# Maxwellovy rovnice pomocí forem

$$\Phi = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dy^\beta \qquad dx^\alpha \wedge dy^\beta = 2!\, dx^{[\alpha} dy^{\beta]} = dx^\alpha dy^\beta - dx^\beta dy^\alpha$$

$$*\Phi = \frac{1}{2} F^*_{\alpha\beta}\, dx^\alpha \wedge dy^\beta, \qquad F^*_{\alpha\beta} \text{ duální k } F_{\alpha\beta}$$

Maxw. r. $d\Phi = 0$,

$d{*\Phi} = 0$. $\quad d(*\phi) = 0$

$\Phi = 2dA$. → vektorový potenciál

– viz dále dole

$d\Phi = 2d^2A \equiv 0$ na další str. 10**

Podrobně:

Tensoru $F_{\alpha\beta}$ přiřadíme

$$\Phi(d\vec{x}, d\vec{y}) = 2!\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]} = 2!\, F_{\alpha\beta} \tfrac{1}{2}\left(dx^\alpha dy^\beta - dx^\beta dy^\alpha\right)$$

vlit podle pracle vždy nejdřív derivační index

Pak série Maxw. rovnic:

$$\boxed{d\Phi = 0}$$

viz $d\Phi = 0 \Rightarrow 3!\, F_{\alpha\beta,\gamma}\, dx^{[\gamma} dy^\alpha dz^{\beta]} = 0$

$$\Rightarrow \left(F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} - F_{\beta\alpha,\gamma} - F_{\alpha\gamma,\beta} - F_{\gamma\beta,\alpha}\right) dx^\gamma dy^\alpha dz^\beta$$

cyklicky a pozn. (2. řádka) vždy prohazuji $\alpha \leftrightarrow \beta$ v každém členu z 1. řádky

sečtu se – viz $F_{\alpha\beta} = -F_{\beta\alpha}$

$$\Rightarrow 2\left(F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta}\right) dx^\gamma dy^\alpha dz^\beta = 0$$

Vhodně zvolujme $dx, dy, dz$, např. $dx = (dx^0, 0, 0, 0)$

$dy = (0, dy^1, 0, 0)$

$dz = (0, 0, dz^2, 0)$

atd.

Odtud dostaneme

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = F_{[\alpha\beta,\gamma]\,\text{cykl.}} = 0$$

tj. 1 sada Maxw. rovnic

Je známo, že $F_{[\alpha\beta,\gamma]\,\text{cykl.}} = 0 \iff F^{*}_{\alpha\beta,}{}^{\beta} = 0$

Podobně ovšem platí $F_{\alpha\beta,}{}^{\beta} = 0 \iff F^{*}_{[\alpha\beta,\gamma]\,\text{cykl.}} = 0$
pro 2. sadu Maxw. rovnic

tedy tato 2. sada pomocí forem je

$$\boxed{d\,{}^{*}\Phi^{*} = 0} \,,$$

kde ${}^{*}\Phi^{*} = 2!\, F^{*}_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]}$

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Zavedení 4-potenciálu $A_\mu \longrightarrow A = A_\mu dx^\mu$
1-forma

$$\Phi = 2dA$$

tj. $\Phi(d\vec{x}, d\vec{y}) = 2!\, A_{\alpha,\beta}\, dx^{[\beta} dy^{\alpha]} =$

$$= 2A_{\alpha,\beta}\,(dx^{\beta} dy^{\alpha} - dx^{\alpha} dy^{\beta}) =$$

$$= 2\,(A_{\beta,\alpha} - A_{\alpha,\beta})\, dx^{\alpha} dy^{\beta}$$

ale $\Phi = 2!\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]} = \cancel{\tfrac{2}{2}}\, F_{\alpha\beta}\, dx^{\alpha} dy^{\beta}$

$\Rightarrow F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$

# Maxwellovy rovnice s pravou stranou

$$F^{\mu\nu}{}_{;\nu} = \frac{4\pi}{c} J^{\mu}, \qquad F_{[\mu\nu;\lambda]\,cykl.} = 0$$

Def. $\boxed{{}^{*}J_{\varrho\sigma\tau} = \varepsilon_{\xi\varrho\sigma\tau} J^{\xi}}$ ✓ $\Rightarrow J^{\xi} = -\frac{1}{3!}\varepsilon^{\xi\varrho\sigma\tau}\,{}^{*}J_{\varrho\sigma\tau}$ (+)

viz $J^{\xi} = -\frac{1}{3!}\varepsilon^{\xi\varrho\sigma\tau}\,{}^{*}J_{\varrho\sigma\tau} =$
$= -\frac{1}{3!}\varepsilon^{\xi\varrho\sigma\tau}\underbrace{\varepsilon_{\alpha\varrho\sigma\tau}}_{= +\varepsilon_{\alpha\varrho\sigma\tau}} J^{\alpha} = \frac{+3!}{3!}\delta^{\xi}_{\alpha}J^{\alpha} = J^{\xi}$ ✓

$$\boxed{{}^{*}F_{\mu\nu} = \frac{1}{2!}\varepsilon_{\mu\nu\varrho\sigma}F^{\varrho\sigma}}$$

