

Fibrovane prostory

Def M P dif. variety

PM je dif. variety

$\pi: PM \rightarrow M$ projekce

PM je lokálne triviálna

$\forall x \in M \exists U$ okolí x

$$\pi^{-1}(U) \cong U \times P$$

$P_x M = \pi^{-1}(x) \cong P$ fiber nad x

P stand. fiber
 PM fiber. prostor

Vektorové bundly

stand. fiber A - vekt. prostor

AM

tenzorová bundle

$$A_x^k M = \underbrace{(A \otimes \dots \otimes A)}_k \otimes \underbrace{(A^* \otimes \dots \otimes A^*)}_l M$$

Def π fiber. prostor

$$\phi: M \rightarrow AM$$

$$\pi \phi = \text{id}$$

$$x \rightarrow \phi(x) \in A_x M$$

Fibrovane prostory

Df M P dif. variety

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P
 PM

stand. fiber
fibr. produkt

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Trivializace vekt. bundlu

Def $A M$ vekt. bundlu dim D

- soubor oblastí $U_{(\alpha)} \subset M$
 pokrývající M

- na $U_{(\alpha)}$ dána báze ${}^{(\alpha)}E_A \quad A=1 \dots D$
 ${}^{(\alpha)}E_A^{-1}$ tedy $\cong AU_{(\alpha)}$

↓
 1) trivializační zobrazení

$${}^{(\alpha)}\Phi_x : A_x M \rightarrow \mathbb{R}^D$$

$${}^{(\alpha)}\Phi_x : \Psi \rightarrow \Psi^A = {}^{(\alpha)}E^A \cdot \Psi$$

2) inverzní triv. zobrazení

$${}^{(\alpha)}\Phi_x^{-1} : \mathbb{R}^D \rightarrow A_x M$$

$$\Psi^A \rightarrow \Psi = \Psi^A {}^{(\alpha)}E_A$$

3) přechodová zobrazení

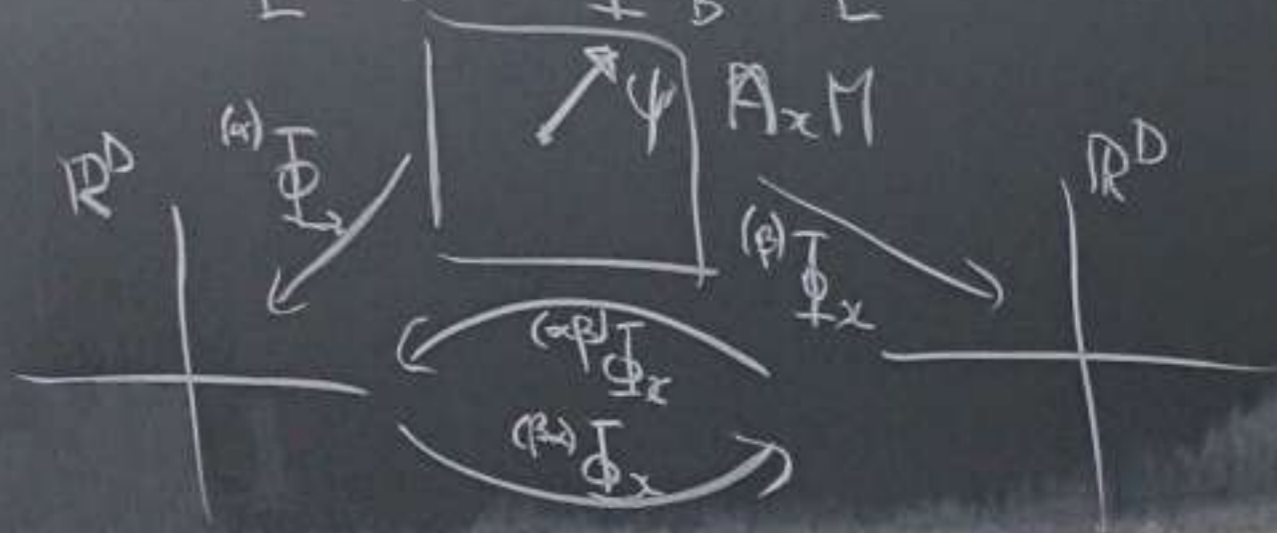
$${}^{(\alpha\beta)}\Phi_x : \mathbb{R}^D \rightarrow \mathbb{R}^D$$

pro $x \in U_{(\alpha)} \cap U_{(\beta)}$

$${}^{(\alpha\beta)}\Phi_x = {}^{(\alpha)}\Phi_x \circ {}^{(\beta)}\Phi_x^{-1}$$

$$\Psi^A \rightarrow {}^{(\alpha\beta)}\Phi_x^A \Psi^B$$

$${}^{(\beta)}\Phi_x^B = {}^{(\alpha)}E_A \cdot {}^{(\alpha\beta)}\Phi_x^A \Psi^B$$



