Lieovy grupy a levoinvariantní pole

Def G je Lieova grupa $\equiv$

G je grupa a diferenc. varieta

$$\left.\begin{array}{lll} G\times G \to G & g,h \to gh \\ G \to G & g \to g^{-1} \\ G \to \{e\} & g \to e \end{array}\right\} \text{ jsou hladké}$$

Pozn: místo trojice zobrazení lze požadovat hladkost zobr.

$G\times G \to G \qquad g,h \to gh^{-1}$

Pozn: hladkost C^0 implikuje existenci analytické struktury

($C^0 \Rightarrow$ anal - těžké - Hilbertův problém, $C^2 \Rightarrow$ anal - jednoduché)

Def: L_g a R_g jsou levé a pravé násobení (posun) na G $\equiv$

$L_g : G \to G \qquad L_g h = gh$

$R_g : G \to G \qquad R_g h = hg$

AD_g je adjoint zobr. (konjugace) G na G $\equiv$

$AD_g : G \to G \qquad AD_g h = g h g^{-1}$ tj. $AD_g = L_g R_{g^{-1}}$

Lemma: $L_g R_h = R_h L_g \qquad \forall g,h \in G$

$\forall g\in G \quad L_g H = H L_g \Rightarrow \exists h\in G \quad H = R_h$

$\forall g\in G \quad R_g H = H R_g \Rightarrow \exists h\in G \quad H = L_h$

D: nechť $h = He$

$Hg = HL_g e = L_g He = L_g h = gh = R_h g \qquad \Rightarrow H = R_h$

Def: tenz. pole A na G je levoinvariantní $\equiv$

$\forall g\in G \qquad L_{g*} A = A$

obdobně pravoinvariantnost

Def: $l[A] = l_A \in \mathbf{T}^a_b G$ je levoinv. roznesení tenzoru $A \in \mathbf{T}^a_{e\,b} G$ do G $\equiv$

$l[A]|_g = L_{g*} A$

obdobně $r[A] = r_A$ značí pravoinv. roznesení

Pozn: zvláště pro $m \in \mathbf{T}_e G$ budeme užívat l_m

Lema: levoinv. roznesení $l[A]$ je levoinvariantní
obdobně pro pravoinv.

D: $(L_{g*} L[A])|_h = L_{g*}(L[A]|_{g^{-1}h}) = L_{g*} L_{g^{-1}h*} A|_e = L_{h*} A|_e = L[A]|_h$

Lema: levo/pravo-invariant-ost komutuje s
$\otimes$, C, $\wedge$, d

D: plyne z vlastností indukc. zobr. (např. $\phi^* d = d\phi^*$)

Lema: $a, b \in \mathcal{T}M$ levoinvariantní vekt. pole $\Rightarrow$
$[a,b]$ je levoinvariantní pole
obdobně pro pravoinv.

D: plyne opět z vl. indukc. zobr. $\phi_*[a,b] = [\phi_* a, \phi_* b]$

Theorem $A \in \mathcal{T}^g_e M$ levo/pravo-invariantní pole
je jednoznačně dáno hodnotou $A|_e$ v e a to
levo/pravo-invariantním roznesením $A = l[A|_e]$ resp. $r[A|_e]$

D: $A|_g = (L_{g*}A)|_g = L_{g*}(A|_e) = l[A]|_g$

Lema: nechť E_α, E^α jsou levoinv. báze, $A = A^{\alpha\dots}_{\beta\dots} E_\alpha \dots E^\beta \dots$
A je levoinv. $\Leftrightarrow$ $A^{\alpha\dots}_{\beta\dots}$ jsou konstanty

Theorem: $A \in \mathcal{T}^g_e G$ $g \in G$

$L_{g*}\, l[A] = l[A]$ $\quad$ $R_{g*}\, r[A] = r[A]$

$R_{g*}\, l[A] = l[AD_{g^{-1}*} A]$ $\quad$ $L_{g*}\, r[A] = r[AD_{g*} A]$

$AD_{g*}\, l[A] = l[AD_{g*} A]$ $\quad$ $AD_{g*}\, r[A] = r[AD_{g*} A]$

$r[A]|_g = l[AD_{g^{-1}*} A]|_g$ $\quad$ $l[A]|_g = r[AD_{g*} A]|_g$

D: $(R_{g*}\, l[A])|_h = R_{g*}\, l[A]|_{hg^{-1}} = R_{g*} L_{hg^{-1}*} A = L_h AD_{g^{-1}*} A = l[AD_{g^{-1}*} A]|_h$

$AD_{g*}\, l[A] = L_{g*} R_{g^{-1}*} l[A] = l[AD_{g^{-1}*} A]$

$r[A]|_g = R_{g*} A = R_{g*}\, l[A]|_e = l[AD_{g^{-1}*} A]|_g \Rightarrow l[A]|_g = r[AD_{g*} A]|_g$

Theorem: $A \in \mathcal{T}^g_e G$ $\quad$ A levoinvariantní
A je bi-invariantní (levo i pravo-inv.) $\Leftrightarrow$
$A|_e$ je AD-invariantní tj. $A|_e = AD_{g*} A|_e$

D: $R_{g*} A = R_{g*}\, l[A|_e] = l[AD_{g^{-1}*} A|_e] = l[A|_e] = A$

Lieova algebra LG a exponenciální zobrazení

Def: $\mathfrak{g}$ je Lieova algebra LG G $\equiv$

$\mathfrak{g} = T_e G$ prostor prvků LA

$[a,b] = [l_a, l_b]|_e$ Lieova závorka na LA

Pozn.: LA $\mathfrak{g}$ je izomorfní s LA levoinv. polí se stand. Lieovou závorkou a to skrze levoinv. rozš. Vztah $a \to l_a$ $[l_a, l_b] = l_{[a,b]}$

Def: $c_{\alpha\beta}{}^{\gamma}$ je strukturní tenzor LA $\mathfrak{g}$ $\equiv$

$[a,b]^{\gamma} = a^{\alpha} b^{\beta} c_{\alpha\beta}{}^{\gamma}$

$c_{\alpha\beta}{}^{\gamma}$ rozneseme na celou G levoinv. rozšíření

komponenty $c_{\alpha\beta}{}^{\gamma}$ vůči levoinv. bázi E_α nazýv. strukt. konst.

