Úloha 2: Tenzorový součin dvou vektorů

Mějme dva vektorové operátory $\hat{U}_k$ a $\hat{V}_l$. Připomeňme, že jejich ireducibilní složky jsou $\hat{V}_0^{(1)} = \hat{V}_z$, $\hat{V}_{\pm 1}^{(1)} = \mp \frac{1}{\sqrt{2}}(\hat{V}_x \pm i\hat{V}_y)$ a podobně $\hat{U}_m^{(1)}$. Zopakujte si definici tenzorového součinu dvou operátorů

$$\hat{W}_m^{(j)} = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | j m \rangle \hat{U}_{m_1}^{(j_1)} \hat{V}_{m_2}^{(j_2)}.$$

Najděte explicitně komponenty $\hat{W}_0^{(0)}$, $\hat{W}_m^{(1)}$, $\hat{W}_m^{(2)}$ a vyjádřete je pomocí kartézských složek vektorů $\hat{U}_k$ a $\hat{V}_l$. Rozmyslete si jak nalezené operátory $\hat{W}_m^{(j)}$ souvisí se skalárním a vektorovým součinem $\hat{U}_k$ a $\hat{V}_l$ a s tenzorem jejich diadického součinu $\hat{T}_{kl} = \hat{U}_k \hat{V}_l$ a jeho rozkladem na izotropní, antisymetrickou a symetrickou bezestopou část.

Poznámka: Potřebné Clebschovy-Gordanovy koeficienty odečtěte ze vzorců pro skládání momentu hybnosti 1+1, které jsme nalezli ve cvičení 4:

$$|2\,2\rangle = |+\rangle|+\rangle$$
$$|2\,1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|0\rangle + |0\rangle|+\rangle\right) \qquad |1\,1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|0\rangle - |0\rangle|+\rangle\right)$$
$$|2\,0\rangle = \frac{1}{\sqrt{6}}\left(|+\rangle|-\rangle + 2|0\rangle|0\rangle + |-\rangle|+\rangle\right) \qquad |1\,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$
$$|2\,{-1}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|-\rangle + |-\rangle|0\rangle\right) \qquad |1\,{-1}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|-\rangle - |-\rangle|0\rangle\right)$$
$$|2\,{-2}\rangle = |-\rangle|-\rangle$$
$$|0\,0\rangle = \frac{1}{\sqrt{3}}\left(|+\rangle|-\rangle - |0\rangle|0\rangle + |-\rangle|+\rangle\right)$$

Vztah separované a kaplované báze
$$|j\,m\rangle = \sum_{m_1, m_2} |j_1 m_1 j_2 m_2\rangle \underbrace{\langle j_1 m_1 j_2 m_2 | j m\rangle}_{C^{jm}_{j_1 m_1 j_2 m_2}}$$

$$|j_1 m_1\rangle |j_2 m_2\rangle \equiv \hat{U}_{m_1}^{(j_1)} \hat{V}_{m_2}^{(j_2)}$$

- Vyjádření $\hat{W}_m^{(j)}$ pomocí $\hat{U}_m^{(1)}$ a $\hat{V}_m^{(1)}$ (ireducibilních složek)

$$\hat{W}_0^{(0)} = \frac{1}{\sqrt{3}}\left(\hat{U}_1^{(1)}\hat{V}_{-1}^{(1)} - \hat{U}_0^{(1)}\hat{V}_0^{(1)} + \hat{U}_{-1}^{(1)}\hat{V}_1^{(1)}\right)$$

$$\hat{W}_1^{(1)} = \frac{1}{\sqrt{2}}\left(\hat{U}_1^{(1)}\hat{V}_0^{(1)} - \hat{U}_0^{(1)}\hat{V}_1^{(1)}\right)$$
$$\hat{W}_0^{(1)} = \frac{1}{\sqrt{2}}\left(\hat{U}_1^{(1)}\hat{V}_{-1}^{(1)} - \hat{U}_{-1}^{(1)}\hat{V}_1^{(1)}\right)$$
$$\hat{W}_{-1}^{(1)} = \frac{1}{\sqrt{2}}\left(\hat{U}_0^{(1)}\hat{V}_{-1}^{(1)} - \hat{U}_{-1}^{(1)}\hat{V}_0^{(1)}\right)$$

