New, Additional pages to strings S27–S30
"old S27 becomes so S31" etc

We shall now rewrite (RS) so that it will lead to the wave equation and constraints.

Regarding the relation $\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t}$ at all points of string (see p. S24), (RS) (multiplied by $\frac{1}{T_0}\sqrt{1-\frac{v_\perp^2}{c^2}}$) and changing $\partial/\partial s$ to $\partial/\partial\sigma$ becomes

$$(\sim) \qquad \frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} = \underbrace{\frac{\sqrt{1-\frac{v_\perp^2}{c^2}}}{\frac{ds}{d\sigma}}}_{=\frac{1}{A(\sigma)}} \frac{\partial}{\partial\sigma}\left[\underbrace{\frac{\sqrt{1-\frac{v_\perp^2}{c^2}}}{\frac{ds}{d\sigma}}}_{=\frac{1}{A(\sigma)}}\frac{\partial \vec{X}}{\partial\sigma}\right]$$

this factor $A(\sigma)$ does not depend on time $t$ – see (+), p. S25. We shall choose the parametrization $\sigma$ so that $A(\sigma) = 1$. Then the above Eq. $(\sim)$ implies the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} = \frac{\partial^2 \vec{X}}{\partial\sigma^2}$$

How to achieve $A(\sigma) = 1$?

assign $\sigma = 0$ to one endpoint and to each piece $ds$ assign

$$d\sigma = \frac{ds}{\sqrt{1-\frac{v_\perp^2}{c^2}}} \Rightarrow A = 1 \quad \text{but also } d\sigma = \frac{1}{T_0}\underbrace{dE}_{\text{energy this piece}}$$

recall (·), p. S25 which implies (after $\times d\sigma = \text{const}$ which is)

$$T_0 \frac{\partial}{\partial t} \underbrace{\left( \frac{ds}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \right)}_{\text{interpret as "conserved energy of piece } d\sigma\text{"}} = 0$$

then integrate $d\sigma = \frac{1}{T_0} dE \xrightarrow{\text{up to point } Q} \sigma(Q) = \frac{E(Q)}{T_0}$

$$\Rightarrow \sigma \in [0, \sigma_1], \quad \boxed{\sigma_1 = \frac{E}{T_0}}$$ ← total energy of the string

The reparametrization (p. S27) $A(\sigma) = 1$ means

$$\left(\frac{ds}{d\sigma}\right)^2 + \frac{1}{c^2} v_\perp^2 = 1$$

Recall $\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t}$, $\frac{\partial \vec{X}}{\partial s}$ is unit (see p. S20)

$$\Rightarrow \left(\frac{\partial \vec{X}}{\partial \sigma}\right)^2 + \frac{1}{c^2}\left(\frac{\partial \vec{X}}{\partial t}\right)^2 = 1$$ ← constraint

<u>4 eqs. for motion of a relativistic string:</u> on $\vec{X}$

1) $\frac{\partial^2 \vec{X}}{\partial \sigma^2} - \frac{1}{c^2} \frac{\partial^2 \vec{X}}{\partial t^2} = 0$ Wave eq. in 2d

2) $\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} = 0$ ← see p. 24

3) $\left(\frac{\partial \vec{X}}{\partial \sigma}\right)^2 + \frac{1}{c^2}\left(\frac{\partial \vec{X}}{\partial t}\right)^2 = 1$

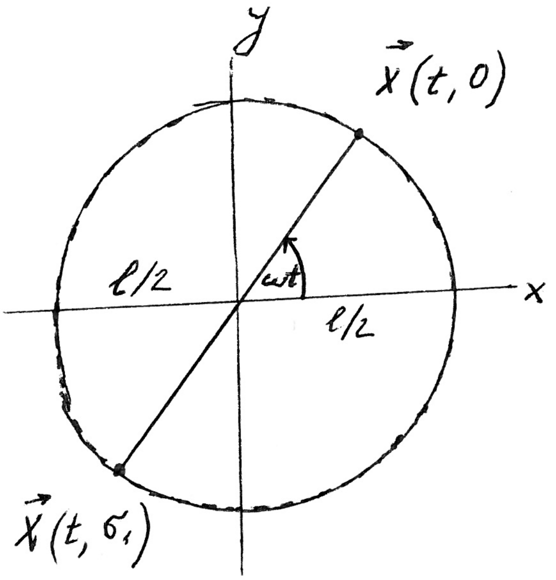
} parametrization conditions

4) $\left.\frac{\partial \vec{X}}{\partial \sigma}\right|_{\sigma=0} = \left.\frac{\partial \vec{X}}{\partial \sigma}\right|_{\sigma=\sigma_1} = 0$ boundary conditions