Trivializace vekt. bundlu

Def A M vekt. bundle dim D

- soubor oblastí $U_{(\alpha)} \subset M$

pokryvajících M

- na $U_{(\alpha)}$ dána báze ${}^{(\alpha)}E_A$ $A=1 \dots D$
 ${}^{(\alpha)}E_A^M$ tedy $\subset AU_{(\alpha)}$

\Downarrow
1) trivializační zobrazení
 ${}^{(\alpha)}\Phi_x : A_x M \rightarrow \mathbb{R}^D$
 ${}^{(\alpha)}\Phi_x : \psi \rightarrow \psi^A = {}^{(\alpha)}E^A \cdot \psi$

2) inverzní triv. zobrazení
 ${}^{(\alpha)}\Phi_x^{-1} : \mathbb{R}^D \rightarrow A_x M$
 $\psi^A \rightarrow \psi = \psi^A {}^{(\alpha)}E_A$

3) přechodové zobrazení

$${}^{(\alpha\beta)}\Phi_x : \mathbb{R}^D \rightarrow \mathbb{R}^D$$

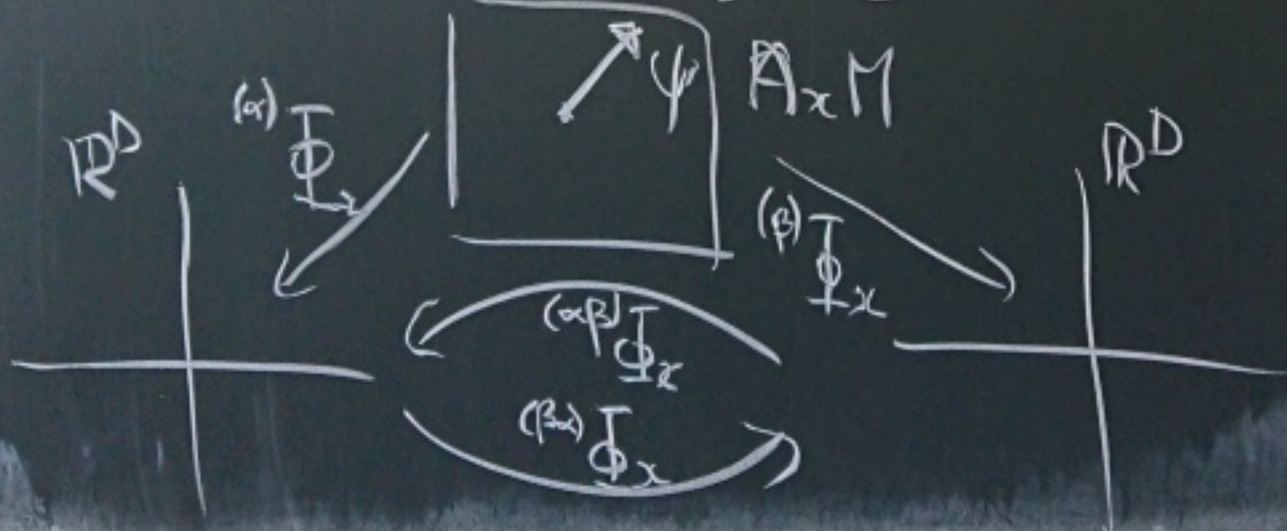
$$\text{pro } x \in U_{(\alpha)} \cap U_{(\beta)}$$

$${}^{(\alpha\beta)}\Phi_x = {}^{(\alpha)}\Phi_x \circ {}^{(\beta)}\Phi_x^{-1}$$

$$\Psi^A \rightarrow {}^{(\alpha\beta)}\Phi_x^A \Psi^B$$

$${}^{(\beta)}E_B = {}^{(\alpha)}E_A \cdot {}^{(\alpha\beta)}\Phi_x^A$$

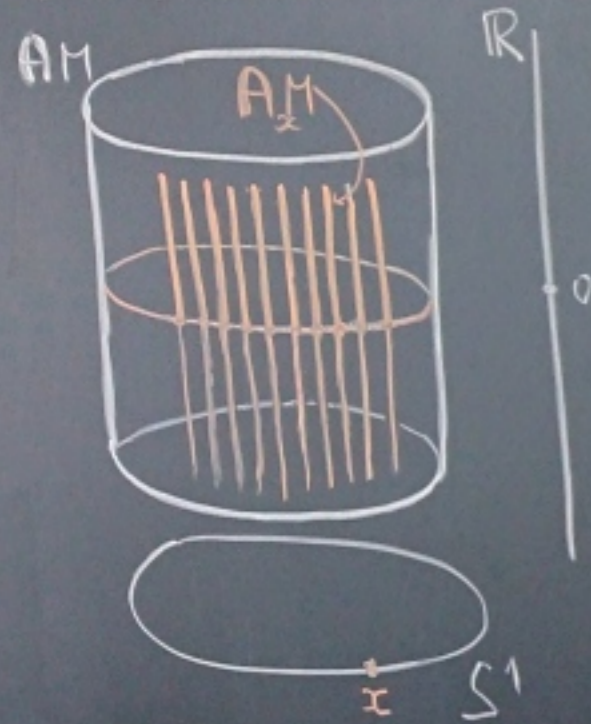
$${}^{(\alpha)}E_A = {}^{(\alpha\beta)}\Phi_x^A \cdot {}^{(\beta)}E_B$$