$$\hat{W}^{(2)}_{2} = \hat{U}^{(1)}_{1}\hat{V}^{(1)}_{1}$$
$$\hat{W}^{(2)}_{1} = \frac{1}{\sqrt{2}}\left(\hat{U}^{(1)}_{1}\hat{V}^{(1)}_{0} + \hat{U}^{(1)}_{0}\hat{V}^{(1)}_{1}\right)$$
$$\hat{W}^{(2)}_{0} = \frac{1}{\sqrt{6}}\left(\hat{U}^{(1)}_{1}\hat{V}^{(1)}_{-1} + 2\hat{U}^{(1)}_{0}\hat{V}^{(1)}_{0} + \hat{U}^{(1)}_{-1}\hat{V}^{(1)}_{1}\right)$$
$$\hat{W}^{(2)}_{-1} = \frac{1}{\sqrt{2}}\left(\hat{U}^{(1)}_{0}\hat{V}^{(1)}_{-1} + \hat{U}^{(1)}_{-1}\hat{V}^{(1)}_{0}\right)$$
$$\hat{W}^{(2)}_{-2} = \hat{U}^{(1)}_{-1}\hat{V}^{(1)}_{-1}$$

- Vyjádření $\hat{W}^{(j)}_{m}$ pomocí $\hat{U}_k$ a $\hat{V}_l$ (kartézských složek)

$$\hat{W}^{(0)}_{0} = \frac{1}{\sqrt{3}}\left\{\frac{-1}{\sqrt{2}}(\hat{U}_x + i\hat{U}_y)\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y) - \hat{U}_z\hat{V}_z + \frac{1}{\sqrt{2}}(\hat{U}_x - i\hat{U}_y)\frac{-1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y)\right\} =$$
$$= \frac{1}{\sqrt{3}}\left\{-\frac{1}{2}\left[\hat{U}_x\hat{V}_x + \hat{U}_y\hat{V}_y + i(\hat{U}_y\hat{V}_x - \hat{U}_x\hat{V}_y)\right] - \hat{U}_z\hat{V}_z - \frac{1}{2}\left[\hat{U}_x\hat{V}_x + \hat{U}_y\hat{V}_y + i(\hat{U}_x\hat{V}_y - \hat{U}_y\hat{V}_x)\right]\right\} =$$
$$= -\frac{1}{\sqrt{3}}\left\{\hat{U}_x\hat{V}_x + \hat{U}_y\hat{V}_y + \hat{U}_z\hat{V}_z\right\} = \underline{-\frac{1}{\sqrt{3}}\hat{U}_k\hat{V}_k}$$

$$\hat{W}^{(1)}_{1} = \frac{1}{\sqrt{2}}\left\{\frac{-1}{\sqrt{2}}(\hat{U}_x + i\hat{U}_y)\hat{V}_z - \hat{U}_z\frac{-1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y)\right\} = \frac{1}{2}\left\{-\hat{U}_x\hat{V}_z - i\hat{U}_y\hat{V}_z + \hat{U}_z\hat{V}_x + i\hat{U}_z\hat{V}_y\right\} =$$
$$= \underline{\frac{1}{2}\left\{\hat{U}_z\hat{V}_x - \hat{U}_x\hat{V}_z + i(\hat{U}_z\hat{V}_y - \hat{U}_y\hat{V}_z)\right\}}$$

$$\hat{W}^{(1)}_{0} = \frac{1}{\sqrt{2}}\left\{\frac{-1}{\sqrt{2}}(\hat{U}_x + i\hat{U}_y)\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y) - \frac{1}{\sqrt{2}}(\hat{U}_x - i\hat{U}_y)\frac{-1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y)\right\} =$$
$$= \frac{1}{2\sqrt{2}}\left\{-\hat{U}_x\hat{V}_x - \hat{U}_y\hat{V}_y - i(\hat{U}_y\hat{V}_x - \hat{U}_x\hat{V}_y) + \hat{U}_x\hat{V}_x + \hat{U}_y\hat{V}_y + i(\hat{U}_x\hat{V}_y - \hat{U}_y\hat{V}_x)\right\} =$$
$$= \underline{\frac{i}{\sqrt{2}}\left(\hat{U}_x\hat{V}_y - \hat{U}_y\hat{V}_x\right)}$$