Def: $k_{\alpha\beta}$ je Killingova bi-lin forma (metrika) $\equiv$

$k_{\alpha\beta} = -\frac{1}{2} c_{\alpha\mu}{}^{\nu} c_{\beta\nu}{}^{\mu}$

Pozn.: koeficient "$-\frac{1}{2}$" je konvenční. Pro kompaktní polprostou bude $k_{\alpha\beta}$ pozitivně definitní.
$k_{\alpha\beta}$ je metrikou (nedegenerovaná) pro polprosté gr.

Lema: $c_{\alpha\beta}{}^{\gamma}$ a $k_{\alpha\beta}$ jsou levoinvariantní

$[l_a, l_b]^{\gamma} = l_a^{\alpha} l_b^{\beta} c_{\alpha\beta}{}^{\gamma}$

Theorem: $c_{\alpha\beta}{}^{\gamma}$ a $k_{\alpha\beta}$ jsou bi-invariantní

D: $Ad_g a = AD_{g*} a$ viz níže, $AD_g e = e$

$Ad_g [a,b] = Ad_g [l_a, l_b]|_e = [Ad_g l_a, Ad_g l_b]|_e = [l_{Ad_g a}, l_{Ad_g b}]|_e = [Ad_g a, Ad_g b]$

$\Rightarrow AD_{g*} c_{\alpha\beta}{}^{\gamma} = c_{\alpha\beta}{}^{\gamma}$ $\Rightarrow$ $c_{\alpha\beta}{}^{\gamma}$ je bi-invariantní

$\Rightarrow k_{\alpha\beta}$ bi-invariantní

Def: exp je exponenc. zobr $\equiv$

$\exp : \mathfrak{g} \to G$

$\exp((\alpha+\beta)m) = \exp(\alpha m)\exp(\beta m)$

$\exp(0) = e \qquad \frac{D}{d\varepsilon}\exp(\varepsilon m)\big|_{\varepsilon=0} = m$

Pozn.: $\exp(\alpha m)$ tak tvoří 1-param. abel. podgrupu G vycházející z e ve směru m

Theorem

$\exp(\alpha m)$ je integr. kř. polí l_m i r_m procházející e

D: nechť $\tilde{l}_m(\alpha), \tilde{r}_m(\alpha)$ jsou integr. kř. polí l_m, r_m skrze e tak že $\tilde{l}_m(0) = \tilde{r}_m(0) = e$

ukážeme, že $\tilde{l}_m(\alpha)$ i $\tilde{r}_m(\alpha)$ splňují vlastnosti $\exp(\alpha m)$

Lema: $\gamma(\alpha)$ je integr. kř. l_m mající společný bod s $\tilde{l}_m(\alpha) \Rightarrow \gamma(\alpha) = \tilde{l}_m(\alpha+\alpha_0)$

plyne z jedn. řešení dif. rovnice prvního řádu

$\tilde{r}_m(0) = e$ - předpoklad $\qquad \frac{D\tilde{r}_m}{d\alpha}\big|_{\alpha=0} = r_m|_e = m$

nechť $k(\alpha) = \tilde{r}_m(\alpha)\tilde{r}_m(\beta) \qquad k(0) = \tilde{r}_m(\beta)$

$\frac{Dk}{d\alpha}\big|_{\alpha_0} = R_{\tilde{r}_m(\beta)*}\frac{D\tilde{r}_m}{d\alpha}\big|_{\alpha_0} = R_{\tilde{r}_m(\beta)*} r_m|_{\tilde{r}_m(\alpha_0)} = r_m|_{\tilde{r}_m(\alpha_0)\tilde{r}_m(\beta)} = r_m|_{k(\alpha_0)}$

$\Rightarrow k(\alpha)$ integr. kř. $r_m \Rightarrow k(\alpha) = \tilde{r}_m(\alpha+\beta) \Rightarrow \tilde{r}_m(\alpha+\beta) = \tilde{r}_m(\alpha)\tilde{r}_m(\beta)$

$r_m|_{\tilde{l}_m(\alpha_0)} = R_{\tilde{l}_m(\alpha_0)} m = R_{\tilde{l}_m(\alpha_0)*}\frac{D\tilde{l}_m}{d\varepsilon}\big|_{\varepsilon=0} = \frac{D}{d\varepsilon}(\tilde{l}_m(\varepsilon)\, l_m(\alpha_0))\big|_{\varepsilon=0} = \frac{D}{d\varepsilon}\tilde{l}_m(\alpha_0+\varepsilon)\big|_{\varepsilon=0} = l_m|_{\tilde{l}_m(\alpha_0)}$

$\Rightarrow \tilde{l}_m(\alpha)$ i $\tilde{r}_m(\alpha)$ jsou int. kř. obou polí l_m i r_m

$\Rightarrow \exp(\alpha m) = \tilde{l}_m(\alpha) = \tilde{r}_m(\alpha)$

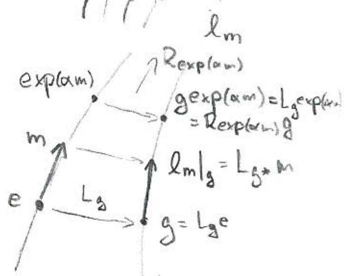
Theorem

l_m je generátor 1-par. gr. diff $R_{\exp(\alpha m)}$

r_m je generátor 1-par. gr. diff $L_{\exp(\alpha m)}$

D: l_m je gener. $R_{\exp(\alpha m)} \Leftrightarrow l_m|_g = \frac{D}{d\varepsilon} R_{\exp(\varepsilon m)}\, g\big|_{\varepsilon=0}$

$\frac{D}{d\varepsilon} R_{\exp(\varepsilon m)}\, g\big|_{\varepsilon=0} = \frac{D}{d\varepsilon}\, g\exp(\varepsilon m)\big|_{\varepsilon=0} = L_{g*}\frac{D}{d\varepsilon}\exp(\varepsilon m)\big|_{\varepsilon=0} = L_{g*} m = l_m|_g$



Lema: $A \in \mathcal{T}^r_s G$

$\mathcal{L}_{l_m} A = -\frac{d}{d\alpha} R_{\exp(\alpha m)*} A\big|_{\alpha=0} \qquad \mathcal{L}_{r_m} A = -\frac{d}{d\alpha} L_{\exp(\alpha m)*} A\big|_{\alpha=0}$

