

Geometrie Lieových grup

$T_e M$ Lieove alg. $[,]$ $C_{\alpha\beta}^{\gamma}$ $K_{\alpha\beta}$ $C_{\alpha\beta\gamma} = C_{[\alpha\beta\gamma]}$

$$\overset{L}{E}_\alpha \quad \overset{L}{E}^\alpha \quad \overset{R}{E}_\alpha \quad \overset{R}{E}^\alpha$$

1) ∇A levoino tenzor

$$\overset{L}{\nabla} A = 0$$

$$A \rightarrow \underbrace{A_{\beta\gamma}^\alpha}_{\text{konst.}} \overset{L}{E}_\alpha \dots \overset{L}{E}^\beta \dots$$

$$\overset{L}{\nabla} \overset{L}{E}_\alpha = 0 \quad \overset{L}{\nabla} \overset{L}{E}^\alpha = 0$$

$$\overset{L}{T}_{\alpha\beta}^{\gamma} = -C_{\alpha\beta}^{\gamma} \quad \overset{L}{R}_{\alpha\beta\gamma}^{\delta} = 0$$

2) ∇A pravoino tenzor

$$\overset{R}{\nabla} A = 0$$

$$\overset{R}{\nabla} \overset{R}{E}_\alpha = 0$$

$$\overset{R}{T}_{\alpha\beta}^{\gamma} = C_{\alpha\beta}^{\gamma} \quad \overset{R}{R}_{\alpha\beta\gamma}^{\delta} = 0$$

$$\overset{R}{\nabla} - \overset{L}{\nabla} = \mathbf{C} \quad C_{\alpha\beta}^{\gamma}$$

$$0 = \overset{R}{\nabla}_\alpha \rho_n^\beta = \overset{L}{\nabla}_\alpha \rho_n^\beta + \overset{R}{A}_{\alpha\gamma}^{\beta} \rho_n^\gamma = (\overset{R}{A}_{\alpha\gamma}^{\beta} - \overset{L}{C}_{\alpha\gamma}^{\beta}) \rho_n^\gamma$$

$$\begin{aligned} \overset{L}{T}_{\alpha\beta}^{\gamma} &= \overset{L}{T}_{\alpha\beta}^{\gamma} \overset{L}{E}_\alpha^{\beta} \overset{L}{E}_\beta^{\gamma} = \\ &= \overset{L}{\nabla}_{\overset{L}{E}_\alpha} \overset{L}{E}_\beta^{\gamma} - \overset{L}{\nabla}_{\overset{L}{E}_\beta} \overset{L}{E}_\alpha^{\gamma} - [\overset{L}{E}_\alpha, \overset{L}{E}_\beta]^{\gamma} \\ &= -C_{\alpha\beta}^{\gamma} \overset{L}{E}_\alpha^{\beta} \overset{L}{E}_\beta^{\gamma} = -C_{\alpha\beta}^{\gamma} \end{aligned}$$

$$\overset{L}{\nabla}_{l_m} \rho_n = \overset{L}{\nabla}_{l_m} \rho_n - \underbrace{\overset{L}{\nabla}_{\rho_n} l_m}_0 - \underbrace{[l_m, \rho_n]}_0$$

$$= l_m \cdot \overset{L}{T} \cdot \rho_n = -l_m \cdot C \cdot \rho_n$$

$$\overset{L}{\nabla} \rho_n = -C \cdot \rho_n$$

Geometrie Lieových grup

$T_e M$ Lieova alg. $[,]$ $C_{\mathbb{R}}^{\infty}$ $K_{\mathbb{R}}$ $C_{\alpha\beta\gamma} = C_{[\alpha\beta\gamma]}$

$$\mathbb{R}^L_{\alpha}$$

$$\mathbb{R}^L_{\alpha}$$

$$\mathbb{R}^D_{\alpha}$$

$$\mathbb{R}^D_{\alpha}$$

Geometrie Lieových grup

$T_e M$ Lieove alg. $[,]$ $C_{\alpha\beta}^{\gamma} \in \mathbb{K}$ $\mathbb{K} = \mathbb{R}$ $C_{\alpha\beta\gamma} = C_{[\alpha\beta\gamma]}$

$$\overset{L}{E}_{\alpha} \quad \overset{L}{E}^{\alpha} \quad \overset{R}{E}_{\alpha} \quad \overset{R}{E}^{\alpha}$$

1) ∇A levoino tenzor

$$\overset{L}{\nabla} A = 0$$

$$A = \underbrace{A^{\alpha}_{\beta}}_{\text{zast.}} \overset{L}{E}_{\alpha} \dots \overset{L}{E}^{\beta}$$

$$\overset{L}{\nabla} \overset{L}{E}_{\alpha} = 0 \quad \overset{L}{\nabla} \overset{L}{E}^{\alpha} = 0$$

$$\overset{L}{T}_{\alpha\beta}^{\gamma} = -C_{\alpha\beta}^{\gamma} \quad \overset{L}{R}_{\alpha\beta\gamma}^{\delta} = 0$$

2) ∇A pravoino tenzor

$$\overset{R}{\nabla} A = 0$$

$$\overset{R}{\nabla} \overset{R}{E}_{\alpha} = 0$$

$$\overset{R}{T}_{\alpha\beta}^{\gamma} = C_{\alpha\beta}^{\gamma} \quad \overset{R}{R}_{\alpha\beta\gamma}^{\delta} = 0$$

$$T_{\alpha\beta}^L = T_{\kappa\lambda}^L E_{\alpha}^{\kappa} E_{\beta}^{\lambda} =$$

$$= \Delta_{\alpha}^{\kappa} E_{\beta}^{\lambda} - \Delta_{\beta}^{\lambda} E_{\alpha}^{\kappa} - [E_{\alpha}, E_{\beta}]^{\kappa}$$

$$= -C_{\kappa\lambda}^{\alpha} E_{\alpha}^{\kappa} E_{\beta}^{\lambda} = -C_{\alpha\beta}^{\kappa}$$

$$\nabla_{l_m}^L \Omega_n = \nabla_{l_m}^L \Omega_n - \underbrace{\nabla_{\Omega_n}^L l_m}_{0} - \underbrace{[l_m, \Omega_n]}_0$$

$$= l_m \cdot \overset{L}{\nabla} \Omega_n = -l_m \cdot C \cdot \Omega_n$$

$$\nabla_{\Omega_n}^L \Omega_n = -C \cdot \Omega_n$$

$$\nabla_{l_m}^L \Omega_n = \nabla_{l_m}^L \Omega_n - \underbrace{\nabla_{\Omega_n}^L l_m}_0 - \underbrace{[l_m, \Omega_n]}_0$$

$$= l_m \cdot \overset{L}{T} \cdot \Omega_n = -l_m \cdot C \cdot \Omega_n$$

$$\nabla^L \Omega_n = -C \cdot \Omega_n$$

$$\nabla^R - \nabla^L = e \quad C_{\alpha\beta}^{\gamma}$$

$$0 = \nabla_{\alpha}^R \Omega_n^F = \nabla_{\alpha}^L \Omega_n^F + A_{\alpha\gamma}^F \Omega_n^{\gamma} = (A_{\alpha\gamma}^F - C_{\alpha\gamma}^F) \Omega_n^{\gamma}$$

Strukturgleichungen pro E^L_α E^R_α

$$d E^L_\alpha + \frac{1}{2} C_{\mu\nu}^\alpha E^L_\mu \wedge E^L_\nu = 0$$

$$d E^R_\alpha - \frac{1}{2} C_{\mu\nu}^\alpha E^R_\mu \wedge E^R_\nu = 0$$

$$\begin{aligned} d_{\mu}^L E^L_\nu &= \underbrace{\nabla_{\mu}^L E^L_\nu}_0 + \Gamma_{\mu\nu}^{\kappa} E^L_\kappa = -C_{\mu\nu}^{\kappa} E^L_\kappa \\ &= -C_{\kappa\mu}^\alpha E^L_\mu E^L_\nu = -\frac{1}{2} C_{\kappa\mu}^\alpha E^L_\mu \wedge E^L_\nu \end{aligned}$$

Strukturgleichungen pro E^L_α E^R_α

$$d E^L_\alpha + \frac{1}{2} C_{\mu\nu}^\alpha E^L_\mu \wedge E^L_\nu = 0$$

$$d E^R_\alpha - \frac{1}{2} C_{\mu\nu}^\alpha E^R_\mu \wedge E^R_\nu = 0$$

$$d E^L_\alpha + \frac{1}{2} C_{\mu\nu}^\alpha E^L_\mu \wedge E^L_\nu = 0$$

$$\begin{aligned} d_\mu E^L_\nu &= \underbrace{\nabla_\mu E^L_\nu}_0 + \frac{1}{2} \Gamma_{\mu\nu}^\kappa E^L_\kappa = -C_{\mu\nu}^\kappa E^L_\kappa \\ &= -C_{\kappa\mu}^\nu E^L_\mu E^L_\nu = -\frac{1}{2} C_{\kappa\mu}^\nu E^L_\mu \wedge E^L_\nu \end{aligned}$$