$$\hat{W}^{(1)}_{-1} = \frac{1}{\sqrt{2}}\left\{\hat{U}_z\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y) - \frac{1}{\sqrt{2}}(\hat{U}_x - i\hat{U}_y)\hat{V}_z\right\} = \frac{1}{2}\left\{\hat{U}_z\hat{V}_x - i\hat{U}_z\hat{V}_y - \hat{U}_x\hat{V}_z + i\hat{U}_y\hat{V}_z\right\} =$$
$$= \underline{\frac{1}{2}\left\{\hat{U}_z\hat{V}_x - \hat{U}_x\hat{V}_z - i(\hat{U}_z\hat{V}_y - \hat{U}_y\hat{V}_z)\right\}}$$

$$\hat{W}^{(2)}_{2} = \frac{-1}{\sqrt{2}}(\hat{U}_x + i\hat{U}_y)\frac{-1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y) = \underline{\frac{1}{2}\left\{\hat{U}_x\hat{V}_x - \hat{U}_y\hat{V}_y + i(\hat{U}_y\hat{V}_x + \hat{U}_x\hat{V}_y)\right\}}$$

$$\hat{W}^{(2)}_{1} = \frac{1}{\sqrt{2}}\left\{\frac{-1}{\sqrt{2}}(\hat{U}_x + i\hat{U}_y)\hat{V}_z + \hat{U}_z\frac{-1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y)\right\} = \underline{-\frac{1}{2}\left\{\hat{U}_x\hat{V}_z + \hat{U}_z\hat{V}_x + i(\hat{U}_y\hat{V}_z + \hat{U}_z\hat{V}_y)\right\}}$$

$$\hat{W}^{(2)}_0 = \frac{1}{\sqrt{6}}\left\{\frac{-1}{\sqrt{2}}(\hat{U}_x + i\hat{U}_y)\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y) + 2\hat{U}_z\hat{V}_z + \frac{1}{\sqrt{2}}(\hat{U}_x - i\hat{U}_y)\frac{-1}{\sqrt{2}}(\hat{V}_x + i\hat{V}_y)\right\} =$$

$$= -\frac{1}{2\sqrt{6}}\left\{\hat{U}_x\hat{V}_x + \hat{U}_y\hat{V}_y + i(\hat{U}_y\hat{V}_x - \hat{U}_x\hat{V}_y) - 4\hat{U}_z\hat{V}_z + \hat{U}_x\hat{V}_x + \hat{U}_y\hat{V}_y + i(\hat{U}_x\hat{V}_y - \hat{U}_y\hat{V}_x)\right\} =$$

$$= \underline{\frac{1}{\sqrt{6}}\left\{2\hat{U}_z\hat{V}_z - \hat{U}_x\hat{V}_x - \hat{U}_y\hat{V}_y\right\}}$$

$$\hat{W}^{(2)}_{-1} = \frac{1}{\sqrt{2}}\left\{\hat{U}_z\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y) + \frac{1}{\sqrt{2}}(\hat{U}_x - i\hat{U}_y)\hat{V}_z\right\} = \underline{\frac{1}{2}\left\{\hat{U}_z\hat{V}_x + \hat{U}_x\hat{V}_z - i(\hat{U}_z\hat{V}_y + \hat{U}_y\hat{V}_z)\right\}}$$

$$\hat{W}^{(2)}_{-2} = \frac{1}{\sqrt{2}}(\hat{U}_x - i\hat{U}_y)\frac{1}{\sqrt{2}}(\hat{V}_x - i\hat{V}_y) = \underline{\frac{1}{2}\left\{\hat{U}_x\hat{V}_x - \hat{U}_y\hat{V}_y - i(\hat{U}_x\hat{V}_y + \hat{U}_y\hat{V}_x)\right\}}$$

- Souvislost $\hat{W}^{(j)}_m$ s $\hat{\vec{U}}\cdot\hat{\vec{V}}$

$$\underbrace{\hat{W}^{(0)}_0 = -\frac{1}{\sqrt{3}}\hat{U}_k\hat{V}_k = -\frac{1}{\sqrt{3}}\hat{\vec{U}}\cdot\hat{\vec{V}}}$$