Theorem: $A \in \mathcal{T}^r_s G$

A je levoinvar $\Leftrightarrow \forall m \;\; \mathcal{L}_{r_m} A = 0$

A je pravoinvar $\Leftrightarrow \forall m \;\; \mathcal{L}_{l_m} A = 0$

Adjoint reprezentace

Def AD_g je adjoint zobr. (konjugace) G na G ≡

$AD_g : G \to G \qquad AD_g h = g h g^{-1}$

Def Ad_g je adjoint zobr (přidružené) G na $\mathfrak{g}$ ≡

$Ad_g : \mathfrak{g} \to \mathfrak{g} \qquad Ad_g = AD_{g*}$

Lema:

$AD_g = L_g R_{g^{-1}} \qquad Ad_g = L_{g*} R_{g^{-1}*}$

Lema:

$Ad_g m = \frac{D}{d\varepsilon} AD_g h(\varepsilon)\Big|_{\varepsilon=0}$ kde $\frac{Dh}{d\varepsilon}\Big|_{\varepsilon=0} = m$

např. pro $h(\alpha) = \exp(\alpha m)$

Theorem

AD_g je homomorf. gr. G $\qquad$ AD je reprez. G na G

Ad_g je homomorf alg. $\mathfrak{g}$ $\qquad$ Ad je reprez. G na $\mathfrak{g}$

D: $AD_g(h_1 h_2) = g h_1 h_2 g^{-1} = g h_1 g^{-1} g h_2 g^{-1} = AD_g h_1 \, AD_g h_2$

$Ad_g [a,b] = [Ad_g a, Ad_g b] \Leftarrow C$ je AD-invariantní

$AD_{g_1 g_2} = AD_{g_1} AD_{g_2} \qquad AD_{g^{-1}} = (AD_g)^{-1} \qquad AD_e = id \qquad$ triviální

$Ad_{g_1 g_2} = Ad_{g_1} Ad_{g_2} \qquad Ad_{g^{-1}} = (Ad_g)^{-1} \qquad Ad_e = \mathbb{1} \qquad \Leftarrow Ad_g = AD_{g*}$

Theorem

$$\exp(Ad_g m) = AD_g \exp m$$

D: $AD_g \exp(\alpha m)$ splňuje vlastn. exp:

$AD_g \exp(\alpha m)\big|_{\alpha=0} = e$

$AD_g \exp((\alpha+\beta) m) = AD_g(\exp(\alpha m) \exp(\beta m)) = (AD_g \exp(\alpha m))(AD_g \exp(\beta m))$

$\frac{D}{d\alpha} AD_g \exp(\alpha m)\big|_{\alpha=0} = Ad_g m$

$\Rightarrow AD_g \exp m = \exp(Ad_g m)$

Def: ad_m je adjoint (přidružené) zobr. $g \to g$ ≡

≡ ad_m je generátor Ad_g tj.

$$ad_m a = \frac{d}{d\varepsilon} Ad_{g_\varepsilon} a \Big|_{\varepsilon=0} \qquad \text{kde } \frac{Dg_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = m \quad g_0 = e$$

např. $g_\varepsilon = \exp(\varepsilon m)$

Theorem:

$$ad_a b = [a,b] \qquad \text{tj.} \quad ad_a{}^F{}_\gamma = a^\alpha c_{\alpha\gamma}{}^\kappa$$

D: $[a,b] = [l_a, l_b]|_e = \mathcal{L}_{l_a} l_b|_e = \frac{d}{d\varepsilon} R_{\exp(-\varepsilon a)*} l_b|_{\varepsilon=0}|_e = l\left[\frac{d}{d\varepsilon} Ad_{\exp(\alpha a)} b\right]\Big|_{\varepsilon=0}\Big|_e = ad_a b$

Theorem: ad je reprezentace $g \to g$ tj.

$$ad_{[a,b]} = [ad_a, ad_b] \equiv ad_a \cdot ad_b - ad_b \cdot ad_a$$

D:

$$ad_{[a,b]} c - [ad_a, ad_b] \cdot c = [[a,b],c] - [a,[b,c]] + [b,[a,c]] =$$
$$= [[a,b],c] + [[b,c],a] + [[c,a],b] \overset{J.I.}{=} 0$$

Theorem:

$$Ad_{\exp a} = \exp(ad_a)$$

kde $\exp M$ pro $M \in g^1_1$ splňuje

$\exp(0) = \delta \qquad \frac{d}{d\alpha}\exp(\alpha M)\Big|_{\alpha=\alpha_0} = M \cdot \exp(\alpha_0 M) \qquad \exp((\alpha+\beta)M) = \exp(\alpha M)\cdot\exp(\beta M)$

a lze napsat jako řadu

$\exp M = \delta + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$

D: o $Ad_{\exp(\alpha a)}$ se musí dokázat vlastnosti $\exp(\alpha ad_a)$

$Ad_{\exp(0a)} = \delta \qquad \frac{D}{d\alpha} Ad_{\exp(\alpha a)}\Big|_{\alpha=0} = ad_a$

$Ad_{\exp((\alpha+\beta)m)} = Ad_{\exp(\alpha m)\exp(\beta m)} = Ad_{\exp(\alpha m)} \cdot Ad_{\exp(\beta m)}$

Theorem

$$[l_a, l_b]^\gamma = l_a^\alpha \, l_b^\beta \, C_{\alpha\beta}{}^\gamma = l^\gamma_{[a,b]}$$

$$[l_a, r_b]^\gamma = 0$$

$$[r_a, r_b]^\gamma = - r_a^\alpha \, r_b^\beta \, C_{\alpha\beta}{}^\gamma = - r^\gamma_{[a,b]}$$

D: $[l_a, l_b] = l_{[a,b]}$ viz výše

$$[l_a, r_b] = \mathcal{L}_{l_a} r_b = - \frac{d}{d\varepsilon} R_{\exp(\varepsilon a)*} r_b \Big|_{\varepsilon=0} = - \frac{d}{d\varepsilon} r_b \Big|_{\varepsilon=0} = 0$$

$$[r_a, r_b] = \mathcal{L}_{r_a} r_b = - \frac{d}{d\varepsilon} L_{\exp(\varepsilon a)*} r_b \Big|_{\varepsilon=0} = - \frac{d}{d\varepsilon} r\left[Ad_{\exp(\varepsilon a)} b \right] \Big|_{\varepsilon=0}$$

$$= - r[ad_a b] = - r_{[a,b]} \qquad R_{g*} C = C$$

$$r^\gamma_{[a,b]}\Big|_g = R_{g*} [a,b]^\gamma = R_{g*}(a^\alpha b^\beta C_{\alpha\beta}{}^\gamma) \overset{\downarrow}{=} r_a^\alpha \, r_b^\beta \, C_{\alpha\beta}{}^\gamma$$

Theorem

$$[a,b] = \frac{D}{d\tau} \left[\exp(\sqrt{\tau} a) \exp(\sqrt{\tau} b) \exp(-\sqrt{\tau} a) \exp(-\sqrt{\tau} b) \right] \Big|_{\tau=0}$$

Def:

G je prostá L. gr. $\equiv$

G je nekomutující

G nemá vlastní invariantní podgrupu, tj.