Př: Lineární bundle nad S^1

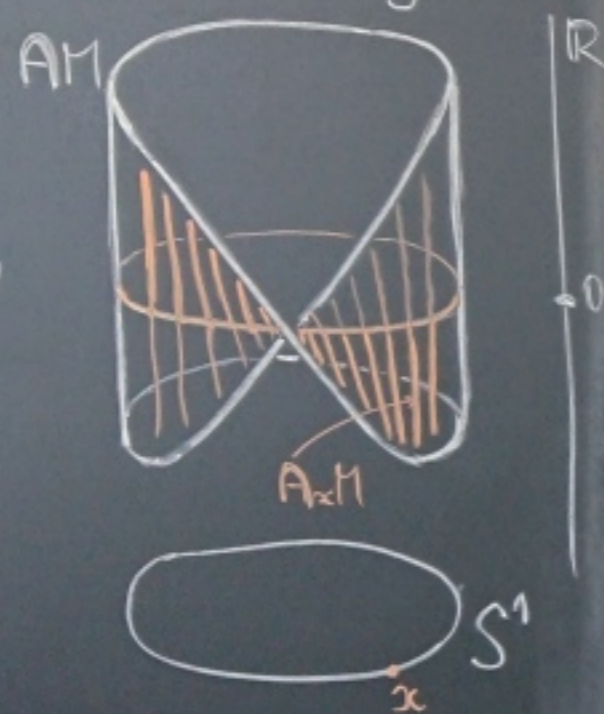
$M = S^1$ $A = \mathbb{R}$

triviální bundle



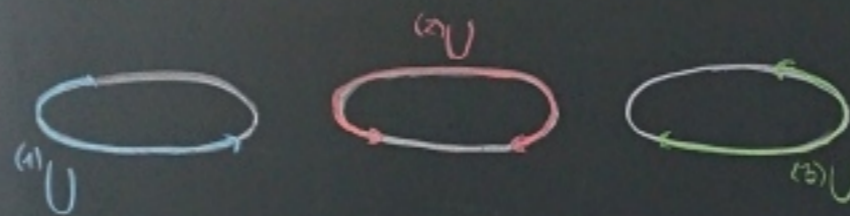
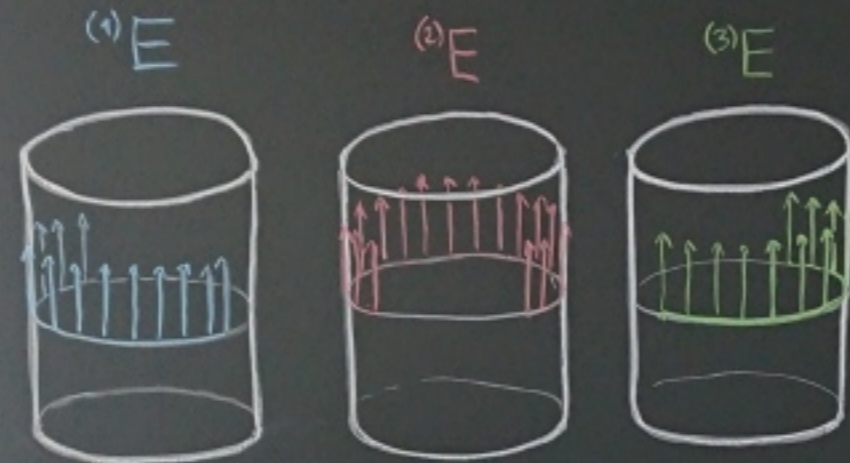
$AM \cong S^1 \times \mathbb{R}$