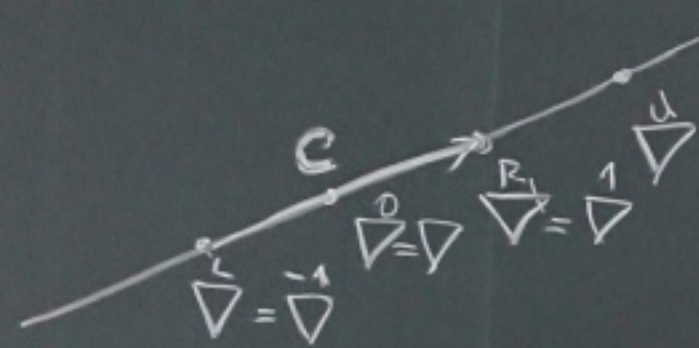
Levi-Civita metrická derivace na Lieově grupě

$$\overset{L}{\nabla} \quad \overset{R}{\nabla}$$

$$\overset{J}{\nabla} = \overset{L}{\nabla} + \frac{J+1}{2} \mathbf{C} =$$

$$= \overset{R}{\nabla} + \frac{J-1}{2} \mathbf{C}$$

$$\begin{aligned} \overset{J}{\nabla}_F a^k &= \overset{L}{\nabla}_F a^k + \frac{J+1}{2} C_{Fk}^k a^k \\ &= \overset{R}{\nabla}_F a^k + \frac{J-1}{2} C_{Fk}^k a^k \end{aligned}$$



$$\overset{J}{\nabla}_F l_m^k = \frac{J+1}{2} C_{Fm}^k l_m^k$$

$$\overset{J}{\nabla}_F \Omega_m^k = \frac{J-1}{2} C_{Fm}^k \Omega_m^k$$

$$\begin{aligned} \overset{J}{\nabla}_{l_m} l_n &= \frac{J+1}{2} l_m \cdot C \cdot l_n = \frac{J+1}{2} [l_m, l_n] \\ &= \frac{J+1}{2} l_{[m,n]} \end{aligned}$$

$$\overset{J}{\nabla}_{\Omega_m} \Omega_n = \frac{1-J}{2} \Omega_{[m,n]}$$

$$\overset{J}{\nabla} c = 0 \quad \overset{J}{\nabla} k = 0$$

$$\begin{aligned} C_{\alpha\beta}^k C_{\beta\gamma}^k &= C_{\alpha\beta}^k C_{\beta\gamma}^k - C_{\alpha\gamma}^k C_{\beta\beta}^k - C_{\alpha\gamma}^k C_{\beta\beta}^k \\ &= C_{\alpha\beta}^k C_{\beta\gamma}^k + C_{\alpha\beta}^k C_{\alpha\gamma}^k + C_{\beta\gamma}^k C_{\alpha\alpha}^k = 0 \end{aligned}$$

$$[[m,n],p] + [(n,p),m] + [(p,m),n] = 0$$

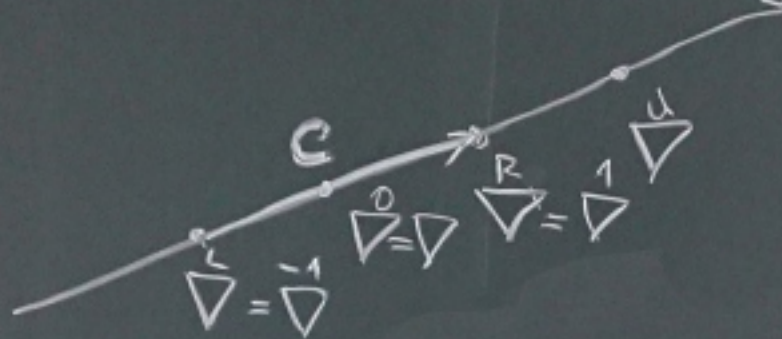
Levi-Civita metrická derivace na Lieově grupě

$$\begin{matrix} L \\ \nabla \end{matrix} \quad \begin{matrix} R \\ \nabla \end{matrix}$$

$$\begin{matrix} J \\ \nabla \end{matrix} = \begin{matrix} L \\ \nabla \end{matrix} + \frac{J+1}{2} \mathbf{C} =$$

$$= \begin{matrix} R \\ \nabla \end{matrix} + \frac{J-1}{2} \mathbf{C}$$

$$\begin{aligned} \begin{matrix} J \\ \nabla \end{matrix} a^k &= \begin{matrix} L \\ \nabla \end{matrix} a^k + \frac{J+1}{2} C_{HK}^K a^k \\ &= \begin{matrix} R \\ \nabla \end{matrix} a^k + \frac{J-1}{2} C_{HK}^K a^k \end{aligned}$$



$$\nabla_{\mu}^{\lambda} l_m^{\nu} = \frac{\lambda+1}{2} C_{\mu\nu}^{\kappa} l_m^{\nu}$$

$$\nabla_{\mu}^{\lambda} \Omega_m^{\kappa} = \frac{\lambda-1}{2} C_{\mu\nu}^{\kappa} \Omega_m^{\nu}$$

$$\begin{aligned} \nabla_{\mu}^{\lambda} l_m l_n &= \frac{\lambda+1}{2} l_m \cdot C \cdot l_n = \frac{\lambda+1}{2} [l_m, l_n] \\ &= \frac{\lambda+1}{2} l_{[m,n]} \end{aligned}$$

$$\nabla_{\mu}^{\lambda} \Omega_m \Omega_n = \frac{1-\lambda}{2} \Omega_{[m,n]}$$

$$\nabla^{\lambda} c = 0$$

$$\nabla^{\lambda} k = 0$$

$$C_{\alpha} C_{\beta\gamma}^{\kappa} = C_{\alpha\lambda}^{\kappa} C_{\beta\gamma}^{\lambda} - C_{\alpha\beta}^{\lambda} C_{\lambda\gamma}^{\kappa} - C_{\alpha\gamma}^{\lambda} C_{\beta\lambda}^{\kappa}$$
$$= C_{\alpha\lambda}^{\kappa} C_{\beta\gamma}^{\lambda} + C_{\alpha\lambda}^{\kappa} C_{\alpha\beta}^{\lambda} + C_{\beta\lambda}^{\kappa} C_{\alpha\gamma}^{\lambda} = 0$$

$$[[m, n], p] + [(n, p), m] + [(p, m), n] = 0$$

Levi-Civita metrická derivace na Lieově grupě

$$\overset{L}{\nabla}, \overset{R}{\nabla}, \overset{\lambda}{\nabla} = \overset{L}{\nabla} + \frac{\lambda+1}{2} C = \overset{R}{\nabla} + \frac{\lambda-1}{2} C$$

$$\overset{\lambda}{T}_{\mu\nu}^{\kappa} = \lambda C_{\mu\nu}^{\kappa} \quad \overset{\lambda}{R}_{\mu\nu}^{\kappa} = -\frac{1-\lambda^2}{2} C_{\mu\nu}^{\alpha} C_{\alpha\kappa}^{\beta}$$

$$\overset{\lambda}{Ric}_{\alpha\beta} = \frac{1-\lambda^2}{2} k_{\alpha\beta}$$

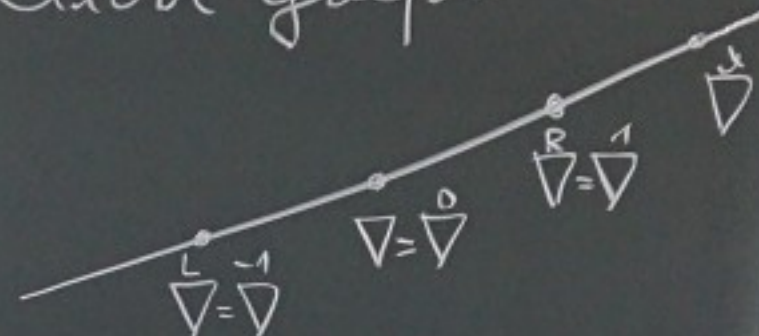
$$\overset{\lambda}{T}_{\mu\nu}^{\kappa} = \overset{L}{T}_{\mu\nu}^{\kappa} + 2 \frac{\lambda+1}{2} C_{[\mu\nu]}^{\kappa} = \lambda C_{\mu\nu}^{\kappa}$$

$$\overset{\lambda}{R}(l_a, l_b) \cdot l_c = \overset{\lambda}{\nabla}_{l_a} \overset{\lambda}{\nabla}_{l_b} l_c - \overset{\lambda}{\nabla}_{l_b} \overset{\lambda}{\nabla}_{l_a} l_c - \overset{\lambda}{\nabla}_{[l_a, l_b]} l_c =$$

