QM II-3 Prostoročasové transformace

OPAKOVÁNÍ:

- grupa prostoročasových transformací $\tau \in G$
  NAPŘ.. translace $\tau = T_{\vec{a}}$ nebo rotace $R(\alpha,\beta,\gamma)$
  s operací skládání zobrazení
- reprezentace na $\mathcal{H}$: .. $\forall \tau \in G \; \exists U(\tau): \mathcal{H} \to \mathcal{H}$ unitární
  $U(\tau_1 \circ \tau_2) = U(\tau_1) U(\tau_2)$
- lieova grupa ... $\tau(\vec{s})$ ... $\vec{s}$.. souřadnice na spoj. grupě
  grupa a souč. diferenc. varieta; {např ... $\vec{s} = \vec{a}$ / $\vec{s} = (\alpha,\beta,\gamma)$}
  - hladké fce: $\vec{s}_3 = f(\vec{s}_1, \vec{s}_2)$ ... pro $\tau(\vec{s}_3) = \tau(\vec{s}_1) \circ \tau(\vec{s}_2)$
    $\vec{s}_2 = h(\vec{s}_1)$ ... $\tau(\vec{s}_2) = \tau(s_1)^{-1}$
- Stoneova věta a generátory:
  - 1-param $\tau(s) \Rightarrow \exists$ souřadnice, že $\hat{U}(s) = e^{i\hat{G}s}$
    kde $\hat{G} = \hat{G}^+$ je tzv. Generátor
  - více param; $\hat{U}(\vec{s}) = \exp\{i \sum_k s_k \hat{G}_k\}$
- lieova algebra a strukturní konstanty:

$$\boxed{[\hat{G}_\nu, \hat{G}_\mu] = i \sum_\lambda C^\lambda_{\mu\nu} \hat{G}_\lambda}$$

- $$\left[e^{i\varepsilon\hat{G}_\mu} e^{i\varepsilon\hat{G}_\nu} e^{-i\varepsilon\hat{G}_\mu} e^{-i\varepsilon\hat{G}_\nu}\right] \sim U(s) \quad \varepsilon \to 0 + \underline{O(\varepsilon^2)}$$

$$= \left[\hat{I} + i\varepsilon\hat{G}_\mu - \frac{1}{2}\varepsilon^2\hat{G}_\mu^2\right] \cdot \left[\hat{I} + i\varepsilon\hat{G}_\nu - \frac{1}{2}\varepsilon^2\hat{G}_\nu^2\right]\left[\hat{I} - i\varepsilon\hat{G}_\mu - \frac{1}{2}\varepsilon^2\hat{G}_\mu^2\right]\left[\hat{I} - i\varepsilon\hat{G}_\nu - \frac{1}{2}\varepsilon^2\hat{G}_\nu^2\right]$$

$$= \hat{I} + i\varepsilon\left[\cancel{\hat{G}_\mu} + \cancel{\hat{G}_\nu} - \cancel{\hat{G}_\mu} - \cancel{\hat{G}_\nu}\right] \qquad \binom{4}{2} = \underline{6}$$

$$+ \varepsilon^2\left[-\cancel{\hat{G}_\mu^2} - \cancel{\hat{G}_\nu^2} + \cancel{\hat{G}_\mu^2} + \cancel{\hat{G}_\nu^2} \underline{- \cancel{G_\mu G_\nu} + \cancel{G_\mu G_\nu} + G_\nu G_\mu - G_\mu G_\nu}\right]$$

$$= \hat{I} + \varepsilon^2 \underline{[\hat{G}_\nu, \hat{G}_\mu]} + O(\varepsilon^2) =$$

• $eeee = U(S) = \exp\{ i \sum_\lambda \bar{C}^\lambda_{\mu\nu} \hat{G}_\lambda \}$

$$= I + i \sum_\lambda \underbrace{\bar{C}^\lambda_{\mu\nu}}\hat{G}_\lambda + \dots \qquad c = f(\varepsilon) \sim \varepsilon^2$$

$$\bar{C}^\lambda_{\mu\nu} = \varepsilon^2 C^\lambda_{\mu\nu}$$

$$\Rightarrow \boxed{[G_\nu, G_\mu] = i \sum_\lambda C^\lambda_{\mu\nu} \hat{G}_\lambda}$$

PŘ: rotace v $\mathbb{R}^3$ $\quad (x,y,z) \equiv \vec{r}$ $\quad \vec{r}\,' = R_{\vec{n}}(\alpha)\vec{r}$

