26.9),

$$\frac{\delta H_G}{\delta \Pi^{\alpha\beta}} = \quad \boxed{\dot{h}_{\alpha\beta} = \frac{\partial \mathcal{H}_g}{\partial \Pi^{\alpha\beta}} = 2N_{(\alpha|\beta)} + \frac{N}{\sqrt{h}}(2\Pi_{\alpha\beta} - \Pi h_{\alpha\beta})\,,} \tag{26.12}$$

which exactly "repeats" the relations (26.7) and (26.8).

The second equation is more labourious; let us treat each of the terms of (26.9) separately:

- In the last term, $-N\sqrt{h}\,({}^{(3)}R - 2\Lambda)$, we use the knowledge from the variational derivation of Einstein equations. We learned there – see equation (23.11) – that if dropping the surface terms (given by behaviour of the metric *derivatives* on the integration-region boundary), then

$$\delta\left[\sqrt{-g}\,(R - 2\Lambda)\right] = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}$$
$$\Longrightarrow \quad \frac{\partial}{\partial g^{\mu\nu}}\left[\sqrt{-g}\,(R - 2\Lambda)\right] = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right).$$

However, we would prefer to know the derivative with respect to *covariant* metric, which reverses the sign, as we know from Section 23.2.1: specifically, we obtained there $\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}$, so

$$\delta\left[\sqrt{-g}\,(R - 2\Lambda)\right] = -\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta}\right)\delta g_{\alpha\beta}\,,$$
$$\Longrightarrow \quad \frac{\partial}{\partial g_{\alpha\beta}}\left[\sqrt{-g}\,(R - 2\Lambda)\right] = -\sqrt{-g}\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta}\right).$$

In the 3D analogy to this result, we can claim that

$$\frac{\partial}{\partial h_{\alpha\beta}}\left[\sqrt{h}\left({}^{(3)}R - 2\Lambda\right)\right] = -\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right). \tag{26.13}$$

- In the middle term of (26.9), $\frac{N}{2\sqrt{h}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right)$, we rewrite

$$2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2 = \Pi^{\kappa\lambda}\Pi^{\mu\nu}\left(2h_{\kappa\mu}h_{\lambda\nu} - h_{\kappa\lambda}h_{\mu\nu}\right)$$

in order to differentiate it by $h_{\mu\nu}$,

$$\frac{\partial}{\partial h_{\alpha\beta}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) = \Pi^{\kappa\lambda}\Pi^{\mu\nu}\left(2\delta^\alpha_\kappa\delta^\beta_\mu h_{\lambda\nu} + 2h_{\kappa\mu}\delta^\alpha_\lambda\delta^\beta_\nu - \delta^\alpha_\kappa\delta^\beta_\lambda h_{\mu\nu} - h_{\kappa\lambda}\delta^\alpha_\mu\delta^\beta_\nu\right) =$$
$$= 4\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - 2\Pi\Pi^{\alpha\beta}\,.$$

H_{34}

Recalling Section 23.2.1 once again, specifically equation (23.6) for the derivative of the metric determinant, $\frac{\partial(-g)}{\partial g_{\mu\nu}} = (-g)g^{\mu\nu}$, we analogously take $\frac{\partial h}{\partial h_{\alpha\beta}} = hh^{\alpha\beta}$ here on Σ_t, so

$$\frac{\partial}{\partial h_{\alpha\beta}}\left(\frac{1}{\sqrt{h}}\right) = -\frac{1}{2h^{3/2}}\frac{\partial h}{\partial h_{\alpha\beta}} = -\frac{1}{2h^{3/2}}\,hh^{\alpha\beta} = -\frac{1}{2\sqrt{h}}\,h^{\alpha\beta}\,.$$

Hence, in total,

$$\frac{\partial}{\partial h_{\alpha\beta}}\left(\frac{2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2}{\sqrt{h}}\right) = \\ = \frac{2}{\sqrt{h}}\left(2\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - \Pi\Pi^{\alpha\beta}\right) - \frac{1}{2\sqrt{h}}\,h^{\alpha\beta}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right). \tag{26.14}$$