$$= \frac{\lambda+1}{2} \left(\overset{\lambda}{\nabla}_{l_a} l_{[b,c]} - \overset{\lambda}{\nabla}_{l_b} l_{[a,c]} - \overset{\lambda}{\nabla}_{l_{[a,b]}} l_c \right) - \frac{1-\lambda^2}{2} \overset{\lambda}{\nabla}_{l_{[a,b]}} l_c$$

$$\Rightarrow \left(\frac{\lambda+1}{2}\right)^2 l([a, [b, c]] - [b, [a, c]] - [[a, b], c]) - \frac{1-\lambda^2}{2} l_{[a,b],c}$$

$$= -\frac{1-\lambda^2}{2} l_{[a,b],c} = -\frac{1-\lambda^2}{2} C_{\alpha\beta}^{\kappa} C_{\mu\gamma}^{\lambda} l_a^{\alpha} l_b^{\beta} l_c^{\gamma}$$



$$\overset{0}{\nabla} = \overset{0}{\nabla} \quad \nabla k = 0 \quad T = 0$$

$$R = -\frac{1}{2} C \cdot C \quad Ric = \frac{1}{2} k$$

$$Ein + \frac{D-2}{4} k = 0 \quad R = \frac{D}{2}$$

orbity l_m Ω_m jsou geodeticky

$$\nabla_{l_m} l_m = l_{[m,m]} = 0$$

exp(αm) geodetika

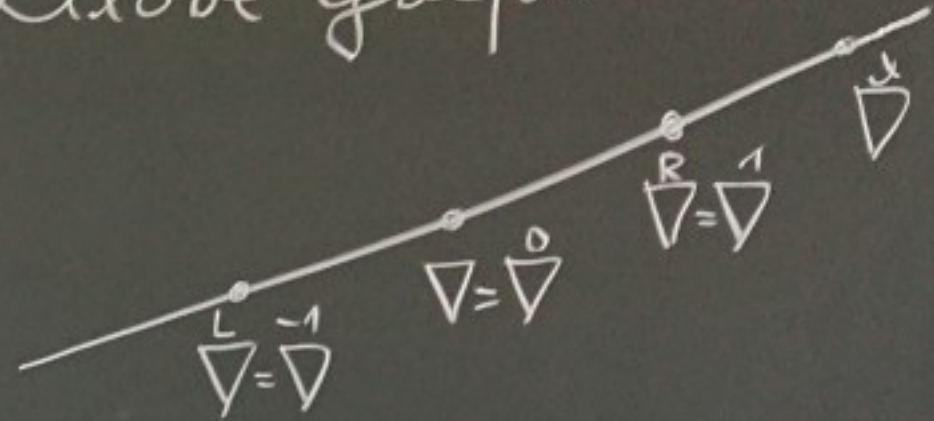
Levi-Civita metrická derivace na Lieově grupě

$$\nabla^L, \nabla^R, \nabla^{\lambda} = \nabla^L + \frac{\lambda+1}{2} \mathbb{C} = \nabla^R + \frac{\lambda-1}{2} \mathbb{C}$$

$$\nabla^{\lambda} \xi^{\mu} = \lambda C_{\mu\nu}^{\mu} \xi^{\nu}$$

$$R_{\mu\nu}^{\mu\alpha} = -\frac{1-\lambda^2}{2} C_{\mu\nu}^{\alpha} C_{\alpha\beta}^{\mu}$$

$$\text{Ric}_{\alpha\beta} = \frac{1-\lambda^2}{2} K_{\alpha\beta}$$

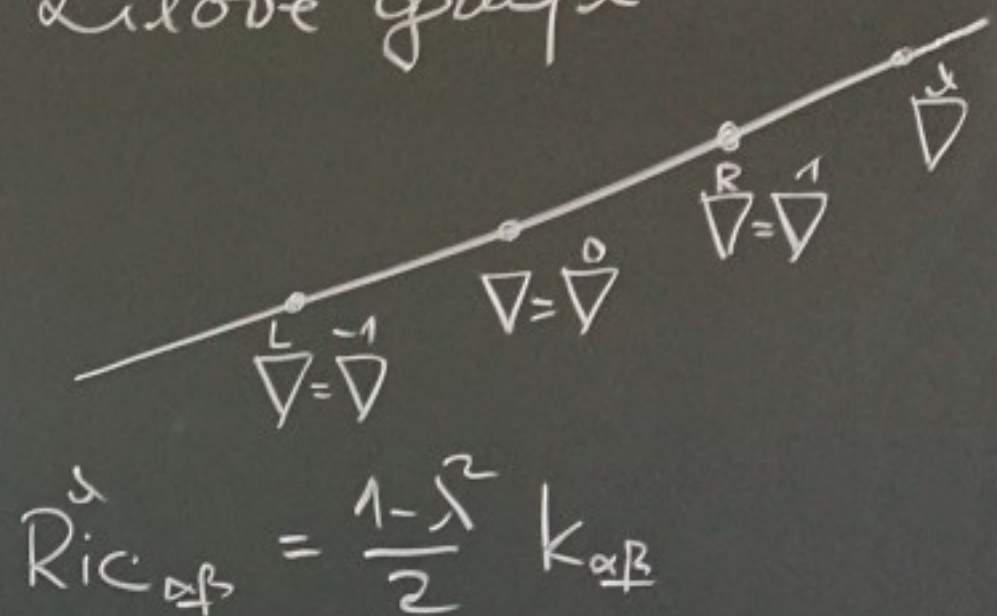


Levi-Civita metrická derivace na Lieově grupě

$$\nabla^L, \nabla^R, \nabla^\lambda = \nabla^L + \frac{\lambda+1}{2} \mathbf{C} = \nabla^R + \frac{\lambda-1}{2} \mathbf{C}$$

$$\overset{\lambda}{T}_{\mu\nu}^{\mu\nu} = \lambda C_{\mu\nu}^{\mu\nu} \quad R_{\mu\nu}^{\mu\nu} = -\frac{1-\lambda^2}{2} C_{\mu\nu}^{\mu\nu} C_{\mu\nu}^{\mu\nu}$$

$$\overset{\lambda}{T}_{\mu\nu}^{\mu\nu} = \overset{L}{T}_{\mu\nu}^{\mu\nu} + 2\frac{\lambda+1}{2} C_{[\mu\nu]}^{\mu\nu} = \lambda C_{\mu\nu}^{\mu\nu}$$

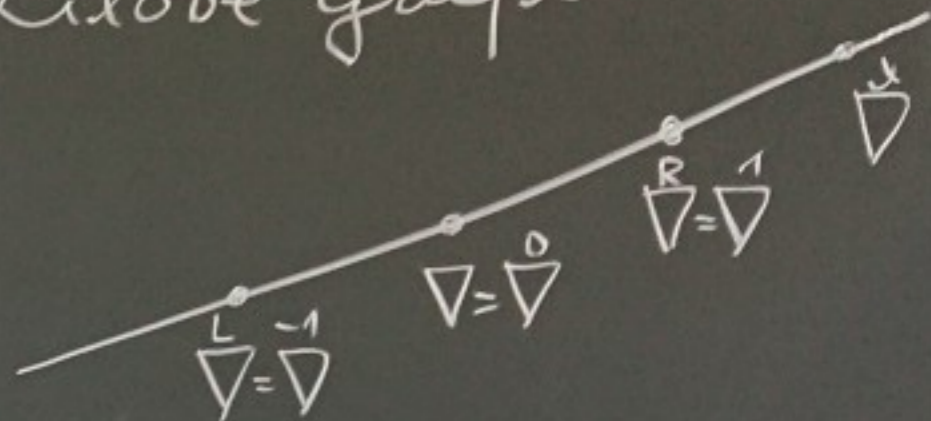


Levi-Civita metrická derivace na Lieově grupě

$$\overset{L}{\nabla}, \overset{R}{\nabla}, \overset{\lambda}{\nabla} = \overset{L}{\nabla} + \frac{\lambda+1}{2} C = \overset{R}{\nabla} + \frac{\lambda-1}{2} C$$

$$\overset{\lambda}{T}_{\mu\nu}^{\kappa} = \lambda C_{\mu\nu}^{\kappa} \quad \overset{\lambda}{R}_{\mu\nu}^{\kappa} = -\frac{1-\lambda^2}{2} C_{\mu\nu}^{\alpha} C_{\alpha\kappa}^{\mu}$$