$\vec{n}_1\ \alpha_1,\ \vec{n}_2\ \alpha_2 \qquad R = R_{n_2}(\alpha_2) R_{n_1}(\alpha)$ ... reprezentace grupy rotací

$SO(3)$ ... matice 3×3; 9 .. $R^T R = R R^T = I$

$\det(R) = +1$

<u>Generátory:</u> $\hat{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = e^{i\alpha \hat{G}_x}$ $\quad$ Také by šlo derivací: $G_x = -i \frac{dR}{d\alpha}$

Taylor: $I + \alpha(-i)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{pmatrix} = I + i\alpha \hat{G}_x$

$$\hat{G}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$\hat{R}_y = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} - G_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \hat{G}_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{R}_{\vec{n}}(\alpha) = \exp\{ i\alpha \vec{n} \cdot \vec{G} \}$$

<u>Jiná reprezentace rotací (grupy $SO(3)$)</u>

otáčení stavů bezstrukt. část. ve 3D ... $\mathcal{H} \equiv L^2(\mathbb{R}^3)$

$|\vec{r}\rangle$ ... $G \equiv SO(3) = \{ \forall R_{\vec{n}}(\alpha) \}$

reprezentace na $\mathcal{H}$ ... $U(R) \equiv \hat{R}_R$

báze v $\mathcal{H}$ ... $|\vec{r}\rangle = (x,y,z)$ $\qquad \hat{R}_R |\vec{r}\rangle = |R\vec{r}\rangle$

působení na $\psi(\vec{r})$??

$$\underbrace{\hat{R}_R|\psi\rangle}_{|\psi'\rangle}=\hat{R}\int d^3r\,\psi(\vec{r})|\vec{r}\rangle=\int d^3r\,\psi(\vec{r})|\underbrace{R\vec{r}}\rangle=$$

$\vec{r}\,'=R\vec{r}$

$\vec{r}=R^{-1}\vec{r}\,'$

$$|\psi'\rangle=\int d^3r'\underbrace{\left|\frac{\partial\vec{r}}{\partial\vec{r}\,'}\right|}_{1=\det R}\psi(R^{-1}\vec{r}\,')|\vec{r}\,'\rangle \quad\longrightarrow\quad \hat{R}_R\psi(\vec{r})=\psi(R^{-1}\vec{r})$$

přezn. ve sféric. souř. $\psi(r,\theta,\varphi)$

$\vec{r}\leftrightarrow r,\theta,\varphi$ $\qquad (r,\theta,\varphi)\xrightarrow{R_z(\alpha)}(r,\theta,\varphi+\alpha)$

$$\hat{R}_{R_z}\psi(r,\theta,\varphi)=\psi(R_z^{-1}(r,\theta,\varphi))=\underbrace{\psi(r,\theta,\varphi-\alpha)}_{\text{Taylor. ..totéž co transl.}}$$

$$\hookrightarrow e^{-\alpha\frac{\partial}{\partial\varphi}}\psi(r,\theta,\varphi) \qquad \hat{L}_z=\underline{-i\hbar\partial_\varphi} \qquad (-i)^2$$

$\Rightarrow \hat{R}_{R_z}=\exp\{-\frac{i}{\hbar}\alpha\hat{L}_z\}$ ... jako transl. $p\to L$

platí: $$\boxed{\hat{R}_{\vec{n}}(\alpha)\psi(\vec{r})=\psi(\underbrace{R_{\vec{n}}^{-1}(\alpha)}_{R_{\vec{n}}(-\alpha)})=e^{-\frac{i}{\hbar}\alpha\vec{n}\cdot\vec{L}}\psi(\vec{r})}$$

rotací

komutační relace generátorů obecné reprezentace na $\mathcal{H}$

$$\underline{R_y(\varepsilon)R_x(\varepsilon)R_y(-\varepsilon)R_x(-\varepsilon)=R_z(-\varepsilon^2)}+O(\varepsilon^2)\quad\xrightarrow[\varepsilon\to0]{SO(3)}$$

$[\exp(i\alpha G_y)$ $\qquad$ } Taylor do 2. ř.

$$=\hat{I}-\varepsilon^2[\hat{G}_y,\hat{G}_x]=\hat{I}-i\varepsilon^2G_z=\exp\{-i\varepsilon G_z\}=R_z(-\varepsilon^2)$$

$$\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iG_z$$

$$\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = G_z$$

Závěry:

