Invariantní symetrické polynomy

vekt. bundle $\mathbb{E}M$ $\quad T_{g}{}^{A}{}_{B}$ $\quad t_{\alpha}{}^{A}{}_{B}$

Lieova alg. gM $\quad Ad_{g}{}^{A}{}_{B}$ $\quad ad_{\alpha}{}^{\mu}{}_{\nu} = c_{\alpha\nu}{}^{\mu}$

reprezentace gM na $\mathbb{E}M$ — prvky z $\mathbb{E}^1_1 M$ tvaru:

$$X^A{}_B = \mathcal{X}^\alpha t_\alpha{}^A{}_B \qquad X \in \mathbb{E}^1_1 M \quad \mathcal{X} \in gM$$

1) multilin. zobr. stupně r

$$P(X_1, \dots, X_r) = P^{A_1}{}_{A_1}{}^{B_r}{}_{A_r} X_1{}^{A_1}{}_{B_1} \dots X_r{}^{A_r}{}_{B_r}$$

$$P(\mathcal{X}_1, \dots, \mathcal{X}_r) = P_{\alpha_1 \dots \alpha_r} \mathcal{X}^{\alpha_1} \dots \mathcal{X}^{\alpha_r}$$

$$P_{\alpha_1 \dots \alpha_r} = t_{\alpha_1}{}^{A_1}{}_{B_1} \dots t_{\alpha_r}{}^{A_r}{}_{B_r} P^{B_1}{}_{A_1}{}^{\dots B_r}{}_{A_r}$$

2) symetrie

$$P(\dots X_i \dots X_j \dots) = P(\dots X_j \dots X_i \dots) \qquad \forall i,j$$

$$P_{\alpha_1 \dots \alpha_r} = P_{(\alpha_1 \dots \alpha_r)}$$

3) ~~polynomy~~

mocnina = homogenní polynom $\quad P(X) = P(X, \dots, X)$ $\quad$ pro sym. P

ekvivalentní informace

$$P(X_1 \dots X_r) = \frac{1}{r!} \frac{\partial^r}{\partial \tau_1 \dots \partial \tau_r} P(t_1 X_1 + \dots t_r X_r)$$

nehomogenní polynom

$$P(X) = \sum_r P_r(X) \qquad P_r \text{ sym. stupně } r$$

(výhoda 1-argum. zápisu)

symetrický součin

$$(P \circ Q)(X) = P(X)\, Q(X)$$

$$(P \circ Q)(X_1 \dots X_{p+q}) = \frac{1}{(p+q)!} \sum_\sigma P(X_{\sigma_1} \dots X_{\sigma_p})\, Q(X_{\sigma_{p+1}} \dots X_{\sigma_{p+q}})$$

$$(P \circ Q)_{\alpha_1 \dots \alpha_{p+q}} = P_{(\alpha_1 \dots \alpha_p} Q_{\alpha_{p+1} \dots \alpha_{p+q})}$$

4) invariance vůči akci gr.

$M = \mathcal{M}\cdot t = t_{\mathcal{M}}$

$\boldsymbol{\mathcal{M}} = \mathcal{M}\cdot ad = ad_{\mathcal{M}}$

$$X \to \tilde{X} = T_{m_\tau}\cdot X\cdot T_{m_\tau}^{-1} \approx X + \tau[M,X]$$

$$\mathcal{X} \to \tilde{\mathcal{X}} = Ad_{m_\tau} X \approx \mathcal{X} + \tau[\mathcal{M},\mathcal{X}] = \mathcal{X} + \tau \boldsymbol{\mathcal{M}}\cdot\mathcal{X}$$

$$P(\tilde{X}_1,\dots,\tilde{X}_k) = P(X_1,\dots,X_k)$$

ekvivalentní podm. v prvním řádu

$\mathbb{M}P = 0$ resp. $\boldsymbol{\mathcal{M}}P = 0$

Důk: inv. v 1. řádu $\Rightarrow$

$$0 = P([M,X_1],X_2,\dots) + P(X_1,[M,X_2],\dots) + \dots$$

$$= P^{B_1B_2\dots}_{A_1A_2\dots}(M^{A_1}{}_N X_1^N{}_{B_1} - X_1^{A_1}{}_N M^N{}_{B_1}) X_2^{A_2}{}_{B_2}\dots + P^{B_1B_2\dots}_{A_1A_2\dots} X_1^{A_1}{}_{B_1}(M^{A_2}{}_N X_2^N{}_{B_2} - X^{A_2}{}_N M^N{}_{B_2})\dots$$