$Ad_G S = S \Rightarrow (S = G \vee S = \{e\})$

G je poloprostá L. gr. $\equiv$

$G = \bigotimes_{k \text{ L.gr.}} G_k$ $\qquad G_k$ prosté

Def:

$\mathfrak{g}$ je prostá L. alg $\equiv$

$\mathfrak{g}$ je neábelovská

$\mathfrak{g}$ nemá vlastní ideál, tj.

$[I, \mathfrak{g}] = I \Rightarrow (I = \mathfrak{g} \vee I = \{0\})$

$\mathfrak{g}$ je poloprostá L. alg. $\equiv$

$\mathfrak{g} = \bigoplus_{k \text{ L.alg.}} \mathfrak{g}_k$ $\qquad \mathfrak{g}_k$ prosté

Theorem:

G L. gr., $\mathfrak{g}$ její L. alg

G je prostá $\Leftrightarrow$ $\mathfrak{g}$ je prostá

G je poloprostá $\Leftrightarrow$ $\mathfrak{g}$ je poloprostá

C

Theorem

$\mathfrak{g}$ prostá L. alg $\Rightarrow$ adj repr. c je ireducibilní

$\mathfrak{g}$ poloprostá L. alg $\Rightarrow$ adj. repr. c je věrná

$\mathfrak{g}$ poloprostá L. alg. $\Rightarrow$ Kill. metrika k je nedegenerovaná

Theorem

G poloprostá L. gr., $\mathfrak{g}$ její L. alg., k Killing. metrika

G kompaktní $\Rightarrow$ k positivně definitní

Def:

$\mathfrak{g}$ poloprostá L. alg., k Killing. metrika

$\mathfrak{g}$ je kompaktní L. alg $\equiv$ k je positivně definitní

Geometrie Lieových grup

Theorem $\overset{L}{E}_\alpha$ a $\overset{R}{E}_\alpha$ levoinv. a pravoinv. báze, $\overset{L}{E}_\alpha|_e = \overset{R}{E}_\alpha|_e = E_\alpha$ báze v $\mathfrak{g} = T_e G$

$k_{\alpha\beta}$ a $c_{\alpha\beta}{}^\kappa$ jsou biinvariantní tenzory na G

$\overset{L}{k}_{\alpha\beta} = \overset{R}{k}_{\alpha\beta} = k_{\alpha\beta}$ jsou konst, kde $\overset{L}{k}_{\alpha\beta}, \overset{R}{k}_{\alpha\beta}$ jsou komponenty vůči $\overset{L}{E}_\alpha, \overset{R}{E}_\alpha$

obdobně $\overset{L}{c}_{\alpha\beta}{}^\kappa = \overset{R}{c}_{\alpha\beta}{}^\kappa = c_{\alpha\beta}{}^\kappa$ a $k_{\alpha\beta}$ jsou komponenty k vůči E_α v $\mathfrak{g}$

Theorem:

G poloprosté $\Leftrightarrow$ $k_{\alpha\beta}$ nedegenerované

G poloprosté kompaktní $\Rightarrow$ $k_{\alpha\beta}$ pos. definitní

Theorem $c_{\alpha\beta\gamma} = c_{[\alpha\beta\gamma]}$ kde $c_{\alpha\beta\gamma} = c_{\alpha\beta}{}^\mu k_{\mu\gamma}$ D: viz LA

Theorem l_m a r_m jsou Killingovy vektory metriky $k_{\alpha\beta}$

D biinvariantnost $k_{\alpha\beta}$ $\Rightarrow$ $\mathcal{L}_{l_m} k = 0$ $\mathcal{L}_{r_m} k = 0$ $\Leftrightarrow$ Killingy

Theorem G poloprostá LG, ε Levi-Civitův tenzor k

ε je biinvariantní, $\overset{?}{k} = |\varepsilon| = \operatorname{Det}^{\frac{1}{2}} k$ je biinv. obj. element

Def $\overset{L}{\nabla}$ kov. derivace zachovávající levoinv. tenzory

$\overset{R}{\nabla}$ kov. derivace zachovávající pravoinv. tenzory

Theorem

$\overset{L}{\nabla}, \overset{R}{\nabla}$ existují a jsou dány jednoznačně

torze $\overset{L}{T}_{\mu\nu}{}^\kappa = -c_{\mu\nu}{}^\kappa$ $\overset{R}{T}_{\mu\nu}{}^\kappa = c_{\mu\nu}{}^\kappa$

křivost $\overset{L}{R}_{\alpha\beta\mu}{}^\nu = \overset{R}{R}_{\alpha\beta\mu}{}^\nu = 0$

jejich rozdíl tenzor je $c_{\mu\nu}{}^\kappa$ tj. $\overset{R}{\nabla} - \overset{L}{\nabla} = \text{Teas}[c] = c$

D: $\overset{L}{\nabla}$ je dáno podm. $\overset{L}{\nabla} l_m = 0$ $\Rightarrow$ nulová der. levoinv. b. $\overset{L}{E}_\alpha$

$\Rightarrow \overset{L}{T}_{\alpha\beta}{}^\mu = \overset{L}{\nabla}_{E_\alpha} \overset{L}{E}_\beta^\mu - \overset{L}{\nabla}_{E_\beta} \overset{L}{E}_\alpha^\mu - [\overset{L}{E}_\alpha, \overset{L}{E}_\beta]^\mu = -c_{\alpha\beta}{}^\kappa \overset{L}{E}_\kappa^\mu \Rightarrow \overset{L}{T}_{\alpha\beta}{}^\mu = -c_{\alpha\beta}{}^\mu$

