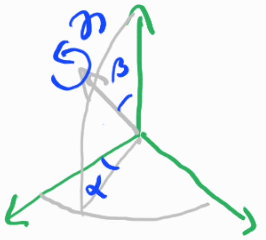
QM II-4 Wignerovy funkce, IRR rotací, tenzorové operátory

OPAKOVÁNÍ:

- grupa rotací $SO(3)$ ... ob matice $R_{\vec{n}}(\alpha)$; $R(\underline{\alpha},\underline{\beta},\gamma)$
- Eulerovy úhly:



$$R(\alpha,\beta,\gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)$$

- Reprezentace na $\mathcal{H}$:

$$\hat{R}(\alpha,\beta,\gamma) = \hat{R}_z(\alpha)\,\hat{R}_y(\beta)\,\hat{R}_z(\gamma)$$

bezstrukt. část... $\hat{R}\psi(\vec{x}) = \psi(R^{-1}\vec{x})$ .. OBECNĚ:

$$= e^{-\frac{i}{\hbar}\alpha \hat{J}_z}\, e^{-\frac{i}{\hbar}\beta \hat{J}_y}\, e^{-\frac{i}{\hbar}\gamma \hat{J}_z}$$

$\hat{J}_k$ ... celkový moment hybnosti

- Wignerovy D-funkce ... vyjádření v bázi J^2, J_z --- $|jma\rangle$

$$\langle jm|\hat{R}(\alpha,\beta,\gamma)|j'm'\rangle = \delta_{jj'}\, D^{(j)}_{mm'}(\alpha,\beta,\gamma) = \delta_{jj'}\, e^{-\frac{i}{\hbar}(\alpha m + \gamma m')}\, d^{(j)}_{mm'}(\beta)$$

"funkce malé d" $\quad d^{(j)}_{mm'}(\beta) = \langle jm|e^{-\frac{i}{\hbar}\beta\hat{J}_y}|jm'\rangle \qquad d^{(0)}_{00}(\beta) = 1 \quad j=0$

PŘ: $\quad d^{(1/2)}_{mm'}(\beta) = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \quad j=\frac{1}{2}$

$j=1$

$$d^{(1)}_{mm'}(\beta) = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}\cos\beta & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}-\frac{1}{2}\cos\beta \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}-\frac{1}{2}\cos\beta & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}+\frac{1}{2}\cos\beta \end{pmatrix}$$

DK:

$$\rightarrow S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{array}{l} m=1 \\ m=0 \\ m=-1 \end{array} \qquad S_y^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad S_y^3 = \frac{\hbar^3}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \hbar^2 S_y$$

$$\Rightarrow \exp\left\{-\frac{i}{\hbar}\beta\hat{S}_y\right\} = I + \left(\cos\frac{\hat{S}_y}{\hbar}\beta - I\right) - i\sin\frac{\hat{S}_y}{\hbar}\beta = \underline{I} + (\cos\beta - 1)\left(\frac{\hat{S}_y}{\hbar}\right)^2 - i\sin\beta\,\frac{\hat{S}_y}{\hbar}$$

OBECNÝ VZOREC: $\quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$$d^{(j)}_{m'm}(\beta) = \sum_k \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,k!\,(k+m'-m)!\,(j-k-m')!}\left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'}\left(\sin\frac{\beta}{2}\right)^{2k-m+m'}(-1)^{k-m+m'}$$

$\forall k$ aby v závorkách ≥ 0

K DŮKAZU: Schwingerův oscilátorový model (Sakurai)

2D LHO: $\quad a_x^+, a_y^+ \quad \ldots \quad \left.\begin{array}{l} J_+ = \hbar a_x^+ a_y \\ J_- = \hbar a_y^+ a_x \end{array}\right\}\ J_\pm = J_x \pm i J_y$

$$J_z = \frac{\hbar}{2}(a_x^+ a_x - a_y^+ a_y) \qquad J^2 = \sum_l J_l^2$$

$[J_k, J_l] = i\hbar\,\varepsilon_{klm} J_m \quad \Rightarrow$ moment. hybn.

