QM II – cv 8

• Dodělávka → δ-krystal

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda \sum_{n\in\mathbb{Z}} \delta_{na}$$

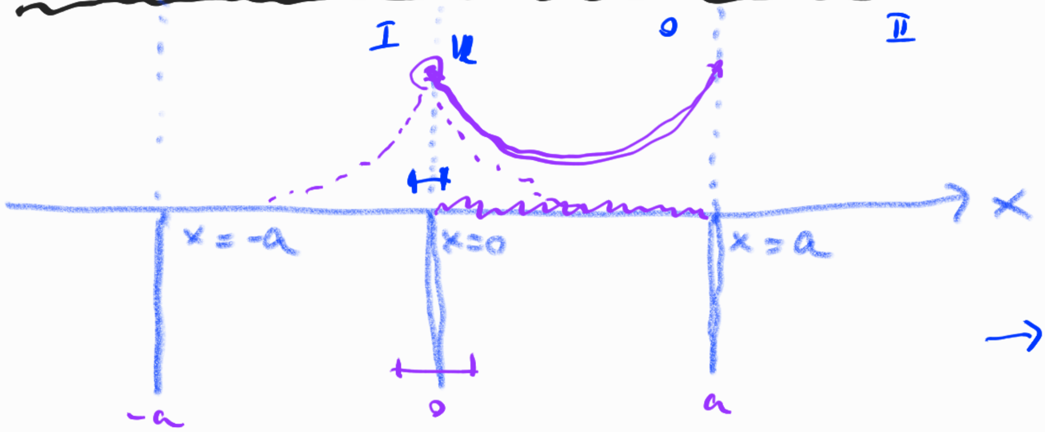
$$H\psi = E\psi$$

Blochův teorém

$$\psi(x) = u_k(x)\, e^{ikx}$$

$$\to \boxed{u_k(x) = u_k(x+a)}$$

$$\forall k \in \left\langle -\frac{a}{\pi}, \frac{a}{\pi} \right\rangle$$



① řešení $x \in (0,a)$ .. $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ $\longrightarrow$ $\psi_0(x) = e^{\varkappa x} + A e^{-\varkappa x}$

$E<0$ .. $E = -\frac{\hbar^2\varkappa^2}{2m}$ $\qquad$ $E>0$ ... $\varkappa \to i\varkappa$

② rozšíření na I .. $x \in (-a, 0\rangle$ $\qquad$ $\psi(x+a) = u_k(x)\, e^{ik(x+a)} = e^{ika}\psi(x)$

$x \in (-a, 0)$ $\qquad$ $\psi_I(x) = e^{-ika}\,\psi_0(x+a) = e^{-ika}\left(e^{\varkappa(x+a)} + A e^{-\varkappa(x+a)}\right)$

③ okraj. podm. spojitost fce ψ v okolí $x = 0$

$$\begin{cases} \psi(0+) = \psi_0(0) = 1 + A \\ \psi(0-) = \psi_I(0) = e^{-ika}\left(e^{\varkappa a} + A e^{-\varkappa a}\right) \end{cases}$$

$$A = \frac{e^{ika} - e^{a\varkappa}}{e^{-a\varkappa} - e^{ika}} \qquad \psi_0 = \frac{e^{-\varkappa a} - e^{\varkappa a}}{e^{-\varkappa a} - e^{ika}}$$

podmínka na derivaci .. δ-chování:

$$\psi'(0-) - \psi'(0+) = \tilde{\lambda}\,\psi_0 \qquad \tilde{\lambda} = \frac{2m\lambda}{\hbar^2}$$

$$\psi_I'(0) - \psi_0'(0) = \varkappa\, e^{-ika}\left[e^{\varkappa a} - A e^{-\varkappa a}\right] - \varkappa(1-A) = \tilde{\lambda}\,\psi_0$$

rovnice pro $\varkappa(k)$ ... $\to$ $E_k = -\frac{\hbar^2\varkappa^2}{2m} < 0$ .. $\left| \begin{matrix} E>0 \\ \varkappa \to i\varkappa \end{matrix} \right.$