- Finally, in the first term of (26.9), $2\Pi^{\mu\nu}N_{\mu|\nu}$, the momenta $\Pi^{\mu\nu}$ are taken as independent of $h_{\alpha\beta}$, and we rewrite

$$N_{\mu|\nu} = h_{\mu\kappa}N^{\kappa}{}_{|\nu} = h_{\mu\kappa}N^{\kappa}{}_{,\nu} + \frac{1}{2}\left(h_{\mu\nu,\lambda} + h_{\lambda\mu,\nu} - h_{\nu\lambda,\mu}\right)N^{\lambda}\,,$$

so

$$\frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta}} = 2\Pi^{\alpha\nu}N^{\beta}{}_{,\nu} = \Pi^{\alpha\nu}N^{\beta}{}_{,\nu} + \Pi^{\beta\nu}N^{\alpha}{}_{,\nu}\,,$$

$$\frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta,\iota}} = \Pi^{\mu\nu}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\iota}_{\lambda} + \delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\iota}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\lambda}\delta^{\iota}_{\mu}\right)N^{\lambda} = \Pi^{\alpha\beta}N^{\iota} + \cancel{\Pi^{\beta\iota}N^{\alpha} - \Pi^{\iota\alpha}N^{\beta}}$$

(the last forms have been claimed on the basis of the necessary symmetry of both the expressions in α and β). Putting the two terms together, we thus have

$$-\frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta}} + \left[\frac{\partial(2\Pi^{\mu\nu}N_{\mu|\nu})}{\partial h_{\alpha\beta,\iota}}\right]_{,\iota} = \\ = \Pi^{\alpha\beta}{}_{,\iota}N^{\iota} + \Pi^{\alpha\beta}N^{\iota}{}_{,\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{,\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{,\nu} = \\ = \left(\Pi^{\alpha\beta}{}_{|\iota} - \cancel{\Gamma^{\alpha}_{\iota\sigma}\Pi^{\sigma\beta}} - \cancel{\Gamma^{\beta}_{\iota\sigma}\Pi^{\alpha\sigma}}\right)N^{\iota} + \Pi^{\alpha\beta}N^{\iota}{}_{,\iota} - \\ - \Pi^{\alpha\nu}\left(N^{\beta}{}_{|\nu} - \cancel{\Gamma^{\beta}_{\nu\sigma}N^{\sigma}}\right) - \Pi^{\beta\nu}\left(N^{\alpha}{}_{|\nu} - \cancel{\Gamma^{\alpha}_{\nu\sigma}N^{\sigma}}\right) = \\ = \Pi^{\alpha\beta}{}_{|\iota}N^{\iota} + \Pi^{\alpha\beta}N^{\iota}{}_{,\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu} = \\ = \Pi^{\alpha\beta}{}_{|\iota}N^{\iota} + \Pi^{\alpha\beta}\sqrt{h}\left(\frac{N^{\iota}}{\sqrt{h}}\right)_{|\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu} = \\ = \sqrt{h}\,(h^{-1/2}\Pi^{\alpha\beta}N^{\iota})_{|\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu}\,.$$

Therefore, collecting the above results for the three Hamiltonian terms, we conclude that