- Prvky $SO(3)$ splňují

$$R_y(\varepsilon)R_x(\varepsilon)R_y(-\varepsilon)R_x(-\varepsilon) = R_z(\varepsilon^2)$$

- $\Rightarrow \forall$ reprezentace $\hat{R}_x(\varepsilon)\ \hat{R}_y(\varepsilon)\ \hat{R}_z(\varepsilon)$ na $\mathcal{H}$ splňuje stejnou relaci
- $\Rightarrow$ generátory $\forall$ reprezentace $[\hat{G}_x, \hat{G}_y] = -i\hat{G}_z$

  (stejně (y,z), (z,x)) $\Rightarrow [\hat{G}_\mu, \hat{G}_\nu] = -i\varepsilon_{\mu\nu\lambda}\hat{G}_\lambda$

- zavedení přeškál. generátor $\hat{J}_\mu = -\hbar G_\mu \to G = \frac{J}{-\hbar}$

  $$\Rightarrow \boxed{[\hat{J}_\mu, \hat{J}_\nu] = i\hbar\varepsilon_{\mu\nu\lambda}\hat{J}_\lambda}$$

PŘ: částice se spinem S ... $\mathcal{H} \equiv \mathbb{C}^{2S+1}$

rotace $\vec{n}$ o α .. $\hat{R}_{\vec{n}}(\alpha) = \exp\left\{-\frac{i}{\hbar}\alpha\vec{n}\cdot\hat{\vec{S}}\right\}$

- pro $s = \frac{1}{2}$ .. $\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$ $\qquad = \exp\left\{-\frac{i}{2}\alpha\vec{n}\cdot\vec{\sigma}\right\}$

Vztah rotací a translací (komut. relace Generátorů)

Grupa translací v $\mathbb{R}^3$ $\quad \vec{r}\,' = \vec{r} + \vec{a} = T_{\vec{a}}\vec{r}$

rotací v $\mathbb{R}^3$ $\quad \vec{r}\,' = R\vec{r}$

- komutační relace

$$\underbrace{e^{\frac{i}{\hbar}\varepsilon\hat{P}_y}\, e^{\frac{i}{\hbar}\varepsilon J_x}\, e^{-\frac{i}{\hbar}\varepsilon\hat{P}_y}\, e^{-\frac{i}{\hbar}\varepsilon J_x}} \simeq \boxed{I - \frac{\varepsilon^2}{\hbar^2}[P_y, J_x]}$$

jaký prvek grupy rototranslací $\quad \vec{r}' = R\vec{r} + \vec{a} \equiv T_{\vec{a}} R \vec{r}$

$$\boxed{T_y(-\varepsilon)\, R_x(-\varepsilon)\, T_y(\varepsilon)\, R_x(\varepsilon)}\, \vec{r} \simeq \boxed{T_z(-\varepsilon^2)}\, \vec{r}$$

$$\vec{r} \xrightarrow{R_x} R_x(\varepsilon)\vec{r} \xrightarrow{T_y(\varepsilon)} R_x(\varepsilon)\vec{r} + \varepsilon\vec{e}_y \xrightarrow{R_x(-\varepsilon)}$$

$$\longrightarrow R_x(-\varepsilon)R_x(\varepsilon)\vec{r} + \varepsilon R_x(-\varepsilon)\vec{e}_y \xrightarrow{T_y(-\varepsilon)}$$

$$\vec{r} \longrightarrow \vec{r} + \underbrace{\varepsilon R_x(-\varepsilon)\vec{e}_y - \varepsilon\vec{e}_y} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \varepsilon^2$$

$$\varepsilon\,[R_x(-\varepsilon) - I]\,\vec{e}_y$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\varepsilon - 1 & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon - 1 \end{bmatrix} \simeq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & -\varepsilon & 0 \end{bmatrix} + O(\varepsilon)$$

$$\vec{r} \to \vec{r} - \varepsilon^2 \vec{e}_z \qquad T_z(-\varepsilon^2) = \exp\left\{-\frac{i}{\hbar}(-\varepsilon)^2 \hat{P}_z\right\}$$

$$\doteq I + \frac{i}{\hbar}\varepsilon^2 \hat{P}_z$$

Závěr: libovolné repr. transl. a rotací na $\mathcal{H}$:

$$[\hat{J}_x, \hat{P}_y] = i\hbar \hat{P}_z \quad \text{... cykl.} \quad x \to y \to z \to x$$

$$\boxed{[\hat{J}_\mu, \hat{P}_\nu] = i\hbar\, \varepsilon_{\mu\nu\lambda} \hat{P}_\lambda}$$