$\Rightarrow$ ~~$F_{\mu\nu} = \ldots$~~ $= \frac{1}{2!}F^{\varrho\sigma}\varepsilon_{\varrho\sigma\mu\nu}$ ✓

$$\varepsilon^{\mu\nu\varrho\sigma}\varepsilon_{\varkappa\lambda\varrho\sigma} = -2\left(\delta^{\mu}_{\varkappa}\delta^{\nu}_{\lambda} - \delta^{\mu}_{\lambda}\delta^{\nu}_{\varkappa}\right) ✓$$

$$\Rightarrow \varepsilon^{\mu\nu\varrho\sigma}\varepsilon_{\mu\nu\varrho\sigma} = -2\Big(\underbrace{\delta^{\mu}_{\mu}\delta^{\nu}_{\nu}}_{16} - \overbrace{\delta^{\mu}_{\nu}\delta^{\nu}_{\mu}}^{\delta^{\mu}_{\mu}}{}_{4}\Big) = -24 ✓$$

$$\Rightarrow \quad \varepsilon^{\alpha\beta\mu\nu}\,{}^{*}F_{\mu\nu} = \frac{1}{2!}\varepsilon^{\alpha\beta\mu\nu}\underbrace{\varepsilon_{\mu\nu\varrho\sigma}}_{=\varepsilon_{\varrho\sigma\mu\nu}}F^{\varrho\sigma}$$

$$= \frac{-2}{2!}\left(\delta^{\alpha}_{\varrho}\delta^{\beta}_{\sigma} - \delta^{\alpha}_{\sigma}\delta^{\beta}_{\varrho}\right)F^{\varrho\sigma} =$$

$$= -\left(F^{\alpha\beta} - F^{\beta\alpha}\right) = -2F^{\alpha\beta}$$

$$F^{\alpha\beta} = -\frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\,{}^{*}F_{\mu\nu} \qquad (\times)$$

$$\Rightarrow \boxed{F_{\alpha\beta} = -\frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\,{}^{*}F^{\mu\nu}} ✓$$

$$F^{\alpha\beta}{}_{;\beta} = \frac{4\pi}{c}J^{\alpha}$$

$(\times) \Rightarrow$ (+) nahoře

$$-\frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\,{}^{*}F_{\mu\nu;\beta} = \frac{4\pi}{c}\left(-\frac{1}{3!}\right)\varepsilon^{\alpha\varrho\sigma\tau}\,{}^{*}J_{\varrho\sigma\tau} \quad \Big/ \times \varepsilon_{\alpha\gamma\delta\varepsilon}$$

$\delta^{\alpha}_{\mu} = -\frac{1}{3!}\varepsilon^{\alpha\beta\lambda\varrho}\varepsilon_{\mu\beta\lambda\varrho}$ ✓, $\quad \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu} = -\frac{1}{2}\varepsilon^{\alpha\beta\lambda\varrho}\varepsilon_{\mu\nu\lambda\varrho}$

NAŠE DEFINICE BYLA: MTW 1

(1) $\Phi = 2!\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]}$

$A = A_\mu\, dx^\mu$ (Srovnání a podle MTW)

$$dA = 2!\, A_{\mu,\nu}\, dx^{[\nu} dy^{\mu]} = 2!\, A_{\beta,\alpha}\, dx^{[\alpha} dy^{\beta]}$$

$$= 2!\, A_{[\beta,\alpha]}\, dx^\alpha dy^\beta =$$

$$= (A_{\beta,\alpha} - A_{\alpha,\beta})\, dx^\alpha dy^\beta$$

$$\Phi = 2!\tfrac{1}{2}(F_{\alpha\beta} - F_{\beta\alpha})\, dx^\alpha dy^\beta = 2F_{\alpha\beta}\, dx^\alpha dy^\beta$$

lib. $dx^\alpha,\, dy^\beta \Rightarrow$ když standardní $A_{\beta,\alpha} - A_{\alpha,\beta} = F_{\alpha\beta}$

pak $\boxed{2dA = \Phi}$

(1) $\Rightarrow \Phi = F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$

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Když nadefinujeme

$F = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$ (souhlasí např. s MTW) 4.25 str. 112-113

tj: $F = \frac{1}{2}\Phi$

a necháme $A = A_\rho\, dx^\rho$

vyjde $\boxed{dA = F}$

$\boxed{\varepsilon_{\alpha\beta\gamma\delta}}$ MTW 2

Přehled vlastností $\varepsilon_{\alpha\beta\gamma\delta} =$ +1 pro $(\alpha\beta\gamma\delta) = 0123$ a sudé permut.