$$= (-\mathbb{M}P)^{B_1B_2\dots}_{A_1A_2\dots} X_1^{A_1}{}_{B_1} X_2^{A_2}{}_{B_2}\dots = -\mathbb{M}P(X_1,\dots,X_k)$$

$$0 = P([\mathcal{M},\mathcal{X}_1],\mathcal{X}_2,\dots) + P(\mathcal{X}_1,[\mathcal{M},\mathcal{X}_2],\dots) + \dots$$

$$= P_{\alpha_1\alpha_2\dots}\mathcal{M}^{\alpha_1}{}_\nu \mathcal{X}_1^\nu \mathcal{X}_2^{\alpha_2}\dots + P_{\alpha_1\alpha_2\dots}\mathcal{X}_1^{\alpha_1}\mathcal{M}^{\alpha_2}{}_\nu \mathcal{X}_2^\nu\dots + \dots$$

$$= -(\boldsymbol{\mathcal{M}}P)_{\alpha_1\alpha_2}\mathcal{X}_1^{\alpha_1}\mathcal{X}_2^{\alpha_2}\dots = -\boldsymbol{\mathcal{M}}P(\mathcal{X}_1,\mathcal{X}_2,\dots)$$

5) konstantnost

kov. der. splňující $\mathcal{D}c = 0$ $\quad \mathcal{D}t = 0$

přirozeně

$\mathbb{D}P = 0 \qquad \mathcal{D}P = 0$

stačí splnit pro jednu der., např. triv. ∂

$\partial P = 0 \Rightarrow \mathcal{D}P = \partial P + \mathbf{A}P = 0$

Poznámka

P je tvořeno pomocí

c, t, kontrakcí

Př: $STR_k(X_1,\dots,X_k) = \frac{1}{k!}\sum_\sigma tr(X_1\cdots X_k) \rightsquigarrow str_k \quad k \sim str_2$

rozšíření na formy na M

$\mathcal{X} \in \Lambda^p \otimes \mathfrak{g}\, M \qquad X \in \Lambda^p \otimes E^1_1 M$

$\mathcal{X}^{\alpha}_{a_1 \dots a_p} \qquad X_{a_1 \dots a_p}{}^{A}{}_{B}$

$$\hat{P}(X_1, X_2 \dots) \equiv P(X_1 \wedge X_2 \wedge \dots)$$

$$= X_1{}^{A_1}{}_{B_1} \wedge X_2{}^{A_2}{}_{B_2} \wedge \dots \; P^{B_1 B_2 \dots}{}_{A_1 A_2 \dots}$$

$$\hat{P}(\mathcal{X}_1, \mathcal{X}_2 \quad) \equiv P(\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \quad)$$

$$= \mathcal{X}_1^{\alpha_1} \wedge \mathcal{X}_2^{\alpha_2} \wedge \dots \; P_{\alpha_1 \dots \alpha_r}$$

$\wedge$ je skew-symetric $\quad \omega \wedge \sigma = (-1)^{pq}\, \sigma \wedge \omega$!

$\hat{P}_r(\mathcal{X})$ netriviální pouze pro $\mathcal{X}$ sudou

$$\hat{P}_r(\mathcal{X}) = \mathcal{X}^{\alpha_1} \wedge \mathcal{X}^{\alpha_2} \wedge \dots \; P_{\alpha_1 \alpha_2 \dots}$$

$$= (-1)^{x}\, \mathcal{X}^{\alpha_2} \wedge \mathcal{X}^{\alpha_1} \wedge \dots \; P_{\alpha_1 \alpha_2 \dots}$$ prohození $\mathcal{X}^{\alpha_1} \wedge \mathcal{X}^{\alpha_2} \to \mathcal{X}^{\alpha_2} \wedge \mathcal{X}^{\alpha_1}$

$$= (-1)^{x}\, \mathcal{X}^{\alpha_1} \wedge \mathcal{X}^{\alpha_2} \wedge \quad P_{\alpha_1 \alpha_2}$$ přeznačení indexů α_1, α_2 + symetrie $P_{\alpha_1 \alpha_2 \dots \alpha_{2n}}$

součin
- v bundle indexech symetrický
- v base indexech skew-symetrický

Chern-Weilova věta

D kov. der. na gM s křivostí $\mathcal{F}$
na EM s křivostí F

$\mathcal{P}$, P inv. sym. polynom na gM resp. E,M

1) $\hat{P}(\mathcal{F})$, resp. $\hat{P}(F)$ jsou uzavřené, tj.