$$\overset{\lambda}{Ric}_{\alpha\beta} = \frac{1-\lambda^2}{2} K_{\alpha\beta}$$



$$\overset{\lambda}{R}(l_a, l_b) \cdot l_c = \overset{\lambda}{\nabla}_{l_a} \overset{\lambda}{\nabla}_{l_b} l_c - \overset{\lambda}{\nabla}_{l_b} \overset{\lambda}{\nabla}_{l_a} l_c - \overset{\lambda}{\nabla}_{[l_a, l_b]} l_c =$$

$$= \frac{\lambda+1}{2} \left(\overset{\lambda}{\nabla}_{l_a} l_{[b,c]} - \overset{\lambda}{\nabla}_{l_b} l_{[a,c]} - \overset{\lambda}{\nabla}_{l_{[a,b]}} l_c \right) - \frac{1-\lambda}{2} \overset{\lambda}{\nabla}_{l_{[a,b]}} l_c$$

$$\rightarrow \left(\frac{\lambda+1}{2} \right)^2 l \left([a, (b, c)] - [b, (a, c)] - [(a, b), c] \right) - \frac{1-\lambda^2}{2} l_{[(a, b), c]}$$

$$= -\frac{1-\lambda^2}{2} l_{[(a, b), c]} = -\frac{1-\lambda^2}{2} C_{\alpha\beta}^{\kappa} C_{\mu\kappa}^{\nu} l_a^{\alpha} l_b^{\beta} l_c^{\nu}$$

$$\nabla = \overset{0}{\nabla} \quad \nabla k = 0 \quad T = 0$$

$$R = -\frac{1}{2} c \cdot c \quad Ric = \frac{1}{2} k$$

$$Ein + \frac{D-2}{4} k = 0 \quad R = \frac{D}{2}$$

orbity l_m Ω_m joon geoditika

$$\nabla_{l_m} l_m = l_{[m,m]} = 0$$

$\exp(\alpha m)$ geodetika

$$\overset{\lambda}{\nabla}_{l_m} = \mathcal{L}_{l_m} + \text{Ad}_{\frac{\lambda-1}{2} m}$$

$$\overset{\lambda}{\nabla}_{R_m} = \mathcal{L}_{R_m} + \text{Ad}_{\frac{\lambda+1}{2} m}$$

$$\text{Ada} \leftrightarrow \begin{matrix} -\alpha^k C_{k\neq}^{\vee} \\ \text{TeG} \end{matrix} = -\text{ada}^{\vee} = -l_{\alpha}^k C_{k\neq}^{\vee}$$

kontinuität

$$\overset{\mathbb{R}}{\nabla}_{l_m} = \mathcal{L}_{l_m} \quad \overset{\mathbb{L}}{\nabla}_{R_m} = \mathcal{L}_{R_m}$$

$$\mathcal{L}_{l_m} = \overset{\lambda}{\nabla}_{l_m} + \Lambda$$

$$\Lambda = -\overset{\lambda}{\nabla}_{l_m} - l_m \overset{\lambda}{T} = -\frac{\lambda+1}{2} C \cdot l_m - \lambda l_m \cdot C = -\frac{\lambda-1}{2} l_m \cdot C$$

$$\nabla_{\ell_m}^{\downarrow} = \mathcal{L}_{\ell_m} + \text{Ad}_{\frac{\downarrow-1}{2}m}$$

$$\nabla_{\Omega_m}^{\downarrow} = \mathcal{L}_{\Omega_m} + \text{Ad}_{\frac{\downarrow+1}{2}m}$$

$$\text{Ad}_a \leftrightarrow \begin{matrix} -\alpha^k C_{k\neq}^{\neq} = -\text{ad}_a^{\neq} \\ \text{TeG} \quad -\ell_a^k C_{k\neq}^{\neq} \end{matrix}$$

Kontinuität \bar{e}

$$\nabla_{\ell_m}^{\mathbb{R}} = \mathcal{L}_{\ell_m}$$

$$\nabla_{\Omega_m}^{\mathbb{L}} = \mathcal{L}_{\Omega_m}$$

$$\vec{\Delta}_{\mathfrak{g}_m}^{\lambda} = \mathcal{L}_{\mathfrak{g}_m} + \text{Ad}_{\frac{\lambda-1}{2}} \mathfrak{g}_m$$

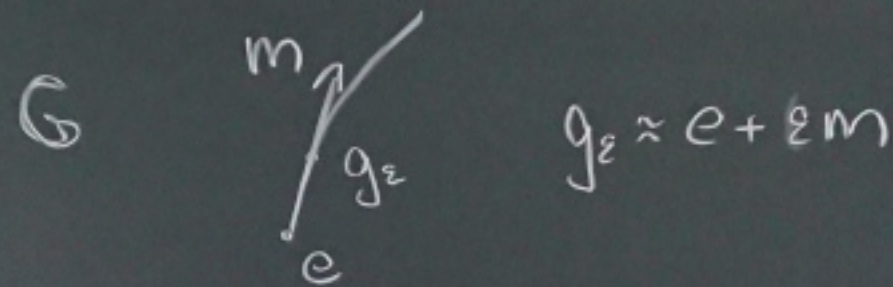
$$\text{Ad}_a \leftrightarrow \begin{array}{l} -\alpha^k C_{\alpha\beta}^{\gamma} \\ \text{TeG} \end{array} = -\text{ad}_{\alpha^k}^{\gamma} \\ -\mathfrak{L}_{\alpha^k} C_{\alpha\beta}^{\gamma}$$

$$\mathcal{L}_{\mathfrak{g}_m} = \vec{\Delta}_{\mathfrak{g}_m}^{\lambda} + \Lambda$$

$$\begin{aligned} \Lambda &= -\vec{\Delta}_{\mathfrak{g}_m}^{\lambda} - \mathfrak{g}_m \cdot \vec{\Delta}_{\mathfrak{g}_m}^{\lambda} \\ &= -\frac{\lambda+1}{2} \mathfrak{g}_m \cdot \mathfrak{g}_m - \lambda \mathfrak{g}_m \cdot \mathfrak{g}_m \\ &= -\frac{\lambda-1}{2} \mathfrak{g}_m \cdot \mathfrak{g}_m \end{aligned}$$

Akce Lieovy grupy na varietě

Def A je akce LG na var M



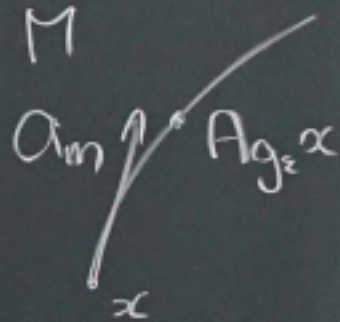
$g \in G \quad A_g : M \rightarrow M$

1) $A_{g_1} \circ A_{g_2} = A_{g_1 g_2}$ levá akce

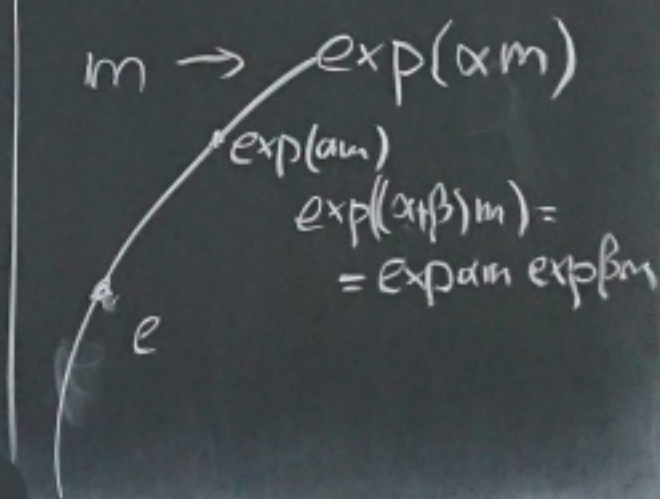
2) $A_{g_1} \circ A_{g_2} = A_{g_2 g_1}$ pravá akce

$A_g x = g x$

$A_g x = x g$



Pr: $L_g \quad R_{g^{-1}} \quad AD_g$ levé akce na G
 $R_g \quad L_{g^{-1}} \quad AD_{g^{-1}}$ pravé akce na G