-- dá se ukázat -- vl. v. $|m_x m_y\rangle$ $j = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots\}$

$\to R_y^{(\alpha)}$ da z konst. pomocí a_x^+, a_y^+ $|jm\rangle$ $\equiv N$ 0 1 2 3

$\frac{1}{\sqrt{m_x!}}(a_x^+)^{m_x}\frac{1}{\sqrt{m_y!}}(a_y^+)^{m_y}|00\rangle$ $\frac{1}{2}$ a_x^+ a_y^+

$\Rightarrow d^j_{mm'}(\beta) = \langle jm|R_y(\beta)|jm'\rangle = \dots$

- Wignerovy D-fce a sfér. harmoniky: $\langle lm|R(\alpha,\beta,\gamma)|lm'\rangle$ $Y_{lm}(\theta,\varphi)$

θ, φ $R(\varphi,\theta,0)$ j_y $D^{(l)}_{mm'}$

$\dfrac{Y(\theta,\varphi)}{Y_{lm}(0,0)}$

$Y_{lm}(\theta,\varphi)$ $\forall\ \theta,\varphi$ dostaneme rotacemi z jediné $Y_{lm}(0,0)$

nechť R $(\theta,\varphi) \xrightarrow{R} (\theta',\varphi')$ $\quad \psi(r)' = \psi(R^{-1}r)$

$(\theta',\varphi') = R(\theta,\varphi)$

$Y_{lm}(\theta',\varphi') = Y_{lm}(R(\theta,\varphi))$

$Y_{lm}(\theta',\varphi') = \hat{R}^{-1}\, Y_{lm}(\theta,\varphi)$

$|lm\rangle = \sum_{m'} \underbrace{\langle lm'|R^{-1}|lm\rangle}_{R^+}\, Y_{lm'}(\theta,\varphi)$

$= \langle lm|R|lm'\rangle^* = D^{(l)\,*}_{mm'}$

$$Y_{lm}(\theta',\varphi') = \sum_{m'} Y_{lm'}(\theta,\varphi)\, D^{(l)}_{mm'}(\alpha,\beta,\gamma)^*$$

$\theta = \varphi = 0$ $\quad (0,0) \longrightarrow (\theta',\varphi')$ $\quad \alpha = \varphi',\ \beta = \theta'$

$$Y_{lm}(\theta',\varphi') = \sum_{m'} Y_{lm'}(0,0)\, D^{(l)}_{mm'}(\varphi',\theta',0)^*$$

① najdeme φ-závislost $(\theta,0) \to (\theta,\varphi)$

$$Y_{lm}(\theta,\varphi') = \sum_{m'} Y_{lm'}(\theta,0)\, D^{(l)}_{mm'}(\varphi',0,0)^*$$

$= \langle lm|e^{-\frac{i}{\hbar}\varphi' J_z}|lm'\rangle^*$

$\delta_{mm'}\, e^{im\varphi'}$

$Y_{lm}(\theta,\varphi') = e^{im\varphi'}\, Y_{lm}(\theta,0)$ ... spojitost na pólu pro $\theta = 0$

$$\boxed{Y_{lm}(0,0) = \delta_{m0}\, c_l}$$

(2) $(0,0) \xrightarrow{R} (\theta', \varphi')$

$$\underbrace{Y_{\ell m}(\theta', \varphi')}= \sum_{m'} \delta_{m'0} c_\ell \; D^{(\ell)*}_{m m'}(\varphi', \theta', 0) = c_\ell \underbrace{D^{(\ell)}_{m0}(\varphi', \theta', 0)^*}$$

porov. s vzorci pro kulové fce $\sqrt{\frac{2\ell+1}{4\pi}}$ ... $\varphi' = \theta' = 0, m = 0$, $P_\ell(1)$

$$\boxed{Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} D^{(\ell)}_{m0}(\varphi, \theta, 0)}$$