$$d\hat{P}(\mathcal{F})=0 \qquad d\hat{P}(F)=0$$

2) D, $\tilde{D}$ dvě kov. der. na EM s křiv. F, $\tilde{F}$

$\hat{P}(\tilde{F})-\hat{P}(F)$ je exaktní, tj.

$$\hat{P}(\tilde{F})-\hat{P}(F)=d\,TP(\tilde{D},D)$$

TP je forma transgrese

Důkaz 1) P homogenní

$$d\hat{P}(\mathcal{F}) = D\wedge\hat{P}(\mathcal{F},\mathcal{F},\dots)$$

$$=\underbrace{\widehat{DP}(\mathcal{F},\mathcal{F},\dots)}_{0 \Leftarrow \text{konst } P}+\underbrace{\hat{P}(D\wedge\mathcal{F},\mathcal{F},\dots)+\hat{P}(\mathcal{F},D\wedge\mathcal{F},\dots)+\dots}_{0 \Leftarrow \text{Bianchi}}$$

$$=0$$

vsuvka

$$\hat{P}(\mathcal{F})=P_{\alpha_1\alpha_2\dots\alpha_n}\mathcal{F}^{\alpha_1}\wedge\mathcal{F}^{\alpha_2}\wedge\dots\wedge\mathcal{F}^{\alpha_n}$$

Důkaz 2)

$\tilde{D}, D$ kov. der na $\mathcal{E}M$

$$\tilde{F} = F + D\wedge\Delta + \underbrace{[\Delta,\Delta]}_{\Delta^{\wedge 2}}$$

$$\Delta = \tilde{D} - D$$

$$D_\tau = \tau\tilde{D} + (1-\tau)D = D + \tau\Delta = \tilde{D} - (1-\tau)\Delta$$

$$F_\tau = \tau\tilde{F} + (1-\tau)F - \tau(1-\tau)\Delta^{\wedge 2} =$$

$$= F + \tau D\wedge\Delta + \tau^2\Delta^{\wedge 2} = \tilde{F} - (1-\tau)\tilde{D}\wedge\Delta + (1-\tau)^2\Delta^{\wedge 2}$$

$$\frac{\partial}{\partial\tau}D_\tau = \Delta$$

$$\frac{\partial}{\partial\tau}F_\tau = \tilde{F} - F + (1-2\tau)\Delta^{\wedge 2} = D\wedge\Delta + 2\tau\Delta^{\wedge 2} = \tilde{D}\wedge\Delta - 2(1-\tau)\Delta^{\wedge 2}$$

$$= D_\tau\wedge\Delta \quad \longleftarrow \quad = D\wedge\Delta + [\tau\Delta,\Delta] = \tilde{D}\wedge\Delta - [(1-\tau)\Delta,\Delta]$$

* (vsuvka o součinech)

$$\frac{\partial}{\partial\tau}\hat{P}(F_\tau) = r\,\hat{P}\left(\frac{\partial}{\partial\tau}F_\tau, F_\tau,\dots\right) = r\,\hat{P}(D_\tau\wedge\Delta, F_\tau,\dots) =$$

homogenní stupně r

$$= r\,D_\tau\wedge\hat{P}(\Delta, F_\tau,\dots) = r\,d\,\hat{P}(\Delta, F_\tau,\dots)$$

konst. P a $D_\tau\wedge F_\tau = 0$

pro polynomiální $P(X) = \sum_r P_r(X)$ definujeme

$$P'(\Delta,X) = \sum_r r\,P_r(\Delta, X,\dots,X) =$$

$$= \sum_r \left(P_r(\Delta,X,\dots X) + P_r(X,\Delta,\dots,X) + \dots + P_r(X,X,\dots\Delta)\right)$$

lze rozšířit na formy, X sudé, Δ libovolné

$\Downarrow$

$$\frac{\partial}{\partial\tau}\hat{P}(F_\tau) = d\,\hat{P}'(\Delta, F_\tau)$$

$$\hat{P}(\tilde{F}) - \hat{P}(F) = d\int_0^1 \hat{P}'(\Delta, F_\tau)\,d\tau$$

$\Downarrow$

forma transgrese

$$TP(\tilde{D}, D) = \int_0^1 \hat{P}'(\Delta, F_\tau)\,d\tau$$

* úmluva o součinech

A, B, Δ 1-formy s hodnotami v $E^1_1 M$, tj. $\Delta_{\underline{m}}{}^{\underline{A}}{}_{\underline{B}}$

$[\,,\,]$ komutátor ve fibrových indexech

$\wedge$ vnější násobení v prostoročasových indexech.