Def A_g je akce G na M

A_m generator akce A_g

$a : \mathfrak{g} \rightarrow \mathcal{T}M$

$\forall m \in \mathfrak{g} \rightarrow A_m \in \mathcal{T}M$

$A_m|_x = \frac{D}{d\epsilon} A_{g_\epsilon} x|_{\epsilon=0}$

$\frac{Dg_\epsilon}{d\epsilon}|_{\epsilon=0} = m \quad g_0 = e$

$\rightarrow A_{\exp(\alpha m)} \rightarrow A_m$

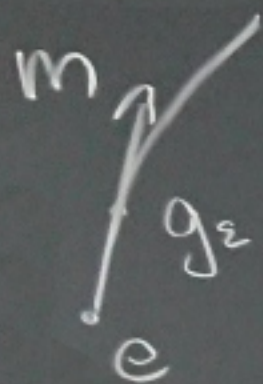
Akce Lieovy grupy na varietě

Def A je akce LG na var M

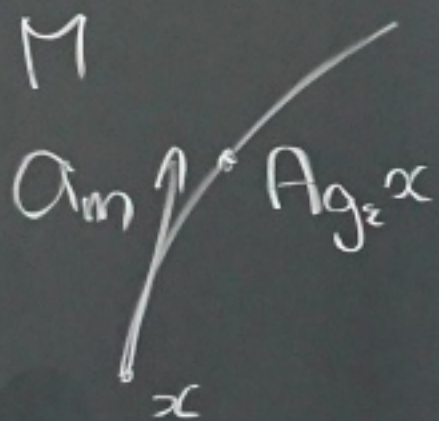
$$g \in G \quad A_g: M \rightarrow M$$

$$\begin{array}{lll} 1) & A_{g_1} \circ A_{g_2} = A_{g_1 g_2} & \text{levá akce} & A_g x = g x \\ 2) & A_{g_1} \circ A_{g_2} = A_{g_2 g_1} & \text{pravá akce} & A_g x = x g \end{array}$$

$$\text{Pr:} \quad \begin{array}{llll} L_g & R_{g^{-1}} & AD_g & \text{levé akce na } G \\ R_g & L_{g^{-1}} & AD_{g^{-1}} & \text{pravé akce na } G \end{array}$$



$$g_\epsilon \approx e + \epsilon m$$



Def A_g je akce G na M

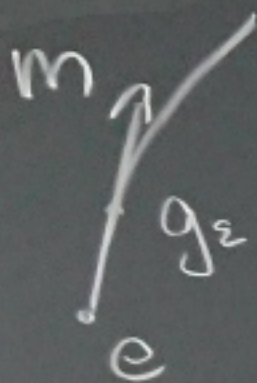
a_m generator akce A_g

$$a : \mathfrak{g} \rightarrow TM$$

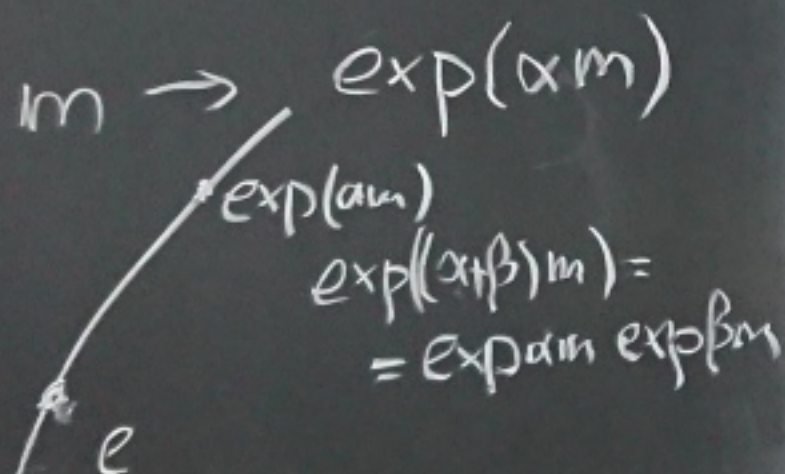
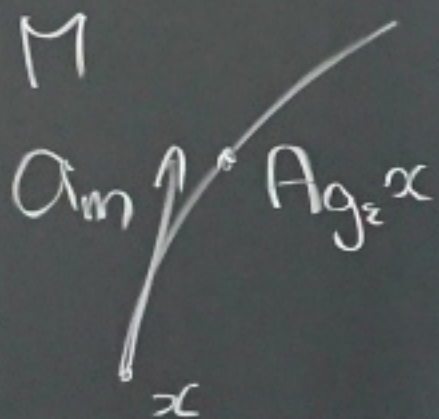
$$\forall m \in \mathfrak{g} \rightarrow a_m \in TM$$

$$a_m|_x = \left. \frac{D}{d\epsilon} A_{g_\epsilon} x \right|_{\epsilon=0}$$

$$\left. \frac{Dg_\epsilon}{d\epsilon} \right|_{\epsilon=0} = m \quad g_0 = e$$



$$g_\epsilon \approx e + \epsilon m$$



Def A_g is a linear map on $T_g M$

A_m generator of A_g

$$A : \mathfrak{g} \rightarrow TM$$

$$\forall m \in \mathfrak{g} \rightarrow A_m \in TM$$

$$A_m|_x = \left. \frac{D}{d\epsilon} A_{g_\epsilon} x \right|_{\epsilon=0}$$

$$\left. \frac{D g_\epsilon}{d\epsilon} \right|_{\epsilon=0} = m \quad g_0 = e$$

$$\rightarrow A_{\exp(\alpha m)} \mapsto A_m$$

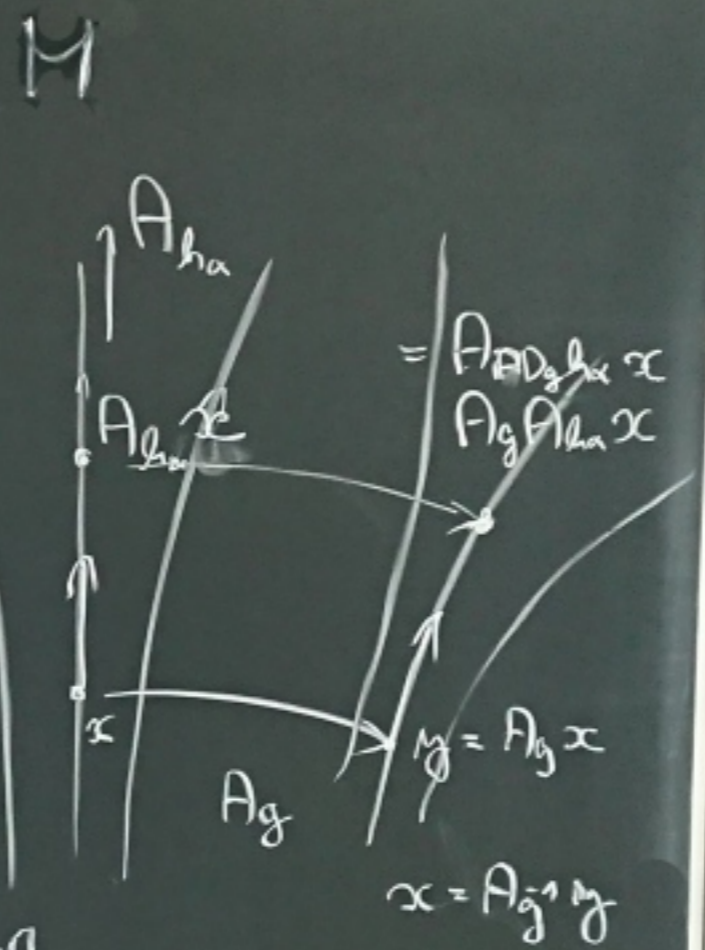
$$a_{m+\alpha n} = a_m + \alpha a_n$$

PF l-akce L_g Ω_m $L_{\exp(\alpha m)}$
 p-akce R_g ℓ_m

Leme

- $A_g * a_m = a_{Ad_g m}$ - A_g levá akce
- $A_{g^{-1}} * a_m = a_{Ad_g m}$ - A_g pravá akce

$$\begin{aligned}
 d_{\alpha} A_g * a_m |_{\alpha=0} &= A_g * \frac{D}{d\alpha} A_{h\alpha} x |_{\alpha=0} = \frac{D}{d\alpha} A_g A_{h\alpha} x |_{\alpha=0} \\
 &= \frac{D}{d\alpha} A_{AD_g h\alpha} \underbrace{A_g x}_y |_{\alpha=0} = a_{Ad_g m} |_{A_g x}
 \end{aligned}$$



Víte

A_g levá akce $[a_m, a_n] = -a_{[m,n]}$

A_g pravá akce $[a_m, a_n] = a_{[m,n]}$

$$\begin{aligned}
 [a_m, a_n] &= \mathcal{L}_{a_m} a_n = -\frac{d}{d\alpha} A_{g\alpha} * a_n \\
 &= -\frac{d}{d\alpha} a_{Ad_{g\alpha} n} |_{\alpha=0} = -a_{\underbrace{ad_{m,n}}_{[m,n]}} \\
 &= -a_{[m,n]}
 \end{aligned}$$

$m = \frac{Dg_\alpha}{d\alpha} |_{\alpha=0}$

$$a_{m+\alpha n} = a_m + \alpha a_n$$

PF

l-akce	L_g	Ω_m	$L_{\exp(\alpha m)}$
r-akce	R_g	I_m	

Lemma

- 1) $A_g * a_m = a_{Ad_g m}$ — A_g levá akce
- 2) $A_{g^{-1}} * a_m = a_{Ad_g m}$ — A_g pravá akce