$\Rightarrow \overset{L}{R}_{\alpha\beta}{}^\mu{}_\nu = \overset{L}{\nabla}_\alpha \overset{L}{\nabla}_\beta \overset{L}{E}_\nu^\mu - \overset{L}{\nabla}_\beta \overset{L}{\nabla}_\alpha \overset{L}{E}_\nu^\mu - \overset{L}{T}_{\alpha\beta}{}^\kappa \overset{L}{\nabla}_\kappa \overset{L}{E}_\nu^\mu = 0 \Rightarrow \overset{L}{R}_{\alpha\beta}{}^\mu{}_\nu = 0$

pro pravoinv. bázi $\overset{R}{E}_\alpha$ platí $[\overset{R}{E}_\alpha, \overset{R}{E}_\beta] = -c_{\alpha\beta}{}^\kappa \overset{R}{E}_\kappa \Rightarrow \overset{R}{T}_{\alpha\beta}{}^\mu = c_{\alpha\beta}{}^\mu$

$\overset{L}{\nabla}_{l_m} r_n = \overset{L}{\nabla}_{l_m} r_n - \underbrace{\overset{L}{\nabla}_{r_n} l_m}_{0} - \underbrace{[l_m, r_n]}_{0} = l_m \overset{L}{T} \cdot r_n = l_m \cdot c \cdot r_n \Rightarrow \overset{L}{\nabla}_\mu r_n^\nu = -c_{\mu\kappa}{}^\nu r_n^\kappa$

nechť $\overset{R}{\nabla} = \overset{L}{\nabla} + A \Rightarrow 0 = \overset{R}{\nabla}_\mu r_m^\nu = \overset{L}{\nabla}_\mu r_m^\nu + A_{\mu\kappa}{}^\nu r_m^\kappa = (A_{\mu\kappa}{}^\nu - c_{\mu\kappa}{}^\nu) r_m^\kappa \Rightarrow A_{\mu\nu}{}^\kappa = c_{\mu\nu}{}^\kappa$

Theorem $\overset{L}{E}{}^\alpha$ a $\overset{R}{E}{}^\alpha$ jsou levo a pravoinv. báze v T^*G, $\overset{R}{E}{}^\alpha|_e = \overset{L}{E}{}^\alpha|_e$

$$d\overset{L}{E}{}^\alpha + \frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{L}{E}{}^\kappa \wedge \overset{L}{E}{}^\lambda = 0$$

$$d\overset{R}{E}{}^\alpha - \frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{R}{E}{}^\kappa \wedge \overset{R}{E}{}^\lambda = 0$$

D. $d_\mu \overset{L}{E}{}^\alpha_\nu = \overset{L}{\nabla}_\mu \wedge \overset{L}{E}{}^\alpha_\nu + \overset{L}{T}_{\mu\nu}{}^\beta \overset{L}{E}{}^\alpha_\beta = -c_{\mu\nu}{}^\beta \overset{L}{E}{}^\alpha_\beta = -c_{\kappa\lambda}{}^\alpha \overset{L}{E}{}^\kappa_\mu \overset{L}{E}{}^\lambda_\nu = -\frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{L}{E}{}^\kappa_\mu \wedge \overset{L}{E}{}^\lambda_\nu$

$d_\mu \overset{R}{E}{}^\alpha_\nu = \overset{R}{\nabla}_\mu \wedge \overset{R}{E}{}^\alpha_\nu + \overset{R}{T}_{\mu\nu}{}^\beta \overset{R}{E}{}^\alpha_\beta = c_{\mu\nu}{}^\beta \overset{R}{E}{}^\alpha_\beta = c_{\kappa\lambda}{}^\alpha \overset{R}{E}{}^\kappa_\mu \overset{R}{E}{}^\lambda_\nu = \frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{R}{E}{}^\kappa_\mu \wedge \overset{R}{E}{}^\lambda_\nu$

Def:

$$\overset{\lambda}{\nabla} = \overset{L}{\nabla} + \frac{\lambda+1}{2}\,\mathbf{c} = \overset{R}{\nabla} + \frac{\lambda-1}{2}\,\mathbf{c} \qquad \nabla \equiv \overset{\lambda=0}{\nabla}$$

tj. $\overset{\lambda}{\nabla}_\mu a^\kappa = \overset{L}{\nabla}_\mu a^\kappa + \frac{\lambda+1}{2} c_{\mu\nu}{}^\kappa a^\nu = \overset{R}{\nabla}_\mu a^\kappa + \frac{\lambda-1}{2} c_{\mu\nu}{}^\kappa a^\nu$

Pozn.: $\overset{\lambda}{\nabla}$ je "interpolace" mezi $\overset{L}{\nabla} = \overset{\lambda=-1}{\nabla}$ a $\overset{R}{\nabla} = \overset{\lambda=1}{\nabla}$

tj. ve smyslu afinní str. prostoru konexí $\overset{\lambda}{\nabla} = \frac{1-\lambda}{2}\overset{L}{\nabla} + \frac{1+\lambda}{2}\overset{R}{\nabla}$

Lemma

$$\overset{\lambda}{\nabla}_\mu l_m^\kappa = \frac{\lambda+1}{2} c_{\mu\nu}{}^\kappa l_m^\nu \qquad \overset{\lambda}{\nabla}_\mu r_m^\kappa = \frac{\lambda-1}{2} c_{\mu\nu}{}^\kappa r_m^\nu$$

souřadnicově

$$\overset{\lambda}{\nabla}_{l_m} l_n = \frac{\lambda+1}{2}\, l_m\cdot c\cdot l_n = \frac{1+\lambda}{2}[l_m, l_n] = \frac{1+\lambda}{2} l_{[m,n]}$$

$$\overset{\lambda}{\nabla}_{r_m} r_n = \frac{\lambda-1}{2}\, r_m\cdot c\cdot r_n = \frac{1-\lambda}{2}[r_m, r_n] = \frac{1-\lambda}{2} r_{[m,n]}$$

D: využití $\overset{L}{\nabla} l_m = 0$ a $\overset{R}{\nabla} r_m = 0$

platí $l_m\cdot c\cdot l_n = [l_m, l_n] = l_{[m,n]}$ a $r_m\cdot c\cdot r_n = -[r_m, r_n] = -r_{[m,n]}$