$\cdot$ kontrakce ve fibrových indexech.

platí:

$$[A_m, B_n] = A_m \cdot B_n - B_n \cdot A_m \underset{\substack{\uparrow \\ \text{pozor!!!}}}{\neq} A_{\underline{m}} \wedge B_{\underline{n}} = A_m \cdot B_n - A_n \cdot B_m$$

$$[A_m \overset{\wedge}{,} B_n] = A_m \wedge B_n - B_n \wedge A_m = A_m \cdot B_n - A_n \cdot B_m - B_n \cdot A_m + B_m \cdot A_n$$

$$= (A \wedge B + B \wedge A)_{mn} = [A_m, B_n] + [B_m, A_n]$$

$$[\Delta_m, \Delta_n] = \Delta_m \cdot \Delta_n - \Delta_n \cdot \Delta_m = \Delta_m \wedge \Delta_n = \Delta^{\wedge 2}_{mn}$$

$$[\Delta_m \overset{\wedge}{,} \Delta_n] = 2[\Delta_{[m}, \Delta_{n]}] = \Delta_m \wedge \Delta_n - \Delta_n \wedge \Delta_m = 2\Delta_m \wedge \Delta_n = 2\Delta^{\wedge 2}_{mn}$$

Charakteristické třídy

P inv. sym. polynom

charakter. třída odpovídající P

$$p = [\hat{P}(F)] \in \mathcal{H}(M)$$

- je prvek kohomologické grupy $\mathcal{H}(M)$
- nezávisí na volbě D
- pro P homogenní stupně r je p $2r$-forma
- pro P nehomogenní označujeme p_{2r} hom. komp.

skutečně, podle Chern-Weilovy věty

$$d\hat{P}(F) = 0$$

$$\hat{P}(\tilde{F}) = \hat{P}(F) + d\,TP(\tilde{D}, D)$$

$$\Rightarrow [\hat{P}(\tilde{F})] = [\hat{P}(F)]$$

kde $[\omega] = \{\omega + d\alpha\}$ pro $d\omega = 0$

integrální charakteristiky (M kompaktní)

$$\int_M p \wedge q \wedge \ldots = \int_M \hat{P}(F) \wedge \hat{Q}(F) \wedge \ldots \qquad p \wedge q \wedge \ldots \text{ stupně } \dim M$$

↑ nezávisí na volbě reprezentanta

$$\int_M \hat{P}(\breve{F}) \wedge \hat{Q}(F) \wedge \ldots = \int_M \hat{P}(F) \wedge \hat{Q}(F) \wedge \ldots + \int_M d\,TP \wedge Q(F) \wedge \ldots$$

$$\hookrightarrow \int_M d(TP \wedge Q(F) \wedge \ldots) = \int_{\partial M} \cancel{TP \wedge Q(F) \wedge} \overset{\nearrow 0}{}$$

při vhodném škálování dává celočíselné hodn.

Chernovy charakteristické třídy

D kov. der. na obecné (komplexní) bundle $E\,M$

(může mít $U(n)$ strukturu či být $GL(n)$ reálný; pro $SO(n)$ ale degenerace → Pontrjaginovy třídy)

Obecná Chernova charakteristická třída je generována inv. sym. polyn.

$$\hat{C}(F) = \hat{\det}\left(\mathbb{1} + \frac{iF}{2\pi}\right)$$

$$c = [\hat{C}(F)] \qquad \text{kde} \qquad c[M]$$

nehomogenní forma

Chernova charakt. třída stupně k – homog. komponenta obecné Chernovy třídy

$$\hat{C}(F) = 1 + \hat{C}_1(F) + \hat{C}_2(F) + \dots$$

$\hat{C}_k(F)$ homog. stupně k $\qquad$ $2k$-forma

$$c_k = [\hat{C}_k(F)]$$

rozpisem determinantu dostaneme $\left(\frac{i}{2\pi} \to \lambda\right)$

$$\hat{\det}(\mathbb{1}+\lambda F) = {}^{[N]}\delta^{B_1 \dots B_N}_{A_1 \dots A_N} (\mathbb{1}+\lambda F)^{A_1}_{B_1} \wedge \dots \wedge (\mathbb{1}+\lambda F)^{A_N}_{B_N}$$

$$= \sum_{k=0}^{N} \lambda^k \underbrace{{}^{[N]}\delta^{B_1 \, B_k \, C_1 \dots C_{N-k}}_{A_1 \, A_k \, C_1 \dots C_{N-k}}}_{\binom{N}{k}^{-1} \, {}^{[k]}\delta^{B_1 \dots B_k}_{A_1 \dots A_k}} \binom{N}{k} F^{A_1}_{B_1} \wedge \dots \wedge F^{A_k}_{B_k}$$