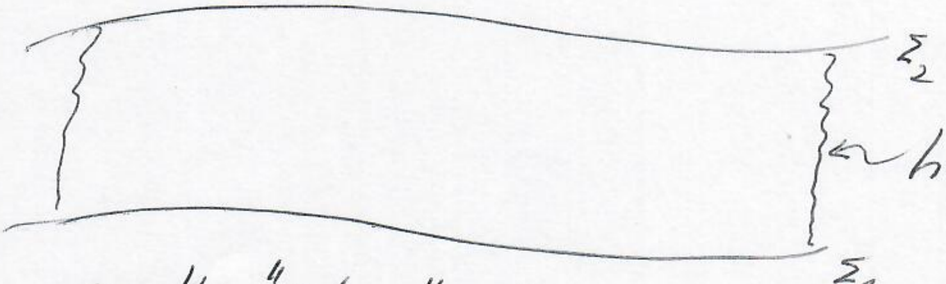
$$\boxed{\begin{aligned}\dot{\Pi}^{\alpha\beta} &:= h^{\alpha}_{\mu}h^{\beta}_{\nu}\pounds_{\mathbf{t}}\Pi^{\mu\nu} = \underline{-\frac{\partial\mathcal{H}_g}{\partial h_{\alpha\beta}} + \left(\frac{\partial\mathcal{H}_g}{\partial h_{\alpha\beta,\iota}}\right)_{,\iota}} \equiv \frac{\delta\mathcal{H}_g}{\delta h_{\alpha\beta}} \\ &= -N\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right) - \\ &\quad - \frac{N}{\sqrt{h}}\left(2\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - \Pi\Pi^{\alpha\beta}\right) + \frac{N}{4\sqrt{h}}\,h^{\alpha\beta}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) + \\ &\quad + \sqrt{h}\,(h^{-1/2}\Pi^{\alpha\beta}N^{\iota})_{|\iota} - \Pi^{\alpha\nu}N^{\beta}{}_{|\nu} - \Pi^{\beta\nu}N^{\alpha}{}_{|\nu}\,.\end{aligned}} \tag{26.15}$$

H30 and bottom

Summary: Constraints (HC), p. H_{30}, ~~[illegible]~~ p. and (4C) ⊛ evolution equations (26.12) and (26.15) $\Longleftrightarrow R_{\mu\nu} = 0$;
„constrained Hamiltonian form of GR"

Appendix to the Hamiltonian formalism for gravitational field: boundary terms & ADM mass-energy

Assume asymptotically flat (AF) spacetimes



here $h_{\alpha\beta}$ are now not fixed

on the "sphere" at spatial infinity we now assume only that $\delta h_{\alpha\beta}$ variations preserve asymptotic flatness

Procedure:

We shall calculate boundary terms by varying H_G and then modify H_G (to H_G') in such a way that we get rid of the boundary terms but EOM remain same.

First, let us introduce in AF spacetime asymptotically Cartesian system, t^α vector will be time translation at ∞, lapse $N \to 1$, shift $N_\alpha \to 0$ when $r = ((x^1)^2 + (x^2)^2 + (x^3)^2) \to \infty$, extrinsic curvature $K_{\alpha\beta} \sim \frac{1}{r^2}$ (so this is the same for momentum)

We have (for the density) $\mathcal{H}_g$ as given in (*) on page H_{30} of lecture 9, i.e.

$$\mathcal{H}_g = \sqrt{h}\left\{N\left[-{}^{(3)}R + \frac{1}{h}\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{1}{2}\frac{1}{h}\pi^2\right] - 2N_\beta\left[D_\alpha\left(\frac{1}{\sqrt{h}}\pi^{\alpha\beta}\right)\right]\right\}$$

simple change of μ,ν to α,β

It then turns out that only the first term $-\sqrt{h}N^{(3)}R$ gives in varying δg on the sphere S with $r \to \infty$ a non-zero contribution (terms with $\pi \sim K \sim \frac{1}{r^2}$ go more rapidly to 0 - viz $\pi^2 \sim \frac{1}{r^4}$)

One gets, when using the parameter λ for variation, viz $\psi = \psi_0 + \underbrace{\frac{d\psi}{d\lambda}\Big|_{\lambda=0} \lambda}_{\delta\psi} + \cdots$

now this total Hamilt. $\int \mathcal{H}_g \, d^3x$

$$\frac{dH_G}{d\lambda} = \int_{\Sigma_t} \Big\{ \Big[\underbrace{A_{\alpha\beta}}_{\frac{\delta H_G}{\delta \pi_{\alpha\beta}}} \delta\pi^{\alpha\beta} - \underbrace{B^{\alpha\beta}}_{-\frac{\delta H_G}{\delta h_{\alpha\beta}}} \delta h_{\alpha\beta} \Big] d^3x - \delta C \Big\}$$

see p. H33

where the boundary term at spatial infinity $r \to \infty$

from

$$\delta C = \lim_{r\to\infty} \int_S r^\alpha h^{\beta\gamma} \left[D_\gamma (\delta h_{\alpha\beta}) - D_\alpha (\delta h_{\beta\gamma}) \right] dS$$

unit normal to S (i.e. $r = \text{const}$)