$\sqrt{\frac{1}{4\pi}}$ $\sqrt{\frac{3}{4\pi}}$ $\sqrt{\frac{5}{4\pi}}$

- Wigner D-fce jako ON báze $L^2\{(\alpha, \beta, \gamma)\}$ tj. $\alpha, \gamma \in (0, 2\pi\rangle$, $\beta \in (0, \pi\rangle$

$$\text{tj. } \forall f(\alpha, \beta, \gamma) = \sum_{\ell, m, m'} c^{(\ell)}_{m m'} D^{(\ell)}_{m m'}(\alpha, \beta, \gamma)$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} D^{j'}_{m' k'}(\alpha, \beta, \gamma)^* D^{j}_{m k}(\alpha, \beta, \gamma)\, d\alpha \sin\beta\, d\beta\, d\gamma = \frac{8\pi^2}{2j+1} \delta_{m m'} \delta_{k k'} \delta_{j j'}$$

- Wignerovy D-fce jako stacionární stavy rotujícího tělesa (QM)

- kvant. rotor. $\mathcal{H}$ $L^2(S_3)$ $\leftarrow$ $\underbrace{\psi(\theta, \varphi)}$ (θ, φ) (n_x, n_y, n_z)

$$H = \frac{\hat{L}^2}{2I} \quad \ldots \quad E_\ell = \frac{\hbar^2 \ell(\ell+1)}{2I}$$

I moment setrvačnosti ... $Y_{\ell m}(\theta, \varphi)$

.. $(2\ell+1)\times$ degener. $m = -\ell, -\ell+1, \ldots, \ell$

- symetrický setrvačník (symetric top)

-- tenzor setrvačnosti

$\to$ vlastní směry -- hodnot

(Rotace molekul)

$I_1 = I_2$; I_3

α, β, γ

$$\boxed{\hat{H} = \frac{1}{2}\left[\frac{\hat{P}_1^2}{I_1} + \frac{\hat{P}_2^2}{I_2} + \frac{\hat{P}_3^2}{I_3}\right]}$$

$\hat{P}_k$ je moment hybnosti v s.s. tělesa $\hat{P}^2, \hat{P}_3$,

body-fixed ang. moment.

symetrický případ $\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2 = J^2 = P^2 = P_1^2 + P_2^2 + P_3^2$

$$\frac{1}{I_1}(P_1^2 + P_2^2) = \frac{1}{I_1}(P^2 - P_3^2)$$

$$\boxed{\hat{H} = \frac{1}{2}\left[\frac{P^2}{I_1} + P_3^2\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\right]}$$

D-fce jako vl. fce momentů hybn.

lab. systém: (space-fixed)

$$\hat{J}_1 = i\hbar\left(\cos\alpha \frac{\cos\beta}{\sin\beta}\partial_\alpha + \sin\alpha\, \partial_\beta - \frac{\cos\alpha}{\sin\beta}\partial_\gamma\right)$$

$$\hat{J}_2 = i\hbar\left(\sin\alpha \frac{\cos\beta}{\sin\beta}\partial_\alpha - \cos\alpha\, \partial_\beta - \frac{\sin\alpha}{\sin\beta}\partial_\gamma\right) \qquad \alpha, \beta, \gamma$$

$$\hat{J}_3 = -i\hbar\, \partial_\alpha$$

v soust. tělesa (body-fixed)

$$\hat{P}_1 = i\hbar\left(\frac{\cos\gamma}{\cos\beta}\partial_\alpha - \sin\gamma\,\partial_\beta - \frac{\cos\beta}{\sin\beta}\cos\gamma\,\partial_\gamma\right)$$