[počet výběrů k součástí v N členech

$$= \sum_{k=0}^{N} \lambda^k \, {}^{[k]}\delta^{B_1 \dots B_k}_{A_1 \dots A_k} F^{A_1}_{B_1} \wedge \dots \wedge F^{A_k}_{B_k} = \sum_{k=0}^{N} \lambda^k F^{[A_1}_{A_1} \wedge \dots \wedge F^{A_k]}_{A_k}$$

$$= 1 + \lambda F^A_A + \frac{1}{2}\lambda^2 \left(F^A_A \wedge F^B_B - F^B_A \wedge B^A_B\right)$$

$$+ \frac{1}{3!}\lambda^3 \left(F^A_A \wedge F^B_B \wedge F^C_C + F^C_A \wedge F^A_B \wedge F^B_C + F^B_A \wedge F^C_B \wedge F^A_C - F^B_A \wedge F^A_B \wedge F^C_C - F^C_A \wedge F^B_B \wedge F^A_C - F^A_A \wedge F^C_B \wedge F^B_C\right) + \dots$$

$$= 1 + \lambda \operatorname{tr} F + \frac{1}{2}\lambda^2 \left((\operatorname{tr} F)^{\wedge 2} - \operatorname{tr}(F^{\wedge 2})\right) + \frac{1}{3!}\lambda^3 \left((\operatorname{tr} F)^{\wedge 3} + 2\operatorname{tr}(F^{\wedge 3}) - 3(\operatorname{tr} F) \wedge \operatorname{tr}(F^{\wedge 2})\right) + \dots$$

$$= 1 + \hat{C}_1[F] + \hat{C}_2[F] + \hat{C}_3[F] + \dots$$

Chernova charakteristika

Obecná Chernova charakteristika (Chern character)

je generována sym. inv. polyn.

$$\hat{Ch}(F) = \mathrm{tr}\, \hat{\exp} \frac{iF}{2\pi} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathrm{tr}\, F^{\wedge k} \qquad \lambda = \frac{i}{2\pi}$$

$$ch = [\hat{Ch}(F)]$$

nehomogenní

Chernova charakteristika stupně k

- homogenní komponenta obecné Chern. charakt.

$$\hat{Ch}(F) = \sum_{k=0}^{\infty} \hat{Ch}_k(F)$$

$$\hat{Ch}_k(F) = \frac{\lambda^k}{k!} \mathrm{tr}\, F^{\wedge k} = \frac{\lambda^k}{k!} \hat{\mathrm{str}}_k(F, \dots, F) \qquad 2k\text{-forma}$$

$$ch_k = [\hat{Ch}_k(F)]$$

často se místo $\hat{Ch}_k(F)$ užívá $\mathrm{tr}\, F^{\wedge k}$

symetrická stopa

$$\mathrm{str}_k(A_1, \dots, A_k) = \sum_{\text{perm. } \sigma} \mathrm{tr}(A_{\sigma_1} \cdot \ldots \cdot A_{\sigma_k})$$

$$\mathrm{str}_k\, \alpha_1 \dots \alpha_k = t_{(\alpha_1}{}^{A_k}{}_{A_1}\, t_{\alpha_2}{}^{A_1}{}_{A_2} \dots t_{\alpha_k)}{}^{A_{k-1}}{}_{A_k}$$

vztah k Chernovým charakt. třídám

viz předchozí výpočet

$$\hat{C}_1 = \hat{Ch}_1$$

$$\hat{C}_2 = -\hat{Ch}_2 + \frac{1}{2} \hat{Ch}_1^{\wedge 2}$$

$$\hat{C}_3 = 2\hat{Ch}_3 - \hat{Ch}_2 \wedge \hat{Ch}_1 + \frac{1}{6} \hat{Ch}_1^{\wedge 3}$$