1) - 4) can be explicitly solved

For (rigidly) rotating string - calculations (relatively) simple but omitted here - lead to the final solution

$$(+) \qquad \vec{X}(t,\sigma) = \frac{\sigma_1}{\pi} \cos \frac{\pi \sigma}{\sigma_1} \left( \cos \frac{\pi c t}{\sigma_1}, \sin \frac{\pi c t}{\sigma_1} \right)$$




$$\frac{\omega}{c} = \frac{\pi}{\sigma_1}$$

since $\sigma_1 = \dfrac{E}{T_0}$ (p. S28)

$$\Rightarrow \frac{\omega}{c} = \frac{\pi T_0}{E}$$

Putting $c = 1$, (+) can be transcribed as

$$\vec{X} = \omega^{-1} \cos \omega \sigma \, (\cos \omega t, \sin \omega t)$$

From $d\sigma = \dfrac{ds}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}$ at the bottom of S27 $\Rightarrow \dfrac{ds/d\sigma}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} = 1$

Angular momentum current

$$\mathcal{M}^{\alpha}_{\mu\nu} \equiv X_\mu P^\alpha_\nu - X_\nu P^\alpha_\mu$$

$\Rightarrow$ total is (using constant $\tau$ lines)

$$M_{\mu\nu} = \int \mathcal{M}^{\tau}_{\mu\nu}(\tau,\sigma)\, d\sigma =$$

$$= \int (X_\mu P^\tau_\nu - X_\nu P^\tau_\mu)\, d\sigma$$

In case of the string rotating rigidly in $(x,y)$-plane only

$$M_{12} = \int_0^{\sigma_1} \left( X_1 P_2^{\tau} - X_2 P_1^{\tau} \right) d\sigma$$

In this we have to use (*) (preceding page) for $X_1$, $X_2$

using also

$$p^{\tau\mu} = \frac{T_0}{c^2} \frac{\frac{ds}{d\sigma}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \frac{\partial X^\mu}{\partial t} = \frac{1}{c^2} T_0 \frac{\partial X^\mu}{\partial t}$$

(see preceding p. for $\frac{ds/d\sigma}{\sqrt{\phantom{x}}} = 1$)

we get $\vec{p}^{\,\tau} = \frac{T_0}{c^2} \frac{\partial \vec{X}}{\partial t} = \frac{T_0}{c} \cos \frac{\pi\sigma}{\sigma_1} \left( -\sin \frac{\pi c t}{\sigma_1}, \cos \frac{\pi c t}{\sigma_1} \right)$

and $M_{12} = \frac{\sigma_1}{\pi} \frac{T_0}{c} \int_0^{\sigma_1} \cos^2 \frac{\pi\sigma}{\sigma_1} d\sigma = \frac{\sigma_1^2 T_0}{2\pi c}$

Angular momentum $J = |M_{12}|$ on p. we have $\sigma_1 = E/T_0$

$$\Rightarrow \quad J = \frac{1}{2\pi T_0 c} E^2, \quad J \sim E^2$$

Define slope parameter $\alpha'$ by

$$\frac{J}{\hbar} = \alpha' E^2 \Rightarrow \alpha' = \frac{1}{2\pi T_0 \hbar c}$$

$$\text{or } T_0 = \frac{1}{2\pi \alpha' \hbar c}$$

Intuition:

rigid rot. $J = I\omega$

$I$ ← moment of inertia; $I \sim ML^2$

$\Rightarrow J = ML^2\omega$, for string we had $M \sim E$, $L \sim E$, $\omega \sim 1/E$

$$\Rightarrow J \sim E^2$$