Theorem:

$$\overset{\lambda}{\nabla} c = 0 \qquad \overset{\lambda}{\nabla} k = 0$$

D: biinvariance c $\Rightarrow$ $\overset{L}{\nabla} c = \overset{R}{\nabla} c = 0$

$\mathbf{c}_\alpha c_{\beta\gamma}{}^\kappa = c_{\alpha\lambda}{}^\kappa c_{\beta\gamma}{}^\lambda - c_{\alpha\beta}{}^\lambda c_{\lambda\gamma}{}^\kappa - c_{\alpha\gamma}{}^\lambda c_{\beta\lambda}{}^\kappa = c_{\alpha\lambda}{}^\kappa c_{\beta\gamma}{}^\lambda + c_{\gamma\lambda}{}^\kappa c_{\alpha\beta}{}^\lambda + c_{\beta\lambda}{}^\kappa c_{\gamma\alpha}{}^\lambda \overset{\text{J.I.}}{=} 0$

$\Rightarrow \overset{\lambda}{\nabla} c = \overset{L}{\nabla} c + \frac{\lambda+1}{2}\mathbf{c}\,c = 0$

$\Rightarrow \overset{\lambda}{\nabla} k = 0$

Theorem

$$\overset{\lambda}{T}_{\mu\nu}{}^\kappa = \lambda\, c_{\mu\nu}{}^\kappa \qquad \overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda = -\frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa \qquad \overset{\lambda}{Ric}_{\alpha\beta} = \frac{1-\lambda^2}{2} k_{\alpha\beta}$$

D: $\overset{\lambda}{T}_{\mu\nu}{}^\kappa = \overset{L}{T}_{\mu\nu}{}^\kappa + 2\frac{\lambda+1}{2} c_{[\mu\nu]}{}^\kappa = \lambda c_{\mu\nu}{}^\kappa$

$\overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda = \overset{L}{R}_{\mu\nu}{}^\kappa{}_\lambda + \overset{L}{\nabla}_\mu \overset{\lambda}{A}_{\nu\lambda}^\kappa - \overset{L}{\nabla}_\nu \overset{\lambda}{A}_{\mu\lambda}^\kappa + \overset{L}{T}_{\mu\nu}{}^\alpha \overset{\lambda}{A}_{\alpha\lambda}^\kappa + \overset{\lambda}{A}_{\mu\alpha}^\kappa \overset{\lambda}{A}_{\nu\lambda}^\alpha - \overset{\lambda}{A}_{\nu\alpha}^\kappa \overset{\lambda}{A}_{\mu\lambda}^\alpha = -\frac{\lambda+1}{2} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa + \left(\frac{\lambda+1}{2}\right)^2 \left(c_{\mu\alpha}{}^\kappa c_{\nu\lambda}{}^\alpha - c_{\nu\alpha}{}^\kappa c_{\mu\lambda}{}^\alpha\right)$

$= \left(\frac{\lambda+1}{2}\right)^2 \left(c_{\mu\nu}{}^\alpha c_{\lambda\alpha}{}^\kappa + c_{\nu\lambda}{}^\alpha c_{\mu\alpha}{}^\kappa + c_{\lambda\mu}{}^\alpha c_{\nu\alpha}{}^\kappa\right) - \frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa \overset{\text{J.I.}}{=} -\frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa$

alternativně

$l_m\cdot \overset{\lambda}{T}\cdot r_n = l_m\cdot\overset{\lambda}{\nabla} r_n - r_n\cdot\overset{\lambda}{\nabla} l_m - [l_m, r_n] = \frac{\lambda-1}{2} l_m\cdot c\cdot r_n - \frac{\lambda+1}{2} r_n\cdot c\cdot l_m = \lambda\, l_m\cdot c\cdot r_n$

$\overset{\lambda}{R}(l_a, l_b)\cdot l_c = \overset{\lambda}{\nabla}_{l_a}\overset{\lambda}{\nabla}_{l_b} l_c - \overset{\lambda}{\nabla}_{l_b}\overset{\lambda}{\nabla}_{l_a} l_c - \overset{\lambda}{\nabla}_{[l_a,l_b]} l_c = \frac{\lambda+1}{2}\left(\overset{\lambda}{\nabla}_{l_a} l_{[b,c]} - \overset{\lambda}{\nabla}_{l_b} l_{[a,c]} - \overset{\lambda}{\nabla}_{l_{[a,b]}} l_c\right) - \frac{1-\lambda}{2}\overset{\lambda}{\nabla}_{l_{[a,b]}} l_c$

$= \left(\frac{1+\lambda}{2}\right)^2 l\big[[a,[b,c]] - [b,[a,c]] - [[a,b],c]\big] - \frac{1-\lambda^2}{4} l_{[[a,b],c]} \overset{\text{J.I.}}{=} -\frac{1-\lambda^2}{4}[[l_a, l_b], l_c]$

$\Rightarrow \overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda\, l_a^\mu l_b^\nu l_c^\lambda = -\frac{1-\lambda^2}{4} l_a^\mu l_b^\nu c_{\mu\nu}{}^\alpha l_c^\lambda c_{\alpha\lambda}{}^\kappa \Rightarrow \overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda = -\frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa$

Theorem

$\nabla \equiv \overset{0}{\nabla}$ je metr. kov. der. k

$$T = 0 \qquad \nabla k = 0 \qquad R = -\frac{1}{4} c\cdot c \qquad Ric = \frac{1}{2} k \qquad Ein + \frac{D-2}{4} k = 0 \qquad R = \frac{D}{2}$$

Theorem

$\exp(\alpha m)$ jsou geodetiky ∇ $\qquad$ orbity l_m a r_m jsou geodetiky ∇

D: $\nabla_{l_m} l_m = \frac{1}{2} l_{[m,m]} = 0$ $\qquad$ $\nabla_{r_m} r_m = \frac{1}{2} r_{[m,m]} = 0$ $\qquad$ $\exp(\alpha m)$ je orbita l_m a r_m

Theorem K levo(pravo) invar. 1-forma $\Rightarrow$ K je Killing-Yano forma vůči $\overset{\lambda}{\nabla}$