$\vdots$

inverzní vztahy

$$\hat{\log}\,\hat{\det}(\mathbb{1}+\lambda F) = \mathrm{tr}(\hat{\log}(\mathbb{1}+\lambda F))$$

$$= \mathrm{tr}\left(\lambda F - \frac{1}{2}\lambda^2 F^{\wedge 2} + \frac{1}{3}\lambda^3 F^{\wedge 3} - \dots\right)$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\lambda^k}{k}\, \mathrm{tr}\, F^{\wedge k}$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} (k-1)!\, \hat{Ch}_k(F)$$

$$= Ch_1 - Ch_2 + 2\,Ch_3 + \dots$$

$$= \hat{\log}\left(1 + \sum_{k=1}^{\infty} \hat{C}_k\right) =$$

$$= (\hat{C}_1 + \hat{C}_2 + \hat{C}_3 + \dots) - \frac{1}{2}(\hat{C}_1 + \hat{C}_2 + \dots)^{\wedge 2} + \frac{1}{3}(\hat{C}_1 + \dots)^{\wedge 3} + \dots$$

$$= \hat{C}_1 + \left(\hat{C}_2 - \frac{1}{2}\hat{C}_1^{\wedge 2}\right) + \left(\hat{C}_3 - \hat{C}_2 \wedge \hat{C}_1 + \frac{1}{3}\hat{C}_1^{\wedge 3}\right) + \dots$$

$$\Downarrow \quad \hat{Ch}_1 = \hat{C}_1$$

$$\hat{Ch}_2 = -\hat{C}_2 + \frac{1}{2}\hat{C}_1^{\wedge 2}$$

$$\hat{Ch}_3 = \frac{1}{2}\hat{C}_3 - \frac{1}{2}\hat{C}_2 \wedge \hat{C}_1 + \frac{1}{6}\hat{C}_1^{\wedge 3}$$

$$\vdots$$

obecně $\hat{Ch}_k$, případně $\hat{C}_k$ tvoří bázi v algebře sym. inv. polyn., tj. každý inv. sym. polyn. lze zapsat jako symetrické fce Chernových charakteristik případně Chernových charakt. tříd

Pontrjaginovy charakteristické třídy

$E \to M$ reálný vekt. bundle s metrikou H

grupa symetrie $O(E, H)$, tj.

$$F = -F^T$$

definujme positivní sym. oper.

$$Q = F \wedge F^T = -F^{\wedge 2}$$

Obecná Pontrjagin-ova třída

$$\hat{PA}(F) = \det{}^{\frac{1}{2}}\left(\mathbb{1} + \frac{Q}{(2\pi)^2}\right) = \det{}^{\frac{1}{2}}\left(\mathbb{1} - \left(\frac{F}{2\pi}\right)^{\wedge 2}\right)$$

$$= \det\left(\mathbb{1} - \frac{F}{2\pi}\right) = \det\left(\mathbb{1} + \frac{F}{2\pi}\right)$$

$$p = [\hat{PA}(F)]$$

pozor, platí $\operatorname{tr} F^{\wedge(2k+1)} = 0$

Pontrjagin-ova charakt. třída stupně $2k$

homogenní komponenta $\sim F^{\wedge 2k}$ ($4k$-forma)

$$\hat{PA}(F) = \sum_{k=0}^{\infty} \hat{PA}_{2k}(F) \qquad \hat{PA}_{2k+1}(F) = 0$$

výpočet obdobný Chern-ové třídě $\left(\lambda \to \frac{1}{(2\pi)^2}\right)$

$$\hat{PA}(F) = 1 + \hat{PA}_2 + \hat{PA}_4 + \hat{PA}_6 + \dots$$

$$= \left[1 + \lambda \operatorname{tr} Q + \frac{1}{2}\lambda^2\left(-\operatorname{tr} Q^{\wedge 2} + (\operatorname{tr} Q)^{\wedge 2}\right) + \frac{1}{3!}\lambda^3\left(2\operatorname{tr} Q^{\wedge 3} - 3\operatorname{tr} Q^{\wedge 2} \wedge \operatorname{tr} Q + (\operatorname{tr} Q)^{\wedge 3}\right) + \dots\right]^{\frac{1}{2}}$$