$$\hat{P}_2 = i\hbar\left(-\frac{\sin\gamma}{\sin\beta}\partial_\alpha - \cos\gamma\,\partial_\beta + \frac{\cos\beta}{\sin\beta}\sin\gamma\,\partial_\gamma\right)$$

$$\hat{P}_3 = -i\hbar\,\partial_\gamma$$

$$[\hat{J}_k, \hat{J}_l] = i\hbar\varepsilon_{klm}\hat{J}_m$$

posun. oper. $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$ $\quad \mathcal{P}_\pm = \mathcal{P}_1 \mp \mathcal{P}_2$ $\quad [\hat{P}_k, \hat{P}_l] = -i\hbar\varepsilon_{klm}\hat{P}_m$

Platí že $D^{*j}_{mm'}(\alpha,\beta,\gamma)$ jsou vl. fce $\underbrace{\hat{J}_z}_{\hbar m}, \underbrace{J^2 = P^2}_{\hbar^2 j(j+1)}, \underbrace{\hat{P}_z}_{\hbar m'}$

posun oper.. $J_\pm$ ... $m \to m\pm1$
$P_\pm$ ... $m' \to m'\pm1$ $\Big\}$ ... $\sqrt{(j-m)(j+m+1)}$

tj $D^{*j}_{mK}(\alpha,\beta,\gamma)$ jsou stac. stavy Q-sym. setrvačníku

$$H = \frac{1}{2}\left[\frac{P^2}{I_1} + P_z^2\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\right] \quad \dots \quad E_{jK} = \frac{\hbar^2 j(j+1)}{2I_1} + k^2\hbar^2\frac{I_1 - I_3}{2I_1 I_3}$$

↳ ∀ hodn je $(2j+1)\times$ degener.
$m = -j, -j+1, \dots, j$

$j = 0, 1, 2, 3, \dots \quad k = -j, -j+1, \dots, j$

$\boxed{\text{část II}}$

- Wignerovy D-funkce jak IRR grupy rotací $G = SO(3)$

ODBOČKA .. doplň teorie repr. grup

- reprezentace grupy $R_1, R_2 \in G$ $\quad R_3 = R_1 R_2$
↳ ∀R .. $\hat{U}(R)$ lin. oper na $\mathcal{H}$ $\quad U(R_1)U(R_2) = U(R_3)$
- ekvivalentní reprezentace $U(R)$ ... $\bar{U}(R)$ ∀ $R \in G$
↳ pokud ∃ unit. operátor $V^+ = V^{-1}$: $\bar{U}(R) = V^+ U(R) V$
stejný ∀ R

Reducibilní reprezentace: ∀ $U(R)$ (tj ∀ $R \in G$)
∃ báze

$$U(R) = \begin{pmatrix} |||| & 0 \\ 0 & |||| \end{pmatrix} \begin{matrix} \}\mathcal{H}_1 \\ \}\mathcal{H}_2 \end{matrix} \qquad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

$$= \left(\begin{array}{c|c} U_1(R) & 0 \\ \hline 0 & U_2(R) \end{array}\right) \quad \ldots \quad \text{2 reprezentace} \quad \begin{array}{c|c} U_1(R) & U_2(R) \\ \mathcal{H}_1 & \mathcal{H}_2 \end{array}$$

Součet reprezentací $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$

$$U(R) = \left(\begin{array}{c|c} U_1 & \\ \hline & U_2 \end{array}\right)$$

- Součin reprezentací: $U_1(R)$ na $\mathcal{H}_1$ $\quad U_2(R)$ na $\mathcal{H}_2$

$$U(R) = U_1(R) \otimes U_2(R) \;\ldots\; \text{na } \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}$$

- Invariantní podprostor $\mathcal{H}_i \subset \mathcal{H}$ .. $\hat{U}(R)$
(reprezentace grupy G)

$$\to \quad \forall |\psi\rangle \in \mathcal{H}_i \; \forall R : \hat{U}(R)|\psi\rangle \subset \mathcal{H}_i$$

reducibilita:

$$\to \left(\begin{array}{c|c} U_1(R) & 0 \\ \hline 0 & U_2(R) \end{array}\right) \Leftrightarrow \begin{array}{l} \} \; \mathcal{H}_i \\ \} \; \mathcal{H}_{i'} \end{array}$$

invar. podprost.