In coordinate components this becomes (remember AF Cart. coord.)

$$\delta C = \lim_{r\to\infty} \sum_{i,k=1}^{3} \int_S \left(\frac{\partial \delta h_{ki}}{\partial x^i} - \frac{\partial \delta h_{ii}}{\partial x^k} \right) r^k \, dS$$

$$= \delta \left\{ \lim_{r\to\infty} \sum_{i,k=1}^{3} \int_S \left(\frac{\partial h_{ki}}{\partial x^i} - \frac{\partial h_{ii}}{\partial x^k} \right) r^k \, dS \right. \quad (\sim)$$

note that $h \sim 1 + \frac{1}{r}$, $\frac{\partial h}{\partial x^i} \sim \frac{1}{r^2}$ but $dS \sim r^2$

so a non zero value

Therefore, we define a new Hamiltonian H'_G by

$$H'_G = H_G + \alpha,$$

where

$$\alpha = \lim_{r\to\infty} \sum_{i,k=1}^{3} \int_S \left(\frac{\partial h_{ki}}{\partial x^i} - \frac{\partial h_{ii}}{\partial x^k} \right) r^k dS$$

then $\frac{\delta H'_G}{\delta \pi_{\alpha\beta}}$ and $-\frac{\delta H'_G}{\delta h_{\alpha\beta}}$ are – for AF spacetimes

really given by expressions on p. H$_{33}$

The numerical value of H'_G gives for AF spacetimes the total mass-energy. For the Schwarzschild as well as for Kerr one gets M!

In fact, it is given so up to a constant. Precisely

$$E = M = \frac{1}{16\pi} \lim_{r\to\infty} \sum_{i,k=1}^{3} \int \left(\frac{\partial h_{ik}}{\partial x^i} - \frac{\partial h_{ii}}{\partial x^k} \right) r^k dS$$

This is the A(rnowitt) D(eser) M(isner) total mass-energy of AF spacetimes at spatial infinity i_0.

It is more general than Komar's expression – no Killing vector is assumed.

For a geometrical formulation at i_0, see A. Ashtekar, „Asymptotic Structure of the Gravitational Field at Spatial Infinity", in General Relativity and Gravitation, Vol. 2., ed. A. Held (Plenum Press, N.Y., 1980)

One can define also the 4-vector $\overset{\text{4-momentum}}{P_\alpha}$ at i_0 involving both total energy and momentum at spatial infinity. Both momentum and angular momentum at i^0 can be obtained by considering more general forms asymptotic of lapse N and shift N^α.

AII

Alternative (more „fundamental") approach to the role of surface integrals in the Hamiltonian formulation of GR

details in Regge & Teitelboim Annls Phys. **88**, 286 (1974)

1967: De Witt: in $\overset{\text{asymp. flat}}{AF}$ spacetime the usual Hamiltonian now denoted H_0 („it was H_g^x")

$$H_0 = \int d^3x \left\{ \underbrace{N(x)\mathcal{H}(x)}_{\text{„it was } \mathcal{H}_g\text{"}} + N^i(x)\mathcal{H}_i(x) \right\} \qquad (1)$$

constraints $\mathcal{H} = \mathcal{H}_i = 0$

should be supplemented by surface integral at ∞

$$E[g_{ij}] = \oint d^2 S_k \left(g_{ik,i} - g_{ii,k} \right)$$

i.e. $H = H_0 + E[g_{ij}]$ only this gives the usual Hamilt. of linearized theory

$g_{ik} \equiv h_{ik}$ in adapt. coord. and AF so all indices down although summation!