D: $\nabla_F K_\varepsilon = \overset{L}{\nabla}_F K_\varepsilon - \frac{\lambda+1}{2} c_{F\varepsilon}{}^{\alpha} K_\alpha = -\frac{\lambda+1}{2} c_{F\varepsilon}{}^{\alpha} K_\alpha = \nabla_{[F} K_{\varepsilon]}$ $\Rightarrow$ $\nabla_F K_\varepsilon = \nabla_{[F} K_{\varepsilon]}$

Theorem

$$\overset{\lambda}{\nabla}_{l_m} = \mathcal{L}_{l_m} + \frac{\lambda-1}{2}\,\mathbf{tens}[l_m\cdot c] \qquad l_m\cdot c = l[ad_m]$$

$$\overset{\lambda}{\nabla}_{\pi_m} = \mathcal{L}_{\pi_m} + \frac{\lambda+1}{2}\,\mathbf{tens}[\pi_m\cdot c] \qquad \pi_m\cdot c = \pi[ad_m]$$

konkrétně

$$\overset{R}{\nabla}_{l_m} = \mathcal{L}_{l_m} \qquad \nabla_{l_m} = \mathcal{L}_{l_m} - \frac{1}{2}\,\mathbf{tens}[l_m\cdot c]$$

$$\overset{L}{\nabla}_{\pi_m} = \mathcal{L}_{\pi_m} \qquad \nabla_{\pi_m} = \mathcal{L}_{\pi_m} + \frac{1}{2}\,\mathbf{tens}[\pi_m\cdot c]$$

D: $\mathcal{L}_{l_m} = \overset{\lambda}{\nabla}_{l_m} + \overset{\lambda}{A}_{l_m}$ $\qquad \overset{\lambda}{A}_{l_m} = -\overset{\lambda}{\nabla} l_m - l_m\cdot\overset{\lambda}{T} = -\frac{\lambda+1}{2}\, c\cdot l_m - \lambda\, l_m\cdot c = -\frac{\lambda-1}{2}\, l_m\cdot c$

$\mathcal{L}_{\pi_m} = \overset{\lambda}{\nabla}_{\pi_m} + \overset{\lambda}{A}_{\pi_m}$ $\qquad \overset{\lambda}{A}_{\pi_m} = -\overset{\lambda}{\nabla}\pi_m - \pi_m\cdot\overset{\lambda}{T} = -\frac{\lambda-1}{2}\, c\cdot\pi_m - \lambda\,\pi_m\cdot c = -\frac{\lambda+1}{2}\,\pi_m\cdot c$

$c = l[c] = \pi[c]$ $\qquad m\cdot c|_e = ad_m$

Def E_α báze v e

$\bar{x}^\alpha$ jsou normální souř. na okolí e $\equiv$

$g = \exp(m^\alpha E_\alpha) \iff \bar{x}^\alpha(g) = m^\alpha$

Theorem $c_{\alpha\beta}{}^{\kappa}$ a $k_{\alpha\beta}$ komponenty c a k vzhledem k E_α v e

$$\bar{k}_{\alpha\beta} = k_{\alpha\beta} - \frac{1}{12} c_{\alpha\kappa}{}^{\mu} c_{\beta\lambda}{}^{\nu} k_{\mu\nu}\,\bar{x}^\kappa\bar{x}^\lambda + \mathcal{O}(\bar{x}^4)$$

$$\bar{\Gamma}^{\mu}_{\alpha\beta} = -\frac{1}{12}\left(c_{\kappa\alpha}{}^{\nu}c_{\nu\beta}{}^{\mu} + c_{\kappa\beta}{}^{\nu}c_{\nu\alpha}{}^{\mu}\right)\bar{x}^\kappa + \mathcal{O}(\bar{x}^3)$$

$$\bar{R}_{\alpha\beta}{}^{\kappa}{}_{\lambda} = -\frac{1}{4} c_{\alpha\beta}{}^{\nu} c_{\nu\lambda}{}^{\kappa} + \mathcal{O}(\bar{x}^2)$$

$\bar{A}^{\alpha\dots}_{\beta\dots}$ komponenty vzhledem k souř. bázi $\frac{\partial}{\partial\bar{x}^\alpha}, d\bar{x}^\alpha$

zřejmě $A^{\alpha\dots}_{\beta\dots} = \bar{A}^{\alpha\dots}_{\beta\dots}|_e$

D: plyne z rozvojů metr., Γ a R v norm. souř.

$$\bar{k}_{\alpha\beta} = \bar{k}_{\alpha\beta}|_e - \frac{1}{3}\bar{R}_{\alpha\kappa\beta\lambda}|_e\,\bar{x}^\kappa\bar{x}^\lambda + \nabla\bar{R}|_e\,\bar{x}^3 + \mathcal{O}(\bar{x}^4)$$

$$\bar{\Gamma}^{\mu}_{\alpha\beta} = -\frac{1}{3}\left(\bar{R}_{\kappa\alpha}{}^{\mu}{}_{\beta} + \bar{R}_{\kappa\beta}{}^{\mu}{}_{\alpha}\right)|_e\,\bar{x}^\kappa + \nabla\bar{R}|_e\,\bar{x}^2 + \mathcal{O}(\bar{x}^3)$$

$$\bar{R}_{\alpha\beta}{}^{\kappa}{}_{\lambda} = \bar{R}_{\alpha\beta}{}^{\kappa}{}_{\lambda}|_e + \nabla\bar{R}|_e\,\bar{x} + \mathcal{O}(\bar{x}^2)$$

využilo se $\bar{A}^{\alpha\dots}_{\beta\dots}|_e = A^{\alpha\dots}_{\beta\dots}$ a $c_{\alpha\beta\gamma} = c_{[\alpha\beta\gamma]}$ – viz LA

Akce LG na varietě

Def: A je akce LG G na varietě M $\equiv$

$A_g : M \to M$ difeomorfismus

$A_{g_1 g_2} = A_{g_1} \circ A_{g_2}$ levá akce – píšeme $A_g x = g x$

nebo

$A_{g_2 g_1} = A_{g_1} \circ A_{g_2}$ pravá akce – píšem $A_g x = x g$

Příkl: Pro $M = G$ máme

$L_g \quad R_{g^{-1}} \quad AD_g$ levé akce na G

$R_g \quad L_{g^{-1}} \quad AD_{g^{-1}}$ pravé akce na G

Def A_g akce G na M

a_m je generátor A_g $\equiv$

$a : \mathfrak{g} \to \mathfrak{T}M$ tj. $\forall m \in \mathfrak{g} \quad a_m \in \mathfrak{T}M$