$\mathcal{H}_i \oplus \mathcal{H}_{i'}$

IRR ≡ ireducibilní reprezentace grupy G

taková, která není reducibilní ... tj. neexistuje unit. V

taková $\forall R$: $U(R) = \left(\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right)$, neexistuje netriviální invariantní podprost.

TVRZENÍ: $\forall$ ireducibilní reprezentace $SO(3)$ je unitárně ekvivalentní nějakému $\left| D^{(j)}_{mm'}(\alpha,\beta,\gamma) \right|$

- $\mathcal{H} \cong \mathbb{C}^{(2j+1)}$ $\quad \big|\;\big|$ matice $(2j+1)\times(2j+1) \longrightarrow \hat{U}^{(j)}(R)$ $\leftarrow$ indentif. o kterou IRR

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$R(\alpha,\beta,\gamma)$

$\Rightarrow \exists$ báze

$$\hat{Q}(R) = \begin{pmatrix} D & & 0 \\ & D^{(j_1)} & \\ 0 & & D^{(j_2)} \end{pmatrix}$$

$$J_z = \left(\diagdown \right)^{2j+1}$$

$J_x \; J_y \qquad J_\pm$

• př.: částice spin 1/2 .. $\mathcal{R}$ je IRR (+) .. $D^{(1/2)}$

- 2 částice spin 1/2 .. 4dim. $\mathcal{H}$

unit. transf. → do kaplované báze → $|jm\rangle = |00\rangle$; $|jm\rangle \begin{cases} |1-1\rangle \\ |10\rangle \\ |1+1\rangle \end{cases}$

$$\to \hat{\mathcal{R}} = e^{J_z} e^{J_y} e^{J_z}$$

$$\begin{pmatrix} D^{(0)} & 0 \\ 0 & D^{(1)} \end{pmatrix}$$

součin IRR .. $\boxed{D^{1/2} \otimes D^{1/2} = D^{(0)} \oplus D^{(1)}}$ → součet IRR $j=0$; $j=1$

~ částice bezstrukt. ve 3D $\mathcal{R}$ .. $e^{iL_z} e^{iL_y} e^{iL_z}$

báze L^2, L_z

$$\underline{\mathcal{R}} = \begin{pmatrix} 1 & & 0 \\ & D^{(1)} & \\ 0 & & \ddots \end{pmatrix} \begin{matrix} \} \ l=0 \\ \} \ l=1 \\ \} \ l=2 \end{matrix}$$

• **Clebsch-Gordanův rozvoj součinu D-fcí**

j_1 j_2 $\quad \mathcal{H} = \mathcal{H}^{(j_1)} \otimes \mathcal{H}^{(j_2)}$ $\quad D = (2j_1+1)(2j_2+1)$ $\quad$ matice $D \times D$

• $\hat{\mathcal{R}}(\alpha,\beta,\gamma) = \mathcal{R}^{(j_1)} \otimes \mathcal{R}^{(j_2)}$

.. v bázi $|j_1 m_1\rangle, |j_2 m_2\rangle$ .. → $D^{(j_1)}_{m_1 m_1'} D^{(j_2)}_{m_2 m_2'} = M_{m_1 m_2, m_1' m_2'}$

$$D^{(j_1)} \otimes D^{(j_2)} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^{(j)}$$

j_{min} ... j_{max}

$$\boxed{D^{(j_1)}_{m_1 m_1'}(R)\, D^{(j_2)}_{m_2 m_2'}(R) = \sum_j \sum_{m m'} \langle j_1 j_2 m_1 m_2 | j m\rangle \langle j_1 j_2 m_1' m_2' | j m'\rangle D^{(j)}_{m m'}(R)}$$