Attitude new:

True phase space of the gravitational field for AF spaces is the space of (g_{ij}, π^{ij}) completed by the introduction of boundary conditions as canonical variables

$\llcorner h_{ij}$

AH(

In vacuum GR a physically „reasonable" solution which is asymptotically flat spacetime behaves at spatial infinity in the Schwarzschild form

$$\boxed{ds^2 \underset{r\to\infty}{\sim} -\left(1-\frac{M}{8\pi r}\right)dt^2 + \left(\delta_{ij} + \frac{M}{8\pi}\frac{x^i x^j}{r^3}\right)dx^i dx^j} \quad (2)$$

- in isotropic coordinates

(Note: in 'correct' units Einstein action reads

$$S_g = \frac{c^3}{16\pi G}\int R\sqrt{-g}\, d^4x$$

units often chosen so that $c=1$, $16\pi G = 1$

the $2GM \to M/8\pi$)

$\Rightarrow$ Any definition of phase space has to contain metric functions behaving at $r\to\infty$ as

$$g_{ij} - \delta_{ij} \sim \frac{1}{r}, \quad g_{ij,k} \sim \frac{1}{r^2}$$

Hamilton's eqs.:

$$\dot{g}_{ij}(x) = \frac{\delta(\text{Ham.})}{\delta\pi^{ij}}$$

$$\dot{\pi}^{ij}(x) = -\frac{\delta(\text{Ham.})}{\delta g_{ij}(x)}$$

general variation of Hamiltonian:

$$\delta(\text{Hamiltonian}) = \int d^3x \left[A^{ij}(x)\,\delta g_{ij}(x) + B_{ij}(x)\,\delta\pi^{ij}(x)\right]$$

$$\frac{\delta(\text{Ham.})}{\delta g_{ij}} = A^{ij}, \quad \frac{\delta(\text{Ham})}{\delta\pi^{ij}} = B_{ij} \quad (3)$$

Now in H_0 in (1), p.

one has

$$\mathcal{H} = \frac{1}{\sqrt{g}}\left(\pi_{ij}\pi^{ij} - \tfrac{1}{2}\pi^2\right) - \sqrt{g}\,R$$

here "$g \equiv h$" in previous lecture pages (det from $h_{\alpha\beta}$)

$$\mathcal{H}_i = -2\,\pi_i{}^j{}_{|j}$$

Now calculating δH_0, keeping all terms, including the boundary terms (cf. also OS-scriptum "issue" of boundary terms):

$$\boxed{\begin{aligned} \delta H_0 = &\int d^3x \left[A^{ij}(x)\,\delta g_{ij}(x) + B_{ij}(x)\,\delta\pi^{ij}(x)\right] \\ &- \oint d^2s_l\, \underline{G^{ijkl}\left(N\,\delta g_{ij|k}\right.} - \left.N_{,k}\,\delta g_{ij}\right) \\ &- \oint d^2s_l \left[2N_k\,\delta\pi^{kl} + \left(2N^k\pi^{jl} - N^l\pi^{jk}\right)\delta g_{jk}\right] \end{aligned}} \qquad (4)$$

here $$G^{ijkl} = \tfrac{1}{2}\sqrt{g}\left(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}\right) \qquad (5)$$

$A^{ij}(x)$ and $B_{ij}(x)$ must be identified with variational derivatives of the Hamiltonian, i.e. with (3) p.

__If__ all surface terms in (4) would vanish H_0 would be correct Hamiltonian - For closed spaces this is the case

But for AF spacetimes the __first $\oint$ in (4)__ does not

For AF spacetimes it is appropriate to assume the following fall-off conditions at ∞: AH

$$\pi^{ij} \sim \frac{1}{r^2}$$

$$N-1 \sim \frac{1}{r}$$

$$N_{,k} \sim \frac{1}{r^2}$$

$$N^i \sim \frac{1}{r}, \quad N^i_{,k} \sim \frac{1}{r^2}$$

The only surviving surface term at ∞:

$$-\oint d^2s_l\, G^{ijkl}\, \delta g_{ij|k} \qquad (N\sim 1)$$

and one gets

$$-\oint d^2s_e\, G^{ijkl}\, \delta g_{ij|k} = -\delta \oint d^2s_k \left(g_{ik,i} - g_{ii,k}\right)$$

$$\equiv -E[g_{ij}]$$

Warning E and even variation of E does not vanish even thoug E is a surface integral

viz for Schw. $E=M$ and general $\delta g_{ij} \Rightarrow \delta M \neq 0$

<u>Conclusion</u> - for asympt. flat spacetimes

correct Hamiltonian is

$$\boxed{H = H_0 + E[g_{ij}]}$$

for constraints satisfied $H=E$