$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$

$$= 1 - \frac{1}{2}\lambda \operatorname{tr} F^{\wedge 2}$$

$$+ \frac{1}{8}\lambda^2\left(-2\operatorname{tr} F^{\wedge 4} + (\operatorname{tr} F^{\wedge 2})^{\wedge 2}\right)$$

$$+ \frac{1}{48}\lambda^3\left(-8\operatorname{tr} F^{\wedge 6} + 6\operatorname{tr} F^{\wedge 4} \wedge \operatorname{tr} F^{\wedge 2} - (\operatorname{tr} F^{\wedge 2})^{\wedge 3}\right) + \dots$$

zřejmě musí

$$4k \leq \dim M$$

$$2k \leq \dim E$$

$n=1$ dim $=2$

$$e = \frac{1}{2} R_{mn}{}^{mn} \varepsilon = \frac{1}{2} R \, \varepsilon = K \varepsilon$$

$$\frac{1}{2\pi} \int_M K \varepsilon = b_0 - b_1 + b_2 \underset{\substack{\uparrow \\ \text{sféra}}}{=} 1 - 0 + 1 = 2$$

sféra: $K = \frac{1}{r^2}$ $|\varepsilon| = r^2 d\omega$

$n=2$ dim $=4$

$$(2\pi)^2 e = \frac{1}{8} \left(R_{kl}{}^{kl} \wedge R_{mn}{}^{mn} \right) \varepsilon =$$

$$= \frac{1}{8} \left(R_{kl}{}^{kl} R_{mn}{}^{mn} + R_{mn}{}^{kl} R_{kl}{}^{mn} - 4 R_{km}{}^{kl} R_{ln}{}^{mn} \right) \varepsilon$$

$$= \frac{1}{8} \Big(\underset{R_{klmn} R^{klmn}}{\mathbf{R}^2} - 4 \underset{Ric_{kl} Ric^{kl}}{Ric^2} + \underset{RR}{R^2} \Big) \varepsilon$$

$$\frac{1}{8} \frac{1}{(2\pi)^2} \int \left(\mathbf{R}^2 - 4 Ric^2 + R^2 \right) \varepsilon = \underset{\text{top. invariant}}{b_0 - b_1 + b_2 - b_3 + b_4}$$

Eulerova třída

tečný bundle – reálný vekt bundle

metrika $\to$ metr. konexe ∇

$\to$ křivost $R_{ab}{}^{m}{}_{n}$

grupa sym – $so(2n)$ $\quad X^{m}{}_{n} \quad X^{mn} = X^{[mn]}$

$\dim M = 2n+1 \qquad e = 0$

$\dim M = 2n$

$$e[M] = \left[\hat{Pf}\left(\frac{R}{2\pi}\right)\right]$$

Pfafián

$$Pf(X) = \frac{1}{2^n n!}\,\varepsilon_{a_1 b_1 \ldots a_n b_n} X^{a_1 b_1} \ldots X^{a_n b_n}$$

$$= X^{12} X^{34} \ldots X^{2n-1\,2n}$$

$$= \xi_1 \xi_2 \ldots \xi_n$$

$$= (\det X)^{\frac{1}{2}}$$

$$\begin{bmatrix} 0 & \xi_1 & & & & & \\ -\xi_1 & 0 & & & & & \\ & & 0 & \xi_2 & & & \\ & & -\xi_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \xi_n \\ & & & & & -\xi_n & 0 \end{bmatrix}$$

Euler. char.

$$\hat{Pf}(R)_{a_1 b_1 \ldots a_n b_n} = \frac{1}{2^n n!}\,\varepsilon_{k_1 l_1 \ldots k_n l_n} R_{a_1 b_1}{}^{k_1 l_1} \wedge \ldots \wedge R_{a_n b_n}{}^{k_n l_n}$$

$$= \frac{1}{2^n n!}\frac{1}{(2n)!}\,\varepsilon^{m_1 n_1 \ldots m_n n_n}\,\varepsilon_{k_1 l_1 \ldots k_n l_n} R_{m_1 n_1}{}^{k_1 l_1} \wedge \ldots \wedge R_{m_n n_n}{}^{k_n l_n}\,\varepsilon_{a_1 b_1 \ldots a_n b_n}$$

$$= \frac{1}{2^n n!} R_{k_1 l_1}{}^{k_1 l_1} \wedge \ldots \wedge R_{k_n l_n}{}^{k_n l_n}\,\varepsilon_{a_1 b_1 \ldots a_n b_n}$$

$$= \frac{(2n)!}{2^n n!} R_{[k_1 l_1}{}^{k_1 l_1} \ldots R_{k_n l_n]}{}^{k_n l_n}\,\varepsilon_{a_1 b_1 \ldots a_n b_n}$$

Gauss-Bonnet theorem

$$\int_M e[M] = \chi(M) = \text{index}_{deRham}(M)$$

kde $\chi(M) = \sum_i (-1)^i b_i$ Eul. charakt.

příklad index theorému

$\int_M e(M)$ dif. geometr. inf. skryté v integr. char. tříd

$\text{index}_{deRham}(M)$ analytické inf. skryté v dimenzích ker a img operátoru d, δ a Δ

$$\chi[M] = \sum_k (-1)^k \underbrace{\dim(H^k M)}_{b_k}$$

b_k Bettiho čísla

topologické informace skryté v dimenzích kohomologických grup – Eulerova charakt.