twistovaný bundle

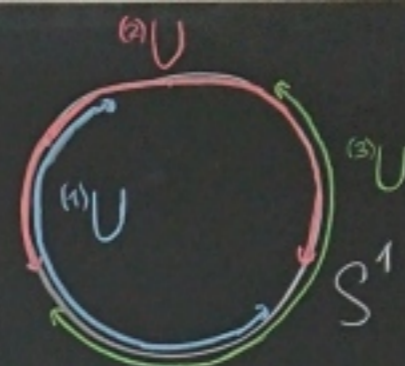


$AM \not\cong S^1 \times \mathbb{R}$

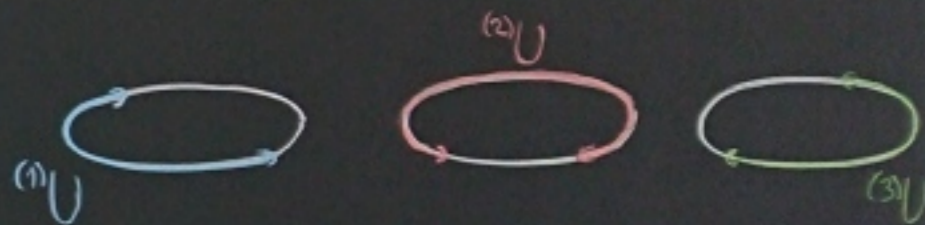
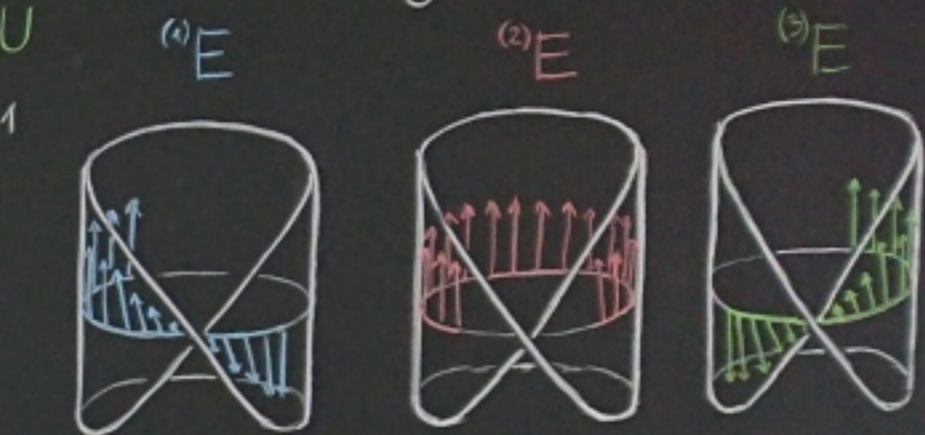
triviální bundle