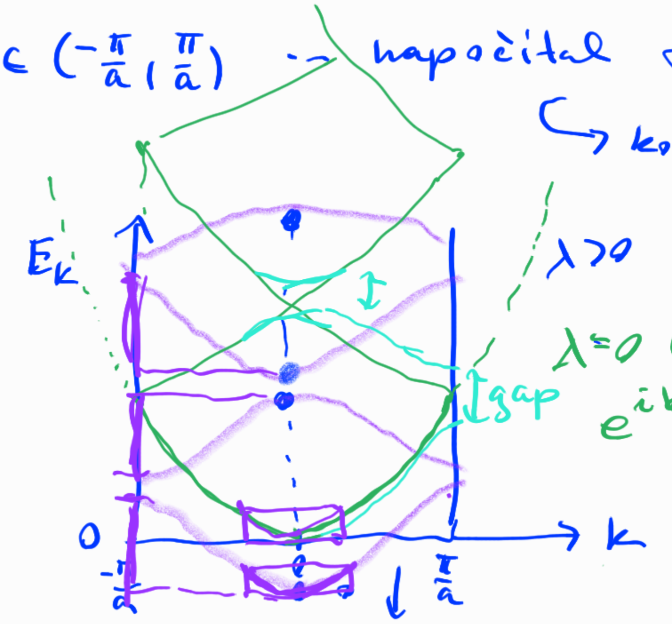
$$\cos ka = \cosh \varkappa a - \frac{\tilde{\lambda}}{4\varkappa}\sinh \varkappa a \quad \dots \quad \text{pro } E_k = -\frac{\hbar^2\varkappa_k^2}{2m} < 0$$

$$\cos ka = \cos \varkappa a - \frac{\tilde{\lambda}}{4\varkappa}\sin \varkappa a \qquad \text{pro } E_k = +\frac{\hbar^2\varkappa^2}{2m} > 0$$

$f(\varkappa) = 0$ $\qquad$ ↑ řešení $\varkappa(k)$ $\qquad$ $\frac{e^{\varkappa a} + e^{-\varkappa a}}{2} = \cosh \varkappa a$

$\forall\, k \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right)$ → napočítal $f(\varkappa)$ $\quad \varkappa = 0.01, 0.02, \ldots$

↪ kořeny $\varkappa_1(k), \varkappa_2(k), \ldots$



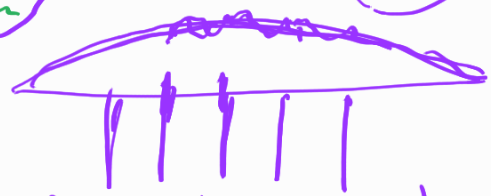
$\lambda = 0$ (volné částice)

e^{ikx} $\quad E_k = \frac{k^2\hbar^2}{2m}$

$\varkappa = k$

dlouhovlnná limita ($k \to 0$)

$\approx$ e^{ikx}



→ efektivní hmotnost

Úloha 1 – Šimon Pajger $\qquad -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\varphi^2} = -\epsilon_1 \frac{d^2}{d\varphi^2}$ .. $E_m = \epsilon_1 m^2$

shrnutí:

	1 částice	rozlišit.	$m^2 + m'^2$	Bosony	Fermi
	0 $\quad \phi_0 = \frac{1}{\sqrt{2\pi}}$	$\phi_0(\varphi_1)\phi_0(\varphi_2)$	0 (1x)	1x	0x
→	ϵ_1 $\quad \phi_{1c} = \frac{1}{\sqrt{\pi}}\cos\varphi,\ \frac{1}{\sqrt{\pi}}\sin\varphi = \phi_{1s}$; $\phi_m = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$ $\quad m = \pm 1$	ϵ $\quad \phi_0(\varphi_1)\phi_{\pm1}(\varphi_2)$, $\phi_{\pm1}(\varphi_1)\phi_0(\varphi_2)$	0 + 1, 1 + 0 (4x)	2x	2x
$m = 2$	$\epsilon_1 4$ .. $\frac{1}{\sqrt{\pi}}\cos/\sin m\varphi$	$2\epsilon_1$ $\quad \phi_1(\varphi_1)\phi_1(\varphi_2)$, $\phi_{-1}(\varphi_1)\phi_{-1}(\varphi_2)$		2x	0x
		(4x) $\quad \phi_1(\varphi_1)\phi_{-1}(\varphi_2)$ $\quad \varphi_2 \quad \varphi_1$		1x	1x

body 1–3

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bod 4 $\qquad H = -\underset{\sim}{\epsilon_1}\left(\frac{\partial^2}{\partial\varphi_1^2} + \frac{\partial^2}{\partial\varphi_2^2}\right) + \underset{\sim}{\lambda}\,\delta(\underbrace{\varphi_1 - \varphi_2}_{\varphi})$