Lovelock gravity

$R^{(q)} = R^{\wedge q}$ $\qquad$ $\mathcal{L}^{(q)} = \frac{1}{(2q)!} G^{(2q)} R^{(q)} = \frac{1}{2^q} R^q + \dots$

$Ric^{(q)} = \frac{1}{(2q-1)!} G^{2q-1} R^{(q)}$ $\qquad$ $G\,Ric^{(q)} = 2q\,\mathcal{L}^{(q)}$

$Lov^{(q)} = Ric^{(q)} - \mathcal{L}^{(q)} g$ $\qquad$ $G\,Lov^{(q)} = -(d-2q)\,\mathcal{L}^{(q)}$

$S^{(q)}_{LOV} = \int \mathcal{L}^{(q)} g^{\frac{1}{2}}$ $\qquad$ $\frac{2}{g^{\frac{1}{2}}} \frac{\delta S^{(q)}_{LOV}}{\delta g^{-1}} = Lov^{(q)}$ $\qquad$ $-\frac{2}{g^{\frac{1}{2}}} \frac{\delta S_{MAT}}{\delta g^{-1}} = T$

$S_{LOV} = \sum_q \alpha_q S^{(q)}_{LOV}$ $\qquad$ $S = S_{LOV} + S_{MAT}$

$\delta S = 0 \quad \Rightarrow \quad \sum_q \alpha_q Lov^{(q)} = T$

$q=0$ $\quad$ $\mathcal{L}^{(0)} = 1$ $\quad$ $Ric^{(0)} = 0$ $\quad$ $Lov^{(0)} = -g$ $\quad$ $S^{(0)}_{LOV} = \int g^{\frac{1}{2}}$

$q=1$ $\quad$ $\mathcal{L}^{(1)} = \frac{1}{2} R$ $\quad$ $Ric^{(1)} = Ric$ $\quad$ $Lov^{(1)} = Ein = Ric - \frac{1}{2} R g$ $\quad$ $S^{(1)}_{LOV} = \frac{1}{2} \int R g$

$q=2$ $\quad$ $\mathcal{L}^{(2)} = \frac{1}{4} R_{GB} = \frac{1}{4}(\mathcal{R}^2 - 4Ric^2 + R^2) = \frac{1}{4}(R_{abcd}R^{abcd} - 4 Ric_{ab} Ric^{ab} + R^2)$

$Ric^{(2)}_{ab} = R\,Ric_{ab} - 2 Ric_{am} Ric_b{}^m - 2 R_{ambn} Ric^{mn} + R_{aklm} R_b{}^{klm}$

$Lov^{(2)} = \frac{1}{2} H = Ric^{(2)} - \mathcal{L}^{(2)} g$

$S^{(2)}_{LOV} = \frac{1}{4} \int (\mathcal{R}^2 - 4 Ric^2 + R^2)\, g^{\frac{1}{2}}$

$q=m$, $d=2m$ $\quad$ $R^{(m)} = \mathcal{L}^{(m)}\, {}^{[2m]}\delta$ $\qquad$ $\mathcal{L}^{(m)} \varepsilon$ $\quad$ Eulerove charakt. forma

$Ric^{(m)} = \mathcal{L}^{(m)} g$

$Lov^{(m)} = 0$

$S^{(m)}_{LOV} = e$ $\qquad$ e $\quad$ Eulerova charakteristika

Einstein $\quad$ $\alpha_0 = -\frac{\Lambda}{\varkappa}$ $\quad$ $\alpha_1 = \frac{1}{\varkappa}$ $\qquad$ $S = \frac{1}{2\varkappa} \int (R - 2\Lambda)\, g^{\frac{1}{2}}$

$Ein + \Lambda g = \varkappa T$

E-G-B $\quad$ $\alpha_0 = -\frac{\Lambda}{\varkappa}$ $\quad$ $\alpha_1 = \frac{1}{\varkappa}$ $\quad$ $\alpha_2 = \frac{2\alpha}{\varkappa}$ $\qquad$ $S = \frac{1}{2\varkappa} \int (R - 2\Lambda + \alpha R_{GB})\, g^{\frac{1}{2}}$

$Ein + \Lambda g + \alpha H = \varkappa T$