- Souvislost $\hat{W}^{(j)}_m$ s $\hat{\vec{U}}\times\hat{\vec{V}}$

$$\hat{\vec{\tilde{W}}} = \hat{\vec{U}}\times\hat{\vec{V}} \qquad \hat{\tilde{W}}_k = \varepsilon_{klm}\hat{U}_l\hat{V}_m$$

$$\left.\begin{aligned}\hat{\tilde{W}}_x &= \hat{U}_y\hat{V}_z - \hat{U}_z\hat{V}_y\\ \hat{\tilde{W}}_y &= \hat{U}_z\hat{V}_x - \hat{U}_x\hat{V}_z\\ \hat{\tilde{W}}_z &= \hat{U}_x\hat{V}_y - \hat{U}_y\hat{V}_x\end{aligned}\right\} \Rightarrow \begin{aligned}\hat{W}^{(1)}_1 &= \frac{1}{2}(\hat{\tilde{W}}_y - i\hat{\tilde{W}}_x) = \frac{i}{\sqrt{2}}\left\{\frac{-1}{\sqrt{2}}(\hat{\tilde{W}}_x + i\hat{\tilde{W}}_y)\right\} = \frac{i}{\sqrt{2}}\hat{\tilde{W}}^{(1)}_1\\ \hat{W}^{(1)}_{-1} &= \frac{1}{2}(\hat{\tilde{W}}_y + i\hat{\tilde{W}}_x) = \frac{i}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}(\hat{\tilde{W}}_x - i\hat{\tilde{W}}_y)\right\} = \frac{i}{\sqrt{2}}\hat{\tilde{W}}^{(1)}_{-1}\\ \hat{W}^{(1)}_0 &= \frac{i}{\sqrt{2}}\hat{\tilde{W}}_z\end{aligned}$$

$$\Rightarrow \underbrace{\hat{W}^{(1)}_k = \frac{i}{\sqrt{2}}\hat{\tilde{W}}^{(1)}_k}$$

- Souvislost $\hat{W}^{(j)}_m$ s $\hat{T}_{kl} = \hat{U}_k\hat{V}_l$ a jeho rozkladem

$$\hat{T}_{kl} = \frac{\hat{T}_{kl} + \hat{T}_{lk}}{2} + \frac{\hat{T}_{kl} - \hat{T}_{lk}}{2} + \frac{\hat{T}_{nn}}{3}\delta_{kl} - \frac{\hat{T}_{nn}}{3}\delta_{kl}$$

$$= \underbrace{\frac{\hat{T}_{nn}}{3}\delta_{kl}} + \underbrace{\frac{\hat{T}_{kl} - \hat{T}_{lk}}{2}} + \underbrace{\frac{\hat{T}_{kl} + \hat{T}_{lk}}{2} - \frac{\hat{T}_{nn}}{3}\delta_{lk}}$$

$$\equiv \underbrace{\hat{T}\delta_{kl}}_{\text{izotropní část}} + \underbrace{\hat{A}_{kl}}_{\text{antisymetrická část}} + \underbrace{\hat{S}_{kl}}_{\text{symetrická bezestopá část}}$$

$$\hat{T} = \frac{\hat{U}_x\hat{V}_x + \hat{U}_y\hat{V}_y + \hat{U}_z\hat{V}_z}{3} = -\frac{1}{\sqrt{3}}\hat{W}_0^{(0)}$$

$$\hat{A}_{xy} = \frac{\hat{U}_x\hat{V}_y - \hat{U}_y\hat{V}_x}{2} = -\frac{i}{\sqrt{2}}\hat{W}_0^{(1)} \qquad \hat{A}_{xx} = 0,\ \hat{A}_{yy} = 0,\ \hat{A}_{zz} = 0$$

$$\hat{A}_{xz} = \frac{\hat{U}_x\hat{V}_z - \hat{U}_z\hat{V}_x}{2} = -\frac{1}{2}\left(\hat{W}_1^{(1)} + \hat{W}_{-1}^{(1)}\right) \qquad \hat{A}_{yx} = -\hat{A}_{xy},\ \hat{A}_{zx} = -\hat{A}_{xz},\ \hat{A}_{zy} = -\hat{A}_{yz}$$

$$\hat{A}_{yz} = \frac{\hat{U}_y\hat{V}_z - \hat{U}_z\hat{V}_y}{2} = -\frac{i}{2}\left(\hat{W}_{-1}^{(1)} - \hat{W}_1^{(1)}\right)$$

$$\hat{S}_{xx} = \hat{U}_x\hat{V}_x - \hat{T} = \frac{1}{3}\left(2\hat{U}_x\hat{V}_x - \hat{U}_y\hat{V}_y - \hat{U}_z\hat{V}_z\right) = \frac{1}{3}\,\frac{1}{2}\left\{3\left(\hat{W}_2^{(2)} + \hat{W}_{-2}^{(2)}\right) - \sqrt{6}\,\hat{W}_0^{(2)}\right\} =$$