$\varphi_1, \varphi_2 \in \langle 0, 2\pi\rangle$

$$\boxed{\begin{aligned}\frac{\varphi_1 + \varphi_2}{2} &= \Phi \\ \varphi_1 - \varphi_2 &= \varphi\end{aligned}}$$

přesné řešení vede úloha →

My: poruchové řešení .. nutno zvlášť jednotlivé hladiny a Bosony / Fermi

- základní stav Bosony: $\psi_0 = \phi_0(\varphi_1)\phi_0(\varphi_2) = \frac{1}{2\pi}$

$E = \underset{0}{E^{(0)}} + E^{(1)}$

$$E^{(1)} = \langle \psi_0 | H_I | \psi_0 \rangle = \frac{\lambda}{(2\pi)^2}\int_0^{2\pi}\underbrace{\int_0^{2\pi}\delta(\varphi_1 - \varphi_2)\,d\varphi_1}_{1}\,d\varphi_2 = \frac{\lambda}{2\pi}$$

$E = 0 + \frac{\lambda}{2\pi}$

• excitovaný stav Bosony: $\psi_\pm = \left[\phi_0(\varphi_1)\,\phi_\pm(\varphi_2) + \phi_0(\varphi_2)\,\phi_\pm(\varphi_1)\right]\frac{1}{\sqrt{2}}$

$E = E^{(0)} + E^{(1)}$ 0. řád: $\psi_\pm = \frac{1}{\sqrt{2}}\frac{1}{2\pi}\left(e^{\pm i\varphi_2} + e^{\pm i\varphi_1}\right) \dots E^{(0)} = \epsilon_1$

matice poruch $H_I \longrightarrow$

$$\begin{array}{c|cc} & \psi_+ & \psi_- \\ \hline \psi_+ & H_{++} & H_{+-} \\ \psi_- & H_{-+} & H_{--} \end{array}$$

$\cdot\ H_{mm'}$

$m = \pm 1$

$$H_{++} = \langle \psi_+ | \underbrace{H_I}_{\lambda\delta(\varphi_1-\varphi_2)} | \psi_+ \rangle = \frac{\lambda}{2}\frac{1}{(2\pi)^2}\iint \left(e^{-i\varphi_2} + e^{-i\varphi_1}\right)\underline{\delta(\varphi_1-\varphi_2)}\left(e^{i\varphi_2} + e^{i\varphi_1}\right) d\varphi_1 d\varphi_2$$

$\cdot\, e^{-i\varphi_1}$ $\cdot\, e^{i\varphi_1}$

$$= \frac{\lambda}{2}\frac{4}{(2\pi)^2}\int_0^{2\pi} d\varphi_1 = \frac{2\lambda}{2\pi} = \frac{\lambda}{\pi}$$

$H_{--} = \frac{\lambda}{\pi}$

$$H_{+-} = \frac{\lambda}{2}\frac{1}{(2\pi)^2}\iint \left(e^{-i\varphi_2} + e^{-i\varphi_1}\right)\delta(\varphi_1-\varphi_2)\left(e^{-i\varphi_2} + e^{-i\varphi_1}\right) d\varphi_1 d\varphi_2$$

$$= \frac{\lambda}{2}\frac{4}{(2\pi)^2}\int_0^{2\pi} e^{-2i\varphi_1} d\varphi_1 = 0$$ ... rotační symetrie

... M .. moment hybn. integrál pohybu

$$H_I \quad \begin{pmatrix} \boxed{} & 0 \\ 0 & \boxed{} \end{pmatrix}$$

$m_1 + m_2$

$0+1$ $0-1$ M $+1$, -1

diskr. rot. sym $\hat{R}_\alpha$ .. $\varphi_1 \to \varphi_1 + \alpha$, $\varphi_2 \to \varphi_2 + \alpha$

$L = -i\hbar\left(\frac{\partial}{\partial_1} + \frac{\partial}{\partial_2}\right)$

$[L, H_I] = [\hat{R}_\alpha, H_I] = 0$

energie $E_\pm = E_0 + \frac{\lambda}{\pi}$ (zůstává degenerovaná v důsl. zach. mom. hybn.)