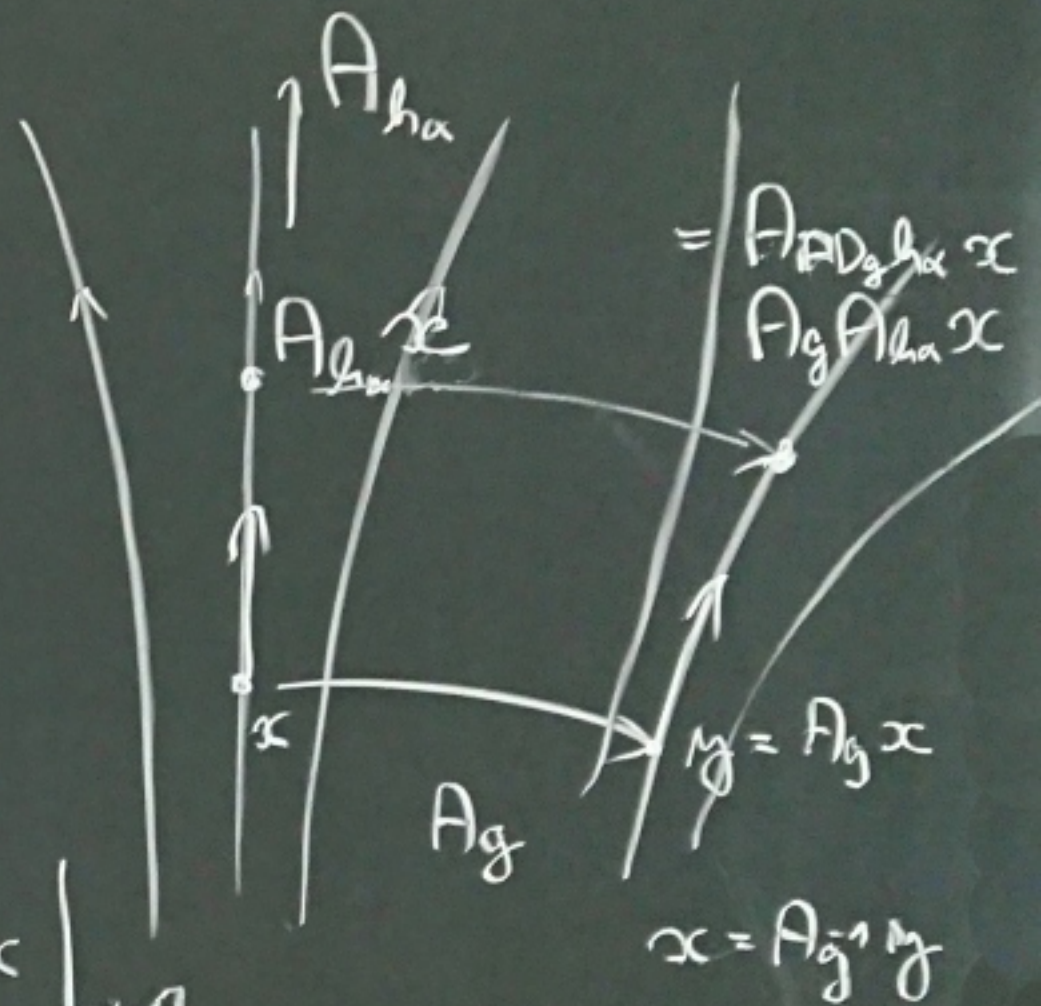
Lens

$$1) A_{g^*} a_m = a_{Adg^m} - A_g \text{ leva' alca}$$

$$2) A_{g^{-1}*} a_m = a_{Adg^m} - A_g \text{ para' alca}$$

$$\text{diz} \quad A_{g^*} a_m|_x = A_{g^*} \frac{D}{d\alpha} A_{h\alpha} x \Big|_{\alpha=0} = \frac{D}{d\alpha} A_g A_{h\alpha} x \Big|_{\alpha=0}$$

$$= \frac{D}{d\alpha} A_{ADg^h\alpha} \underbrace{A_g x}_y \Big|_{\alpha=0} = a_{Adg^m} |_{A_g x}$$



Víte

A_3 levá strana $[a_m, a_n] = -a_{[m, n]}$

A_3 pravá strana $[a_m, a_n] = a_{[m, n]}$

Niite

$$A_g \text{ levá atce } [a_m, a_n] = -a_{[m,n]}$$

$$A_g \text{ pravé atce } [a_m, a_n] = a_{[m,n]}$$

$$[a_m, a_n] = \int a_m a_n = -\frac{d}{d\alpha} A_{g_\alpha} a_n \Big|_{\alpha=0}$$

$$= -\frac{d}{d\alpha} a_{Ad_{g_\alpha} n} \Big|_{\alpha=0} = -a_{\underbrace{ad_m n}_{[m,n]}}$$
$$m = \frac{Dg_\alpha}{d\alpha} \Big|_{\alpha=0}$$

$$= -a_{[m,n]}$$

Reprezentace L.G. a L.A. na vektorovém prost. V

Def $T: G \rightarrow \text{Lin } V \quad g \rightarrow Tg \in V_1$

$$Tg_2 = Tg \cdot Tg_2 \quad Tg_{\mathbb{B}}^{\mathbb{A}} = Tg_{\mathbb{M}}^{\mathbb{A}} \cdot Tg_{\mathbb{R}}^{\mathbb{B}}$$

Def $t: \mathfrak{g} \rightarrow \text{Lin } V \quad m \rightarrow t_m \in V_1$

$$t_{[m,n]} = [t_m, t_n] = t_m t_n - t_n t_m$$

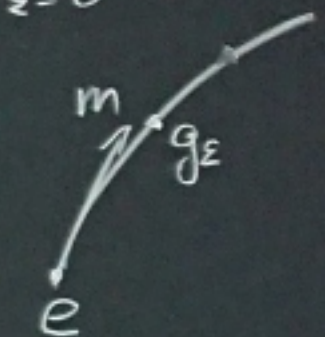
$$t_m^{\mathbb{A}}_{\mathbb{B}} = m^{\alpha} t_{\alpha}^{\mathbb{A}}_{\mathbb{B}}$$

$$C_{\mu\nu}^{\alpha} t_{\alpha}^{\mathbb{K}}_{\mathbb{L}} = t_{\mu}^{\mathbb{K}}_{\mathbb{M}} t_{\nu}^{\mathbb{M}}_{\mathbb{L}} - t_{\nu}^{\mathbb{K}}_{\mathbb{M}} t_{\mu}^{\mathbb{M}}_{\mathbb{L}}$$

$$e + \varepsilon m \quad T_{e+\varepsilon m} = \delta + \varepsilon t_m$$

Def T rep. LG na V
 \downarrow t generátor této rep.

$$t_m = \left. \frac{d}{d\varepsilon} Tg_{\varepsilon} \right|_{\varepsilon=0}$$

$$T_{\exp m} = \exp t_m$$


$$T_{\exp 0} = \delta$$

$$\left. \frac{d}{d\varepsilon} T_{\exp(\varepsilon m)} \right|_{\varepsilon=\varepsilon_0} = \left. \frac{d}{d\varepsilon} T_{\exp(\varepsilon m)} \right|_{\varepsilon=0} T_{\exp \varepsilon_0 m}$$

$$= t_m \cdot T_{\exp(\varepsilon_0 m)}$$

$$\Rightarrow T_{\exp \varepsilon m} = \exp(\varepsilon t_m)$$

Reprezentace L.G. a L.A. na vektorovém prost. V

Def $T: G \rightarrow \text{Lin } V \quad g \rightarrow T_g \in V_1^1$
 $T_{g_2} = T_{g_1} \cdot T_{g_2} \quad T_{g_2}^A{}_B = T_{g_1}^A{}_M \cdot T_{g_2}^M{}_B$

Def $t: \mathfrak{g} \rightarrow \text{Lin } V \quad m \rightarrow t_m \in V_1^1$
 $t_{[m,n]} = [t_m, t_n] = t_m t_n - t_n t_m$

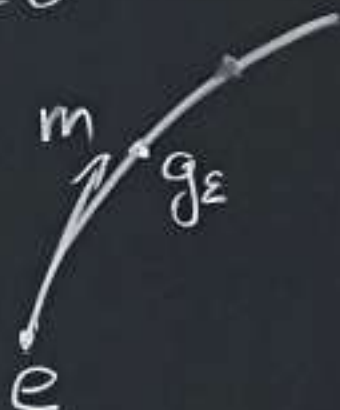
$t_m^A{}_B = m^\alpha t_\alpha^A{}_B$
 $C_{K^\alpha} t_{\alpha^\beta}{}^\gamma{}_\delta = t_{K^\beta}{}^\alpha{}_\delta t_{\alpha^\gamma}{}^\delta{}_\delta - t_{\alpha^\gamma}{}^\beta{}_\delta t_{\alpha^\delta}{}^\delta{}_\delta$

$$e + \varepsilon m \quad T_{e + \varepsilon m} = \delta + \varepsilon t_m$$

D-f

\downarrow T rep. LG $m \in V$
 t generator této rep.