${}^{(12)}\Phi = [1]$
 ${}^{(23)}\Phi = [1]$
 ${}^{(31)}\Phi = [1]$



twistovaný bundle

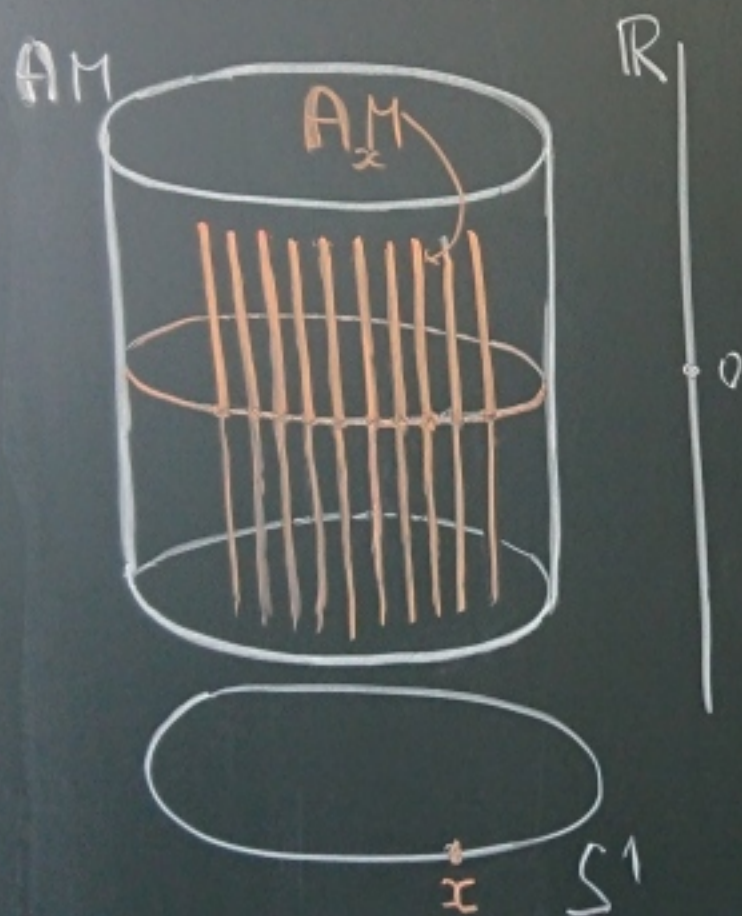


${}^{(12)}\Phi = [1]$
 ${}^{(23)}\Phi = [1]$
 ${}^{(31)}\Phi = [-1]$

Př: Lineární bundle nad S^1

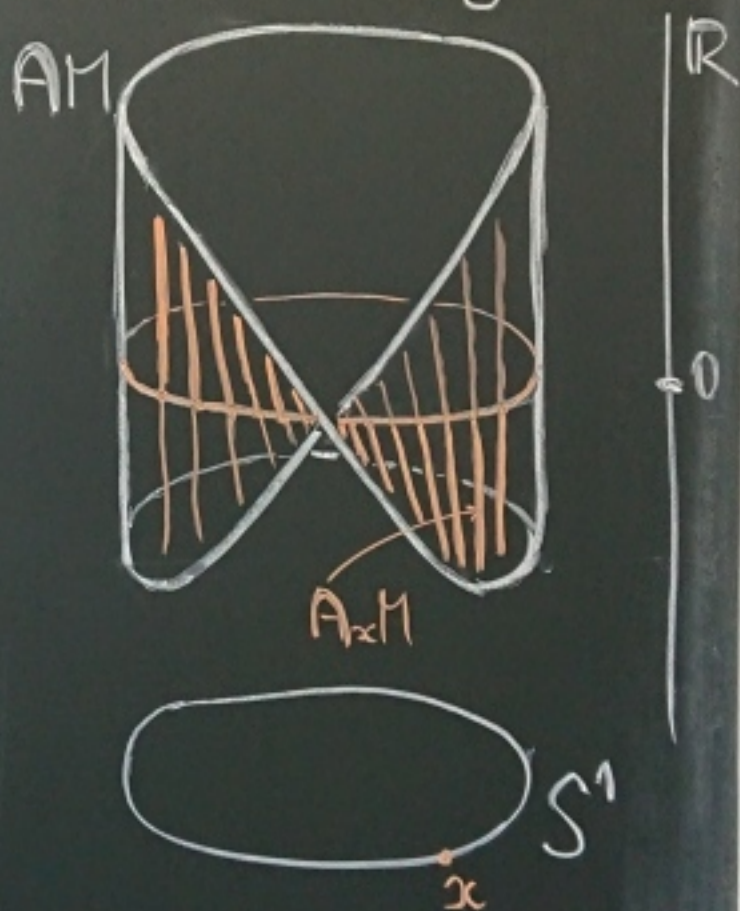
$$M = S^1 \quad A = \mathbb{R}$$

triviální bundle



$$AM \cong S^1 \times \mathbb{R}$$

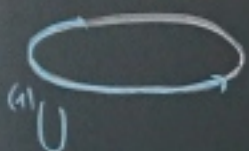
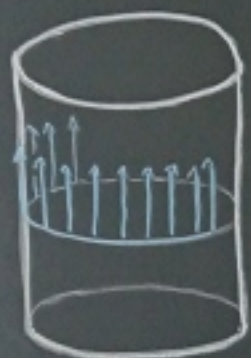
twistoraný bundle



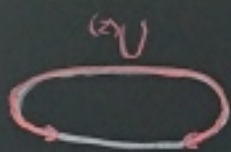
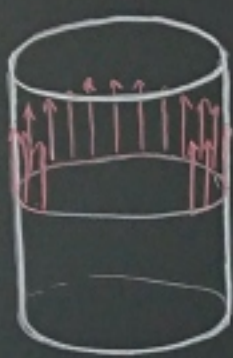
$$AM \not\cong S^1 \times \mathbb{R}$$

trivialní bundle

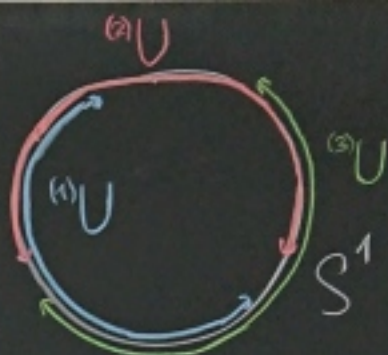
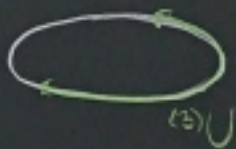
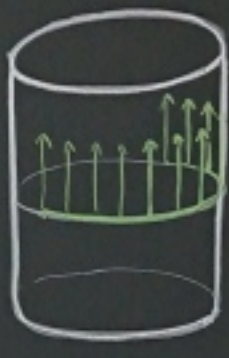
⁽¹⁾E