−1 pro liché perm. 0123

0 když se lib. indexy rovnají

ale $\varepsilon^{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\gamma\delta}$ ... myšleno „po indexech"

tj. $\varepsilon_{0123} = +1$ $\quad \varepsilon^{0123} = -1$ a vše ostatní permutacemi

Platí:

1)
$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\kappa\lambda} = -\begin{vmatrix} \delta^\alpha_\mu & \delta^\alpha_\nu & \delta^\alpha_\kappa & \delta^\alpha_\lambda \\ \delta^\beta_\mu & \delta^\beta_\nu & \delta^\beta_\kappa & \delta^\beta_\lambda \\ \delta^\gamma_\mu & \delta^\gamma_\nu & \delta^\gamma_\kappa & \delta^\gamma_\lambda \\ \delta^\delta_\mu & \delta^\delta_\nu & \delta^\delta_\kappa & \delta^\delta_\lambda \end{vmatrix}$$

2)
$$\varepsilon^{\alpha\beta\gamma\sigma}\varepsilon_{\mu\nu\kappa\sigma} = -\begin{vmatrix} \delta^\alpha_\mu & \delta^\alpha_\nu & \delta^\alpha_\kappa \\ \delta^\beta_\mu & \delta^\beta_\nu & \delta^\beta_\kappa \\ \delta^\gamma_\mu & \delta^\gamma_\nu & \delta^\gamma_\kappa \end{vmatrix} \equiv -\delta^{\alpha\beta\gamma}_{\mu\nu\kappa}$$

v přednášce označeno

3)
$$\varepsilon^{\alpha\beta\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = -2\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right)$$

4)
$$\varepsilon^{\alpha\lambda\rho\sigma}\varepsilon_{\mu\lambda\rho\sigma} = -6\,\delta^\alpha_\mu = -3!\,\delta^\alpha_\mu$$

5)
$$\varepsilon^{\alpha\lambda\rho\sigma}\varepsilon_{\alpha\lambda\rho\sigma} = -24 = -4!$$

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu}\, 2!\, du^{[\mu} dv^{\nu]}$$

$$= F_{\mu\nu}\, du^{[\mu} dv^{\nu]} = \frac{1}{2} \Phi$$

$$dF = \frac{1}{2} d\Phi = \frac{1}{2}\, 3!\, F_{\mu\nu,\alpha}\, du^{[\alpha} dv^\mu dw^{\nu]}$$

$$= \frac{1}{2} F_{\mu\nu,\alpha}\, dx^\alpha \wedge dx^\mu \wedge dx^\nu$$

$$^*F = \frac{1}{2} F^*_{\mu\nu}\, dx^\mu \wedge dx^\nu = \dots = \frac{1}{2}\, {}^*\Phi$$

$$d\,{}^*F = \frac{1}{2}\, 3!\, F^*_{\mu\nu,\alpha}\, du^{[\alpha} dv^\mu dw^{\nu]}$$

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z minulé přednášky $\Rightarrow$ povynásobení vztahu $-\frac{1}{2} \varepsilon^{\alpha\beta\mu\nu}\, {}^*F_{\mu\nu,\beta} = \frac{4\pi}{c}\left(-\frac{1}{3!}\right) \varepsilon^{\alpha\varrho\sigma\tau}\, {}^*J_{\varrho\sigma\tau}$ $\Big/ \times \varepsilon_{\alpha\gamma\delta\varepsilon}$

z 10*** dole: $\quad {}^*F_{[\mu\nu,\alpha]} = \frac{4\pi}{c} \frac{1}{3!}\, {}^*J_{[\mu\nu\alpha]} \quad \Big/ \times du^\alpha dv^\mu dw^\nu \ \frac{1}{2} 3!$

$$d\,{}^*F = \frac{4\pi}{c} \left( \frac{1}{2}\, {}^*J_{\mu\nu\alpha}\, du^{[\alpha} dv^\mu dw^{\nu]} \right) \equiv {}^*J$$

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$${}^*J = \frac{1}{2}\, {}^*J_{\mu\nu\alpha}\, du^{[\alpha} dv^\mu dw^{\nu]} \qquad \Rightarrow \boxed{d\,{}^*F = \frac{4\pi}{c}\, {}^*J}$$

$$= \frac{1}{2} \varepsilon_{\varrho\mu\nu\alpha} J^\varrho\, du^{[\alpha} dv^\mu dw^{\nu]}$$

$$= \frac{1}{2} J^\varrho \varepsilon_{\varrho\mu\nu\alpha} \frac{1}{3!} \big( du^\alpha dv^\mu dw^\nu + du^\mu dv^\nu dw^\alpha + du^\nu dv^\alpha dw^\mu$$
$$- du^\mu dv^\alpha dw^\nu - du^\nu dv^\mu dw^\alpha - du^\alpha dv^\nu dw^\mu \big)$$
$$+ \dots$$

Souhlasí s MTW (4.31)