$a_m|_x = \frac{D}{d\varepsilon} A_{g_\varepsilon} x \Big|_{\varepsilon=0}$ kde $\frac{Dg_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = m \quad g_0 = e$, např. $g_\varepsilon = \exp(\varepsilon m)$

Theorem: a_m generator akce A_g

A_g levá akce $\qquad A_{g*} a_m = a_{Ad_g m}$

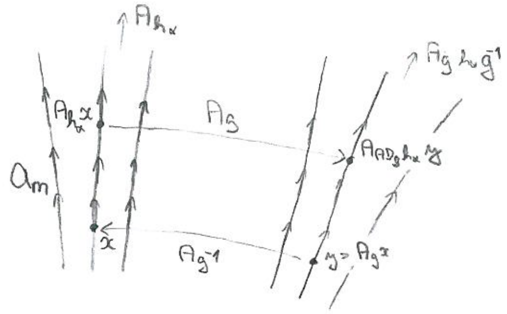
A_g pravá akce $\qquad A_{g^{-1}*} a_m = a_{Ad_g m}$

D: A_g levá akce

$A_{g*}(a_m|_x) = A_{g*} \frac{D}{d\alpha} A_{h_\alpha} x \Big|_{\alpha=0} = \frac{D}{d\alpha} A_g A_{h_\alpha} x \Big|_{\alpha=0}$

$= \frac{D}{d\alpha} A_{AD_g h_\alpha} A_g x \Big|_{\alpha=0} = a_{Ad_g m}\Big|_{A_g x}$

kde $h_0 = e \quad \frac{Dh_\alpha}{d\alpha}\Big|_{\alpha=0} = m$



Theorem a_m generátor akce A_g

A_g levá akce $\qquad [a_m, a_n] = - a_{[m,n]}$

A_g pravá akce $\qquad [a_m, a_n] = a_{[m,n]}$

D: A_g levá akce

$[a_m, a_n] = \mathcal{L}_{a_m} a_n = -\frac{d}{d\alpha} A_{g_\alpha *} a_n \Big|_{\alpha=0} = -\frac{d}{d\alpha} a_{Ad_{g_\alpha} n} \Big|_{\alpha=0} = -a_{ad_m n} = -a_{[m,n]}$

Př: Akce L_g a R_g na G mají generátory r_m a l_m

Def: varieta M s metrikou g

grupa isometrií $Iso(M,g) \equiv$

$\varphi \in Iso(M,g) \Leftrightarrow \varphi: M \to M$ difeomorf.

$\varphi^* g = g$

$\varphi_1 \varphi_2 = \varphi_1 \circ \varphi_2$ (skládání zobr.)

Lieovu alg. LG $Iso(M,g)$ označíme $iso(M,g)$

Lema $Iso(M,g)$ je podgrupa $Diff(M)$

pro varietu M konečné dim D je $Iso(M,g)$ konečné dimenze maximálně $\frac{(D+1)D}{2}$

Pozn:

- Podobné definice lze provést i pro jiné objekty než g

Def: $\varphi \in Iso(M,g)$ $\quad s \in iso(M,g)$

působení φ lze chápat jako levou akci $Iso(M,g)$ na M

$\Xi_\varphi x = \varphi x$

označíme

$\xi_s = \mathcal{S} \in \mathcal{T}M$ generátor isometr. asoc. s s

$\exp(\tau s) = \varphi_\tau$ 1-par. gr. isometrií asoc. s s

Lema: ξ_s je Killingův vektor na M metr. g

Lema: $[\xi_m, \xi_n] = -\xi_{[m,n]}$

zvolíme-li bázi $e_\alpha \in iso(M,g)$ a označíme $\xi_\alpha = \xi_{e_\alpha}$

$[\xi_\alpha, \xi_\beta] = -c_{\alpha\beta}{}^{\gamma} \xi_\gamma$

kde $c_{\alpha\beta}{}^{\gamma}$ jsou str. konst. vůči e_α

D: Ξ_φ je levá akce - viz obecně

Posun Killingových vektorů

nechť R, K_μ jsou Kill. vektory
s kom. relacemi

$$[R, K_\mu] = -c_{R\mu}{}^\nu K_\nu$$

Pak R-posun vektoru $V = V^\mu K_\mu$ lze
řešit maticově

$$V_\varphi = R_{\varphi *} V_0 \qquad R_\varphi \text{ posunutí podél } R$$

$$\frac{d}{d\varphi} V_\varphi = -\mathcal{L}_R V_\varphi = -[R, K_\mu] V_\varphi^\mu = c_{R\mu}{}^\nu V_\varphi^\mu K_\nu$$

řešíme v podobě $V_\varphi = V_\varphi^\nu K_\nu \implies$

$$\frac{d}{d\varphi} V_\varphi^\nu = c_{R\mu}{}^\nu V_\varphi^\mu \qquad c_{R\mu}{}^\nu = \mathrm{ad}_R{}^\nu{}_\mu$$

$$V_\varphi^\nu = \exp(\varphi c_R)^\nu{}_\mu V_0^\mu \qquad \exp(\varphi c_R)^\nu{}_\mu = \mathrm{Ad}_{R_\varphi}{}^\nu{}_\mu$$

Př: $[X, Y] = 0 \qquad c_X = [0] \qquad \exp(x c_X) = [1] \qquad Y_{x*} = Y$

Př:

$$[R, X] = -Y \qquad c_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad c_R^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[R, Y] = X$$

$$V_0 = a_0 X + b_0 Y \qquad V_\varphi = R_{\varphi *} V_0 = a_\varphi X + b_\varphi Y$$

$$\exp(\varphi c_R) = (\cos\varphi)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\sin\varphi)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{bmatrix} a_\varphi \\ b_\varphi \end{bmatrix} = \exp(\varphi c_R) \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} \cos\varphi\, a_0 - \sin\varphi\, b_0 \\ \sin\varphi\, a_0 + \sin\varphi\, b_0 \end{bmatrix}$$

$$V_\varphi = (\cos\varphi\, a_0 - \sin\varphi\, b_0) X + (\sin\varphi\, a_0 + \cos\varphi\, b_0) Y$$