⁽²⁾E

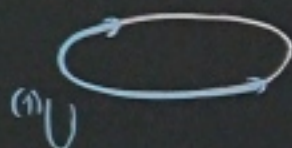


⁽³⁾E

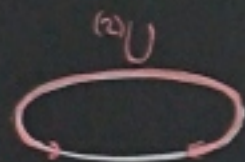


twistovaná bundle

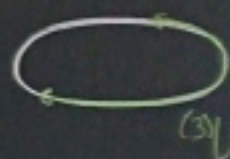
⁽¹⁾E



⁽²⁾E



⁽³⁾E



$${}^{(12)}\Phi = [1]$$

$${}^{(23)}\Phi = [1]$$

$${}^{(31)}\Phi = [1]$$

$${}^{(12)}\Phi = [1]$$

$${}^{(23)}\Phi = [1]$$

$${}^{(31)}\Phi = [-1]$$

Kovariantní derivace

na vektorovém bundlu

$$\phi, \psi \in \text{Sect } AM \quad f \in FM$$

AM

$$\xi, \zeta \in TM$$

D_ξ kovar. der na AM
ve směru $\xi \in TM$

$$D_\xi : \text{Sect } AM \rightarrow \text{Sect } AM$$

$D_\xi \phi$ je dáno zmalostí ϕ na
 x libov. okolí U bodu $x \in M$

$$D_\xi(\phi + \psi) = D_\xi \phi + D_\xi \psi$$

$$D_\xi(f\phi) = f D_\xi \phi + (\xi \cdot df) \phi$$

$$D_{f\xi + g\zeta} \phi = f D_\xi \phi + g D_\zeta \phi$$

Kovariantní diference

$$D_\xi \phi = \xi \cdot D\phi$$

$$D_\xi \phi^A = \xi^m D_m \phi^A$$

$$D_m \phi^A$$

rozšíření na tens. bundle

$$D(\phi \otimes \psi) = (D\phi) \otimes \psi + \phi \otimes (D\psi)$$

$$D C \phi = C D \phi$$

$$Df = df$$

Kovariantní derivace na vektorovém bundlu

$\phi, \psi \in \text{Sect } AM$ $f \in FM$
 $\xi, \zeta \in TM$

D_ξ kovar. der. na AM
ve směru $\xi \in TM$

$$D_\xi : \text{Sect } AM \rightarrow \text{Sect } AM$$

$D_\xi \phi|_x$ je dáno znalostí ϕ na
libov. okolí U bodu $x \in M$

$$D_\xi(\phi + \psi) = D_\xi \phi + D_\xi \psi$$

$$D_\xi(f\phi) = f D_\xi \phi + (\xi \cdot df) \phi$$

$$D_{\xi + f\zeta} \phi = D_\xi \phi + f D_\zeta \phi$$

Kovariantní diference.

$$D_{\xi} \phi = \xi \cdot D \phi$$

$$D_{\xi} \phi^A = \xi^B D_B \phi^A$$

$$D_B \phi^A$$

ποσεισμένοι με tens. bundle

$$D(\phi \otimes \psi) = (D\phi) \otimes \psi + \phi \otimes (D\psi)$$

$$Dc\phi = cD\phi$$

$$Df = df$$

NOTA: kov. derivace

\tilde{D}, \tilde{D} kov. der. na AM

$$\tilde{D}\phi - D\phi = A \cdot \phi$$

$$\tilde{D}_m \phi^A - D_m \phi^A = A_{m\ B}^A \phi^B$$

$$\tilde{D}T - DT = AT$$

$$A_{\ B}^T{}^{AB} = A_{\ B}^A T_{\ C}^{MB} + A_{\ C}^B T_{\ B}^{AC} - A_{\ B}^M T_{\ C}^{AB} - A_{\ C}^M T_{\ B}^{AC}$$

$$P_T: A_m M^A_B = A_{m\ N}^A M^N_B - A_{m\ B}^N M^A_N$$

$$A_m M = [A_m, M]$$

trivializace

E_A báze v AM

$$\partial E_A = 0 \quad \partial E^A = 0$$

$$\phi = \phi^A E_A$$

$$\partial\phi = (d\phi^A) \otimes E_A$$

$$\partial_m \phi^A = (\partial\phi)_m^A = \phi^A_{,m}$$

$$D = \partial + A \quad A_{m\ B}^A$$

M pseudoderivace $A_{\ B}^{\otimes 2} M$

$M: \text{Vect } A_{\ B}^{\otimes 2} M \rightarrow \text{Vect } A_{\ B}^{\otimes 2} M$

$$M(\phi + \psi) = M\phi + M\psi$$

$$M(\phi\psi) = (M\phi)\psi + \phi(M\psi)$$

$$M c\phi = c M\phi$$

$$Mf = 0$$

$$M T_{\ B \dots}^A \dots = M^A_N T_{\ B \dots}^N \dots + \dots - M^N_B T_{\ N \dots}^A \dots$$

$$M\phi^A = M^A_{\ N} \phi^N$$

Notah kov. derivaci

\tilde{D}, \tilde{D} kov. der. na AM

$$\tilde{D}\phi - D\phi = A \cdot \phi$$

$$\tilde{D}_m \phi^A - D_m \phi^A = A_m^A{}_B \phi^B$$

$$\tilde{D}T - DT = AT$$

$$A_m^A{}_B T^{AB}{}_{CD} = A_m^A{}_M T^{MB}{}_{CD} + A_m^B{}_M T^{AM}{}_{CD} + \dots \\ - A_m^M{}_B T^{AB}{}_{CD} - A_m^M{}_C T^{AB}{}_{CM} - \dots$$