$$= \frac{1}{2}\left(\hat{W}_2^{(2)} + \hat{W}_{-2}^{(2)}\right) - \frac{1}{\sqrt{6}}\hat{W}_0^{(2)}$$

$$\hat{S}_{yy} = \hat{U}_y\hat{V}_y - \hat{T} = \frac{1}{3}\left(-\hat{U}_x\hat{V}_x + 2\hat{U}_y\hat{V}_y - \hat{U}_z\hat{V}_z\right) = \frac{1}{3}\,\frac{1}{2}\left\{-3\left(\hat{W}_2^{(2)} + \hat{W}_{-2}^{(2)}\right) - \sqrt{6}\,\hat{W}_0^{(2)}\right\}$$

$$= -\frac{1}{2}\left(\hat{W}_2^{(2)} + \hat{W}_{-2}^{(2)}\right) - \frac{1}{\sqrt{6}}\hat{W}_0^{(2)}$$

$$\left(\hat{S}_{zz} = -\hat{S}_{xx} - \hat{S}_{yy} = \sqrt{\frac{2}{3}}\,\hat{W}_0^{(2)}\right)$$

$$\hat{S}_{xy} = \frac{\hat{U}_x\hat{V}_y + \hat{U}_y\hat{V}_x}{2} = -\frac{i}{2}\left(\hat{W}_2^{(2)} - \hat{W}_{-2}^{(2)}\right) \qquad \hat{S}_{yx} = \hat{S}_{xy},\ \hat{S}_{zx} = \hat{S}_{xz},\ \hat{S}_{zy} = \hat{S}_{zy}$$

$$\hat{S}_{xz} = \frac{\hat{U}_x\hat{V}_z + \hat{U}_z\hat{V}_x}{2} = \frac{1}{2}\left(\hat{W}_{-1}^{(2)} - \hat{W}_1^{(2)}\right)$$

$$\hat{S}_{yz} = \frac{\hat{U}_y\hat{V}_z + \hat{U}_z\hat{V}_y}{2} = \frac{i}{2}\left(\hat{W}_1^{(2)} + \hat{W}_{-1}^{(2)}\right)$$

$$\overleftrightarrow{\hat{T}} = \vec{\hat{U}}\vec{\hat{V}} = \begin{pmatrix} -\frac{1}{\sqrt{3}}\hat{W}_0^{(0)} + 0 + \frac{1}{2}\left(\hat{W}_2^{(2)} + \hat{W}_{-2}^{(2)}\right) - \frac{1}{\sqrt{6}}\hat{W}_0^{(2)} & 0 - \frac{i}{\sqrt{2}}\hat{W}_0^{(1)} - \frac{i}{2}\left(\hat{W}_2^{(2)} - \hat{W}_{-2}^{(2)}\right) & 0 - \frac{1}{2}\left(\hat{W}_1^{(1)} + \hat{W}_{-1}^{(1)}\right) + \frac{1}{2}\left(\hat{W}_{-1}^{(2)} - \hat{W}_1^{(2)}\right) \\ 0 + \frac{i}{\sqrt{2}}\hat{W}_0^{(1)} - \frac{i}{2}\left(\hat{W}_2^{(2)} - \hat{W}_{-2}^{(2)}\right) & -\frac{1}{\sqrt{3}}\hat{W}_0^{(0)} + 0 - \frac{1}{2}\left(\hat{W}_2^{(2)} + \hat{W}_{-2}^{(2)}\right) - \frac{1}{\sqrt{6}}\hat{W}_1^{(2)} & 0 - \frac{i}{2}\left(\hat{W}_{-1}^{(1)} - \hat{W}_1^{(1)}\right) + \frac{i}{2}\left(\hat{W}_1^{(2)} + \hat{W}_{-1}^{(2)}\right) \\ 0 + \frac{1}{2}\left(\hat{W}_1^{(1)} + \hat{W}_{-1}^{(1)}\right) + \frac{1}{2}\left(\hat{W}_{-1}^{(2)} - \hat{W}_1^{(2)}\right) & 0 + \frac{i}{2}\left(\hat{W}_{-1}^{(1)} - \hat{W}_1^{(1)}\right) + \frac{i}{2}\left(\hat{W}_1^{(2)} + \hat{W}_{-1}^{(2)}\right) & -\frac{1}{\sqrt{3}}\hat{W}_0^{(0)} + 0 + \sqrt{\frac{2}{3}}\,\hat{W}_0^{(2)} \end{pmatrix}$$