$$t_m = \left. \frac{d}{d\varepsilon} T g_\varepsilon \right|_{\varepsilon=0}$$



Def

\downarrow T rep. LG $m \in V$
 t generator t is rep.

$$t_m = \left. \frac{d}{d\varepsilon} T g_\varepsilon \right|_{\varepsilon=0}$$

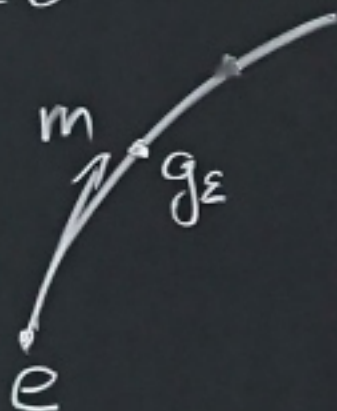
$$T \exp m = \exp t_m$$

$$T \exp 0 = \delta$$

$$\left. \frac{d}{d\varepsilon} T \exp(\varepsilon m) \right|_{\varepsilon=\varepsilon_0} = \left. \frac{d}{d\varepsilon} T \exp(\varepsilon m) \right|_{\varepsilon=0} T \exp \varepsilon_0 m$$

$$= t_m \cdot T \exp(\varepsilon m)$$

$$\Rightarrow T \exp \varepsilon m = \exp(\varepsilon t_m)$$



$$[a, b] = \frac{D}{d\tau} \left(\exp(\sqrt{\tau} a) \exp(\sqrt{\tau} b) \exp(-\sqrt{\tau} a) \exp(-\sqrt{\tau} b) \right) \Big|_{\tau=0}$$

$$\frac{d}{d\tau} T \exp(\sqrt{\tau} a) \exp(\sqrt{\tau} b) \exp(-\sqrt{\tau} a) \exp(-\sqrt{\tau} b) \Big|_{\tau=0} = t_{[a, b]} \stackrel{?}{=} [t_a, t_b]$$

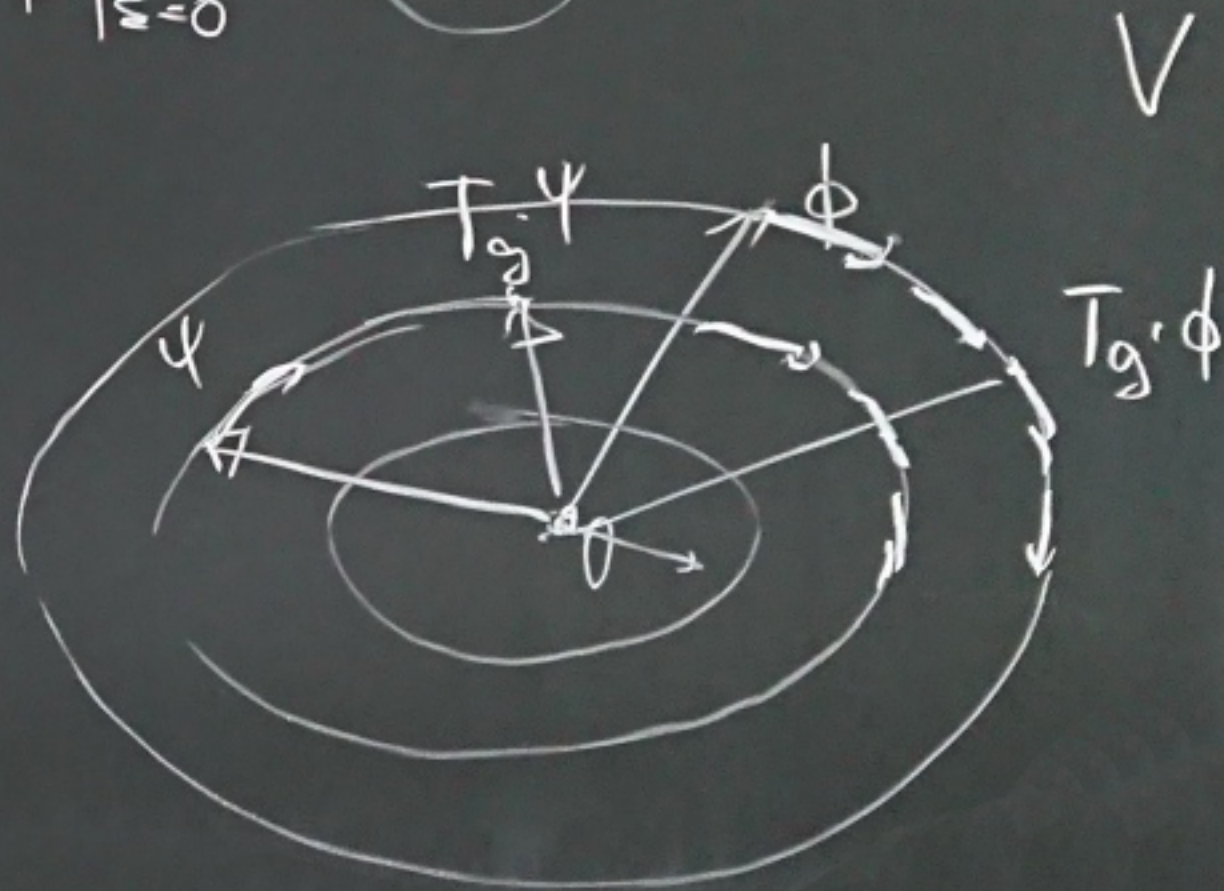
$$\begin{aligned} T \exp(\sqrt{\tau} a) &= \exp(t \sqrt{\tau} a) = \delta + \sqrt{\tau} t_a + \frac{1}{2} \tau t_a^2 + \dots \\ &= \left(\delta + \sqrt{\tau} t_a + \frac{1}{2} \tau t_a^2 + \dots \right) \left(\delta + \sqrt{\tau} t_b + \frac{1}{2} \tau t_b^2 + \dots \right) \left(\delta - \sqrt{\tau} t_a + \frac{1}{2} \tau t_a^2 + \dots \right) \left(\delta - \sqrt{\tau} t_b + \frac{1}{2} \tau t_b^2 + \dots \right) = \\ &= \delta + \sqrt{\tau} (t_a + t_b - t_a - t_b) \\ &\quad + \tau \left(\cancel{t_a^2} + \cancel{t_b^2} + t_a \cdot t_b - \cancel{t_a} t_b - t_b \cancel{t_a} + \cancel{t_a} t_b \right) = \tau [t_a, t_b] \end{aligned}$$

$$t_{[a, b]} = [t_a, t_b]$$

T repr. LG. ma V

$$g \quad T_g \quad \phi \rightarrow T_g \cdot \phi \quad \phi^A = T_g^A_B \phi^B$$

$$m \rightarrow t_m \quad t_m|_\phi = \frac{D}{d\varepsilon} T_{g_\varepsilon} \cdot \phi|_{\varepsilon=0} = \textcircled{t_m} \cdot \phi$$