Př.: $A_m M^A{}_B = A_m^A{}_N M^B{}_N - A_m^N{}_B M^A{}_N$

$$A_m M = [A_m, M]$$

trivializace

E_A báze v AM

$$\partial E_A = 0 \quad \partial E^A = 0$$

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$$\partial \phi = (d\phi^A) \otimes E_A$$

$$\partial_m \phi^A = (\partial \phi)_m^A = \phi^A_{|m}$$

$$D = \partial + A \quad A_{m \ B}^A$$

M pseudodifferential $A_e^s M$

$M : \text{Sect } A_e^s M \rightarrow \text{Sect } A_e^s M$

$$M(\phi + \psi) = M\phi + M\psi$$

$$M(\phi \psi) = (M\phi)\psi + \phi(M\psi)$$

$$M \partial \phi = \partial M\phi$$

$$Mf = 0$$

$$M T_{B \dots}^A \dots = M_{N \dots}^A T_{B \dots}^N \dots + \dots - M_{B \dots}^N T_{N \dots}^A \dots$$

$$M \phi^A = M_{N \dots}^A \phi^N$$

Kovariantní vnější der.
na vekt. bundle

$$\Lambda^p \otimes A^z M$$

$$\phi_1 \wedge \phi_2 \wedge \dots = \binom{p}{p_1 p_2 \dots} A(\phi_{1} \otimes \phi_{2} \otimes \dots)$$

$$p_1 \binom{p}{p_1} \quad p_2 \binom{p}{p_2}$$

$$\begin{aligned} \phi_M^M \wedge \psi_N^N &= 2 \phi_M^M \psi_N^N \\ &= \phi_M^M \otimes \psi_N^N - \phi_N^N \otimes \psi_M^M \end{aligned}$$

D_d kov. vnější der na $\Lambda^p A^z M$

$$(D_d \phi)_{a_0 a_1 \dots a_p} = (p+1) D_{[a_0} \phi_{a_1 \dots a_p]}$$

$$D_d : \text{Sect } \Lambda^p A^z M \rightarrow \Lambda^{p+1} A^z M$$

$$D_d(\phi + \psi) = D_d \phi + D_d \psi$$

$$D_d(\phi \wedge \psi) = (D_d \phi) \wedge \psi + (-1)^p \phi \wedge D_d \psi$$

$$D_d \phi^p = D\phi \quad \text{pro } \phi \in \text{Sect } A^z M \quad p=0$$

$$D_d \phi = d\phi \quad \text{pro } \phi \in \text{Sect } \Lambda^p M \quad z_l=0$$

$$\phi = \sum_{a_1 < \dots < a_p} \phi_{a_1 a_p} dx^{a_1} \wedge \dots \wedge dx^{a_p} \quad \phi_{a_1 \dots a_p} \in \text{Sect } A^z M$$

$$D_d \phi = \sum_{a_1 < \dots < a_p} D_d \phi_{a_1 \dots a_p} \wedge dx^{a_1} \wedge \dots \wedge dx^{a_p} = (p+1) \sum_{a_0 < a_1 < \dots < a_p} D_{[a_0} \phi_{a_1 \dots a_p]} dx^{a_0} \wedge dx^{a_1} \wedge \dots \wedge dx^{a_p}$$

Kovariantní vnější der.
na vekt. bundle

$$\Lambda^p \otimes A^q M$$

$$\underbrace{\phi_1}_{p_1 \ell_1} \wedge \underbrace{\phi_2}_{p_2 \ell_2} \wedge \dots = \begin{pmatrix} p \\ p_1 \ p_2 \ \dots \end{pmatrix} A(\phi_1 \otimes \phi_2 \otimes \dots)$$

$$\begin{aligned} \phi_{13}^H \wedge \psi_{12}^N &= 2 \phi_{13}^H \psi_{12}^N \\ &= \phi_{13}^H \otimes \psi_{12}^N - \phi_{12}^H \otimes \psi_{13}^N \end{aligned}$$

$\mathbb{D}d$ kov. vnejší derivace na $\Lambda^p A_x^z M$

$$\mathbb{D}d : \text{Sect } \Lambda^p A_x^z M \rightarrow \Lambda^{p+1} A_x^z M$$

$$\mathbb{D}d(\phi + \psi) = \mathbb{D}d\phi + \mathbb{D}d\psi$$

$$\mathbb{D}d(\phi \wedge \psi) = (\mathbb{D}d\phi) \wedge \psi + (-1)^p \phi \wedge \mathbb{D}d\psi$$

$$\mathbb{D}d \overset{\uparrow p}{\phi} = \mathbb{D}\phi \quad \text{pro } \phi \in \text{Sect } A_x^z M \quad p=0$$

$$\mathbb{D}d\phi = d\phi \quad \text{pro } \phi \in \text{Sect } \Lambda^p M \quad z, l=0$$