$\alpha\beta\gamma$

DK: $\hat{\mathcal{R}}^{(j_1)}(R) \otimes \hat{\mathcal{R}}^{(j_2)}(R) = \hat{\mathcal{R}}(R)$ $\quad |j_1 m_1'\rangle |j_2 m_2'\rangle$

$\langle j_1 m_1| \langle j_2 m_2| \hat{\mathcal{R}}(R) |j_1 m_1'\rangle |j_2 m_2'\rangle$

C-G koef. $\langle j_1 j_2 m_1 m_2 | j m\rangle$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |jm\rangle \underbrace{\langle jm | j_1 m_1\rangle |j_2 m_2\rangle}$$

$$\sum_{jm} \sum_{\cancel{j'}m'} \langle j_1 j_2 m_1 m_2 | jm\rangle \underbrace{\langle jm | R | j'm'\rangle}_{\cancel{\delta_{jj'}}\, D^{(j)}_{mm'}(R)} \langle j_1 j_2 m_1' m_2' | j'm'\rangle$$

Praktická aplikace: často potřebujeme $\langle \overset{Y}{\psi} | \overset{Y}{H} | \overset{Y}{\psi} \rangle$

$\psi(\vec r_1, \vec r_2)$

$$\int Y^*_{\cdot}\, \underbrace{Y\, Y}_{\sum C Y}\, d\Omega = ??$$

$$\int Y^*_{lm} Y_{l'm'} d\Omega \sim \delta_{ll'}\delta_{mm'} \qquad Y_{l_1m_1}(r_1)\, Y_{l_2m_2}(r_2)$$

Gauntova formule:

$$\int d\Omega\, Y_{lm}(\theta,\varphi)^* Y_{l_1m_1}(\theta,\varphi) Y_{l_2m_2}(\theta,\varphi) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi\,(2l+1)}}\, \langle l_1 l_2 0 0 | l 0\rangle \langle l_1 l_2 m_1 m_2 | l m\rangle$$

DK: $\sqrt{\frac{2l_1+1}{4\pi}}\, D^{(l_1)}_{m_1 0}(\varphi,\theta,0)^* \sqrt{\frac{2l_2+1}{4\pi}}\, D^{(l_2)}_{m_2 0}(\varphi,\theta,0)^*$ $\quad |l_1-l_2| \le l \le l_1+l_2$

m, $m' = 0 \rightarrow D^{(l)*}_{m0} \sim Y_{lm}$

$$\sum \int Y_{lm}^* Y_{lm}\, d\Omega$$

• VEKTOROVÉ A TENZOROVÉ OPERÁTORY V QM

v klas. mech.. $\vec r = (x,y,z)$ -- rotace ... $\vec r' = R\vec r$
(x_1,x_2,x_3) $\qquad x'_k = \sum_l R_{kl} x_l$

vektor veličiny (v_1,v_2,v_3) $\qquad v'_k = \sum_l R_{kl} v_l$

v QM: .. vekt.. $(\hat v_1,\hat v_2,\hat v_3)$ .. $|\psi\rangle$ $\qquad v_k = \langle\psi|\hat V_k|\psi\rangle$
(v_1,v_2,v_k) $\qquad v'_k = \sum_l R_{kl} v_l$

důsledky: $v_k = \langle\psi|\hat V_k|\psi\rangle \qquad |\psi'\rangle = \hat R(R)|\psi\rangle$

$$v'_k = \langle\psi'|\hat V_k|\psi'\rangle = \langle\psi|\hat R^+ \hat V_k \hat R|\psi\rangle \quad \forall |\psi\rangle\ \forall R$$

$$v'_k = \sum_l R_{kl} v_l$$

$$\langle\psi|\hat R^+ \hat V_k \hat R|\psi\rangle = \sum_l R_{kl} \langle\psi|\hat V_k|\psi\rangle$$