T репр. LG на V

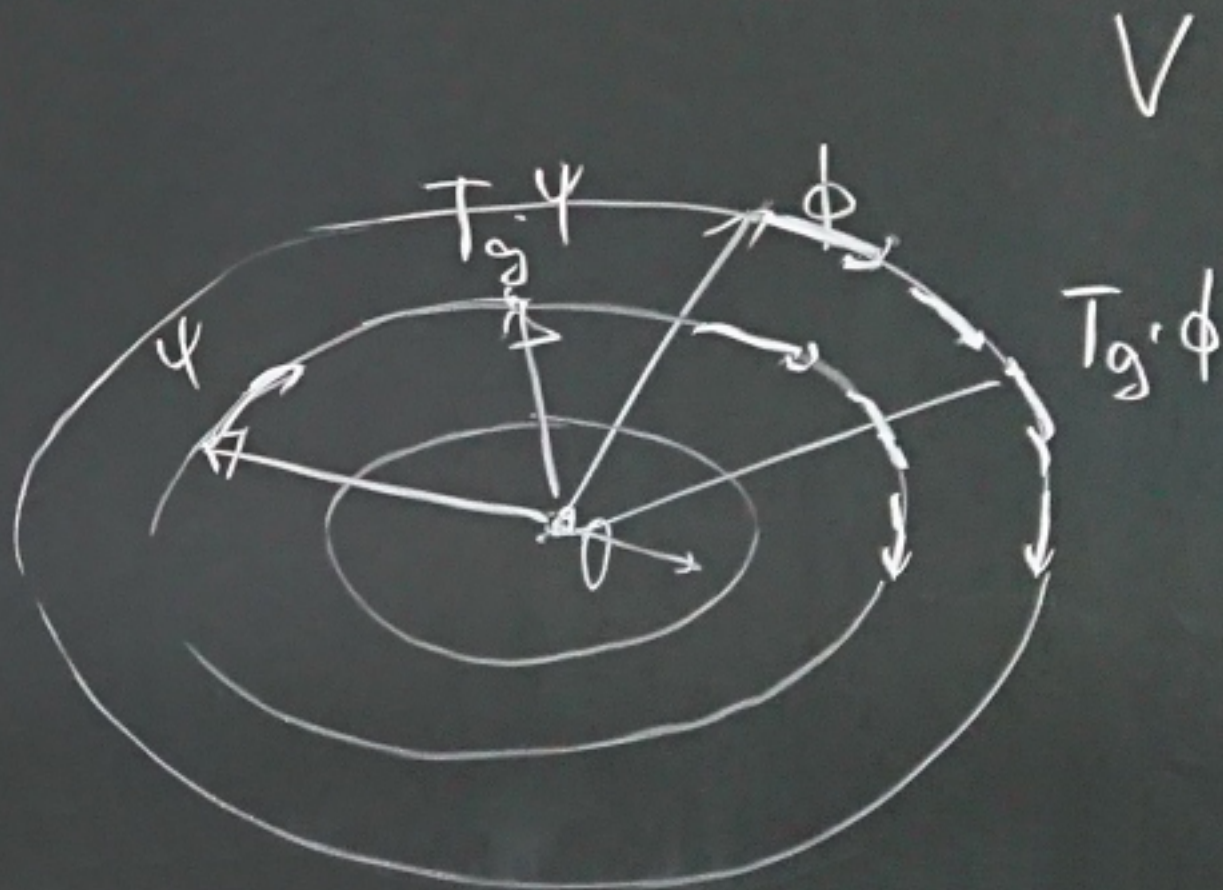
$$g \quad T_g \quad \phi \rightarrow T_g \cdot \phi \quad \phi^A = T_g^A_B \phi^B$$

$$m \rightarrow t_m \quad t_m|_\phi = \left. \frac{d}{d\varepsilon} T_{g_\varepsilon} \cdot \phi \right|_{\varepsilon=0} = t_m \cdot \phi$$

T репр. LG. на V

$$g \quad T_g \quad \phi \rightarrow T_g \cdot \phi \quad \phi^A = T_g^A_B \phi^B$$

$$m \rightarrow t_m \quad t_m|_\phi = \frac{D}{d\varepsilon} T_{g_\varepsilon} \cdot \phi|_{\varepsilon=0} = \textcircled{t_m} \cdot \phi$$



Grupa isometrií

$$\text{Iso}(M, g) \quad \varphi: M \rightarrow M$$

$$\varphi^* g = g$$

$$\varphi_1 \circ \varphi_2 = \varphi_1 \varphi_2$$

$\text{Iso}(M, g)$ tvoří L.G.

$\text{iso}(M, g)$ přísluší LA

M dim D

$\text{Iso}(M, g)$ má maximálně dim $\frac{D(D+1)}{2}$

$$\text{Iso}(M, g) \quad M$$

$$\varphi \rightarrow \square_{\varphi}$$

$$\square_{\varphi} x = \varphi x$$

$$\text{iso}(M, g)$$

$$S \rightarrow \xi_s \in \tau M$$

$$\xi_s \equiv S$$

$$\exp(\tau s) = \varphi_s$$

$$\mathcal{L}_{\xi_s} g = 0 \quad \varphi_* g = g$$

ξ_s je Killingův vektor

$$[\xi_m, \xi_n] = -\{[m, n]\}$$

běžně Kill vekt. ξ_α

$$[\xi_\alpha, \xi_\beta] = -\underbrace{\{[\xi_\alpha, \xi_\beta]\}}_{C_{\alpha\beta}{}^\mu \xi_\mu} = -C_{\alpha\beta}{}^\mu \xi_\mu$$

Grupa isometrií

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Grupa isometrií

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$$\varphi^* g = g$$

$$\varphi_1 \circ \varphi_2 = \varphi_1 \varphi_2$$

$\text{Iso}(M, g)$ tvoří L.G.

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$$\xi_s \equiv s$$

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ξ_s je Killingův vektor

$$[\xi_m, \xi_n] = - \{ [m, n] \}$$

benutz Kill wert. ξ_α

$$[\xi_\alpha, \xi_\beta] = - \underbrace{\{ [\xi_{\alpha_i}, \xi_\beta] \}}_{C_{\alpha\beta}^{\mu} \xi_\mu} = - C_{\alpha\beta}^{\mu} \xi_\mu$$

Akce symetrie na generátorech symetrie

Killingovy vektor R K_M

$$\mathcal{R} \in \text{iso}(M, g) \quad \mathcal{R}_\varphi = \exp(\varphi \mathcal{R}) \in \text{Iso}(M, g)$$

$$\xi_{\mathcal{R}} = R \quad \prod \mathcal{R}_\varphi = \mathcal{R}_\varphi$$

$$[R, K_M] = -C_{RM}{}^\nu K_\nu$$

$$V_\varphi = \mathcal{R}_\varphi^* V_0 \quad V = V^M K_M \quad \uparrow \text{konstantní na } M$$

$$\frac{d}{d\varphi} V_\varphi = -\mathcal{L}_R V_\varphi = -[R, V_\varphi] = -[R, K_M] V_\varphi^M = C_{RM}{}^\nu V_\varphi^M K_\nu$$

$$\frac{d}{d\varphi} V_\varphi^\nu = C_{RM}{}^\nu V_\varphi^M \quad V_\varphi^\nu = \exp(C_R)^\nu{}_\mu V_0^\mu \quad C_{RM}{}^\nu = \text{ad}_{R_M}{}^\nu$$

Akce symetrie na generátorech symetrie

Killingovy vektor R K_M

$$\mathcal{R} \in \text{iso}(M, g) \quad \mathcal{R}_\varphi = \exp(\varphi \mathcal{R}) \in \text{Iso}(M, g)$$

$$\{\mathcal{R}\} = \mathcal{R} \quad \prod \mathcal{R}_\varphi = \mathcal{R}_\varphi$$

$$[R, K_M] = -C_{RM}^V K_V$$

$$V_\varphi = \mathcal{R}_\varphi^* V_0 \quad V = V^M K_M$$

↑ *translatici na M*

$$V_\varphi = R_\varphi^* V_0$$

$$V = V^M K_T$$

↑
transformation M

$$\frac{d}{d\varphi} V_\varphi = -\int_R V_\varphi = -[R, V_\varphi] = -[R, K_M] V_\varphi^M = C_{RM}^\vee V_\varphi^M K_\vee$$

$$\frac{d}{d\varphi} V_\varphi^\vee = C_{RM}^\vee V_\varphi^M$$

$$V_\varphi^\vee = \exp(C_R)^\vee V_0^M$$

$$C_{RM}^\vee = \text{ad}_{RM}^\vee$$

isometrie v E^2

$$R \times Y$$

$$[R, X] = -Y$$

$$[R, Y] = X$$

$$[X, Y] = 0$$

$$C_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad C_R^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\exp(\varphi C_R) = \cos \varphi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin \varphi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$V_0 = a_0 X + b_0 Y$$

$$V_\varphi = \exp(\varphi C_R)^\top \cdot V_0 = \begin{bmatrix} \cos \varphi a_0 - \sin \varphi b_0 \\ \sin \varphi a_0 + \cos \varphi b_0 \end{bmatrix}$$

X, Y, Z transact of E^3

$$[X, Y] = [Y, Z] = [Z, X] = 0$$

$$C_X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \exp(x C_X) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_0 = a_0 X + b_0 Y$$

$$V_x = \exp(x C_X) \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

X, Y, Z translate of E^3

$$[X, Y] = [Y, Z] = [Z, X] = 0$$

$$C_x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \exp(x C_x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_0 = a_0 X + b_0 Y$$

$$V_x^R = \exp(x C_x) \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

isométrie de E^2

R, X, Y

$$[R, X] = -Y$$

$$[R, Y] = X$$

$$[X, Y] = 0$$

$$C_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad C_R^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\exp(\varphi C_R) = \cos \varphi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin \varphi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$V_0 = a_0 X + b_0 Y$$

$$V_\varphi^M = \exp(\varphi C_R)^M \cdot V_0^V = \begin{bmatrix} \cos \varphi a_0 - \sin \varphi b_0 \\ \sin \varphi a_0 + \cos \varphi b_0 \end{bmatrix}$$