D_d kov. vnejší derivace na $\Lambda^p A_x^z M$ $(D_d \phi)_{a_0 a_1 \dots a_p} = (p+1) D_{[a_0} \phi_{a_1 \dots a_p]}$

$$D_d : \text{Sect } \Lambda^p A_x^z M \rightarrow \Lambda^{p+1} A_x^z M$$

$$D_d(\phi + \psi) = D_d \phi + D_d \psi$$

$$D_d(\phi \wedge \psi) = (D_d \phi) \wedge \psi + (-1)^p \phi \wedge D_d \psi$$

$$D_d \phi \stackrel{1_p}{=} D\phi \quad \text{pro } \phi \in \text{Sect } A_x^z M \quad p=0$$

$$D_d \phi = d\phi \quad \text{pro } \phi \in \text{Sect } \Lambda^p M \quad z, l=0$$

$$\phi = \sum_{a_1 < \dots < a_p} \phi_{a_1 a_p} dx^{a_1} \wedge \dots \wedge dx^{a_p} \quad \phi_{a_1 \dots a_p} \in \text{Sect } A_x^z M$$

$$D_d \phi = \sum_{a_1 < \dots < a_p} D_d \phi_{a_1 \dots a_p} \wedge dx^{a_1} \wedge \dots \wedge dx^{a_p} = (p+1) \sum_{a_0 < a_1 < \dots < a_p} D_{[a_0} \phi_{a_1 \dots a_p]} dx^{a_0} \wedge dx^{a_1} \wedge \dots \wedge dx^{a_p}$$

D na AM

∇ na TM T_{mn}^a

D rozšířený D na TM

$$D(A \otimes \phi) = \underbrace{(DA)}_{\nabla A} \otimes \phi + A \otimes \underbrace{(D\phi)}_{D\phi}$$

\uparrow TM AM ∇A $D\phi$

$$D_{a_0} \phi_{a_1 \dots a_p}^A = D_{a_0} \wedge \phi_{a_1 \dots a_p}^A + T_{a_0 a_1}^a \wedge \phi_{[a_1 a_2 \dots a_p]}^A$$

$$= (p+1) D_{[a_0} \phi_{a_1 \dots a_p]}^A + \binom{p+1}{2} T_{[a_0 a_1}^a \phi_{|a_1 a_2 \dots a_p]}^A$$

D_d kov. vnější derivace na $\Lambda^p A_x^z M$

$$(D_d \phi)_{a_0 a_1 \dots a_p} = (p+1) D_{[a_0} \phi_{a_1 \dots a_p]}$$

$$D_d : \text{Sect } \Lambda^p A_x^z M \rightarrow \Lambda^{p+1} A_x^z M$$

$$D_d(\phi + \psi) = D_d \phi + D_d \psi$$

$$D_d(\phi \wedge \psi) = (D_d \phi) \wedge \psi + (-1)^p \phi \wedge D_d \psi$$

$$D_d \phi^1_p = D\phi \quad \text{pro } \phi \in \text{Sect } A_x^z M \quad p=0$$

$$D_d \phi = d\phi \quad \text{pro } \phi \in \text{Sect } \Lambda^p M \quad x_l=0$$

$$\phi = \sum_{a_1 < \dots < a_p} \phi_{a_1 \dots a_p} dx^{a_1} \wedge \dots \wedge dx^{a_p} \quad \phi_{a_1 \dots a_p} \in \text{Sect } A_x^z M$$

$$D_d \phi = \sum_{a_1 < \dots < a_p} D_d \phi_{a_1 \dots a_p} \wedge dx^{a_1} \wedge \dots \wedge dx^{a_p} = (p+1) \sum_{a_0 < a_1 < \dots < a_p} D_{[a_0} \phi_{a_1 \dots a_p]} dx^{a_0} \wedge dx^{a_1} \wedge \dots \wedge dx^{a_p}$$

D ma AM

∇ ma TM T_{mn}^a

D rozšířen D ma TM

$$D(A \otimes \phi) = \underbrace{(DA)}_{\nabla A} \otimes \phi + A \otimes \underbrace{(D\phi)}_{D\phi}$$

\uparrow \uparrow
 TM AM

$$D \lrcorner_{\alpha_0} \phi_{\alpha_1 \dots \alpha_p}^A = D_{\alpha_0} \wedge \phi_{\alpha_1 \dots \alpha_p}^A + T_{\alpha_0 \alpha_1}^n \wedge \phi_{|\alpha_1| \alpha_2 \dots \alpha_p}^A$$
$$= (p+1) D_{[\alpha_0} \phi_{\alpha_1 \dots \alpha_p]}^A + \binom{p+1}{2} T_{[\alpha_0 \alpha_1}^n \phi_{|\alpha_1| \alpha_2 \dots \alpha_p]}^A$$