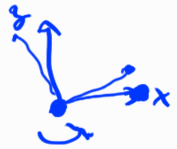
$$\Rightarrow \forall R: \boxed{\hat{R}^+(R)\,\hat{V}_k\,\hat{R}(R) = \sum_\ell R_{k\ell}\,\hat{V}_\ell} \leftarrow (*)$$

Def: Řekneme, že veličina $\hat{V}_k$ je vektor pokud $\forall R$ platí $(*)$

speciálně jen malé rotace kolem osy $\vec{n}$ $\hookrightarrow \vec{e}_z$, $\quad R_{\vec{n}}(\alpha)$ .. malý úhel $\alpha = \varepsilon$

$$\hat{R} = \exp\{-\tfrac{i}{\hbar}\varepsilon\,\vec{n}\cdot\vec{J}\} \sim I - \tfrac{i}{\hbar}\varepsilon\,\vec{n}\cdot\vec{J}$$

$$R_z(\varepsilon) = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon & 0 \\ \sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$



$$(*) \quad \left(\hat{I} + \tfrac{i}{\hbar}\varepsilon\hat{J}_z\right) V_k \left(\hat{I} - \tfrac{i}{\hbar}\varepsilon\hat{J}_z\right) = \sum_\ell R_{k\ell}V_\ell =$$

$k = x \qquad V_x + \frac{i}{\hbar}\varepsilon\left(J_zV_x - V_xJ_z\right) = V_x - \varepsilon V_y + O(\varepsilon^2)$

$k = y \qquad V_y + \frac{i}{\hbar}\varepsilon\left(J_zV_y - V_yJ_z\right) = \varepsilon V_x + V_y + O(\varepsilon^2)$

$k = z \qquad V_z + \frac{i}{\hbar}\varepsilon\left(J_zV_z - V_zJ_z\right) = V_z + O(\varepsilon^2)$

člen v ε^1: def $\boxed{[\hat{J}_k, \hat{V}_\ell] = i\hbar\,\varepsilon_{k\ell m}\hat{V}_m}$

důsl. J je vektor $V = J$

$\hat{x}_k, \hat{p}_\ell \quad \hat{L}_k$

Jednodušší definice vektorové veličiny

Dá se DK, že obě def jsou ekvivalentní

Baker-Kambel-Hausdorff lemma:

$$\underbrace{e^{\frac{i}{\hbar}\phi\hat{J}}\,\hat{V}\,e^{-\frac{i}{\hbar}\phi\hat{J}}}_{\hat{V}(\phi)} = \hat{V} + \frac{1}{1!}\left[\tfrac{i}{\hbar}\phi\hat{J}, \hat{V}\right] + \frac{1}{2!}\left[\tfrac{i}{\hbar}\phi\hat{J}, \left[\tfrac{i}{\hbar}\phi\hat{J}, V\right]\right] + \dots$$

$$\partial_\phi V = [iJ, V]$$

$$\phi\left[\tfrac{i}{\hbar}\hat{J}, \underset{V}{\bullet}\right] \quad \text{superoperátor} \quad \text{oper} \xrightarrow{S} \text{oper}$$

Tenzorové veličiny: klasicky $T'_{ijk\ldots} = \sum_{i'j'k'\ldots} R_{ii'}R_{jj'}\cdots T_{i'j'k'}$

$\forall\psi\ \forall R$ $\langle\psi'|T_{ijk}|\psi'\rangle$ transf $\longrightarrow$ def. tenzorové veličiny

$$\boxed{\hat{R}^+(R)\,\hat{T}_{ijk\ldots}\,\hat{R}(R) = \sum_{i'j'k'\ldots} R_{ii'}R_{jj'}R_{kk'}\ldots T_{i'j'k'}}$$