$\boxed{\text{QMII-6}}$ Identické částice – formalismus II. kvantování

OPAKOVÁNÍ

$$\mathcal{H}^{(N)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)} \otimes \dots \otimes \mathcal{H}^{(1)} = \mathcal{H}^{(S)} \oplus \mathcal{H}^{(A)} \oplus \dots \text{ ostatní invar. pr.}$$

výměnná symetrie $\quad P_{ij} |m_1 \dots m_i \dots m_j \dots\rangle = |m_1 \dots m_j \dots m_i \dots\rangle$

$\forall i j \quad P_{ij} |sym\rangle = |sym\rangle \quad \dots$ Bosony $\quad$ celý spin

$\qquad P_{ij} |asym\rangle = -|asym\rangle \quad \dots$ Fermiony $\quad$ polospin

- operátory: $[\hat{A}, P_{ij}] = 0 \longrightarrow P_{ij} \hat{A} P_{ij} = \hat{A}$
- projektory: $\mathcal{H} \to \mathcal{H}^{(S)} \quad \dots \quad \hat{S} = \frac{1}{N!} \sum_n \hat{P}_n \qquad S P_{ij} = P_{ij} S = S \qquad S = S^2 = S^+$

  $\mathcal{H} \to \mathcal{H}^{(A)} \quad \dots \quad \hat{A} = \frac{1}{N!} \sum_n (-1)^n \hat{P}_n \qquad A P_{ij} = P_{ij} A = -A \qquad A = A^2 = A^+$

$\hat{A}\, \phi_1(x_1) \phi_2(x_2) \dots \phi_n(x_n) =$ Slater determinant

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- **Reprezentace obsazovacích čísel** $\quad \dots \mathcal{H}^{(S)} \subset \mathcal{H}^{(N)}$

$\mathcal{H}^{(N)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)} \otimes \dots \otimes \mathcal{H}^{(1)} \qquad \dots \mathcal{H}^{(1)} \dots \quad \hat{C}|m\rangle = c_m |m\rangle$

$\hookrightarrow \{|m_1 m_2 \dots m_N\rangle\}_{\vec{m}} \quad \dots$ báze v $\mathcal{H}^{(N)} \qquad \{|m\rangle\}_{m=1}^{\infty} \dots$ báze

$\hat{I} = \sum_{m_1} \sum_{m_2} \dots |m_1 m_2 \dots m_N\rangle\langle m_1 m_2 \dots m_N| \qquad$ nejvýš N_m

báze v $\mathcal{H}^{(S)} \quad \dots \quad \hat{S} |m_1 m_2 \dots m_N\rangle \qquad \vec{m} = (m_1, m_2 \dots m_N)$

opakování $\qquad$ např $\hat{S}|m_1 m_2 \dots m_N\rangle = \hat{S}|m_2 m_1 \dots m_N\rangle$

Def: OBSAZOVACÍ ČÍSLA $\dots$ pro stav $|m\rangle$ def obs. číslo N_m

$\hookrightarrow$ pro stav $\hat{S}|m_1 m_2 \dots m_N\rangle \quad \dots \quad N_m =$ počet výskytů čísla $m \in \{m_1, m_2, \dots m_N\}$

* př: $N=5 \qquad |\underbrace{m_1 m_2 m_3 m_4 m_5}\rangle = |1\,1\,3\,3\,1\,6\rangle$

$N_1 \quad m=1$ v $\dots$ 2 $\quad \dots \quad N_1 = 2;\ N_2 = 0;\ N_3 = 2,\ N_4 = N_5 = 0$

$N_6 = 1,\ N_k = 0 \quad k > 6$

platí $\boxed{N = \sum_k N_k}$

platí, že $\quad |m_1 m_2 \dots m_N\rangle$ a $|m_1' m_2' \dots m_N'\rangle$ mají stejná N_k

pak $\quad S|m_1 m_2 \dots m_N\rangle = S|m_1' m_2' \dots m_N'\rangle$

def bázi pomocí N_k: $\quad {}_{\mathcal{H}^{(S)}} |N_1 N_2 \dots\rangle_N = \eta\, \hat{S} |m_1 m_2 \dots m_N\rangle_n \qquad x^N$

η … normalizace

* $|2\,0\,2\,0\,0\,1\,0\,0 \dots\rangle_N = \eta\, \underline{\hat{S}}\, |1\,1\,3\,3\,6\rangle$

$$n = \sqrt{\frac{N!}{N_1!\,N_2!\cdots}}$$

$$\frac{1}{N!}\left(|\alpha\beta\gamma\rangle + |\beta\alpha\gamma\rangle \cdots \right)$$

$$\left[|\alpha\alpha\gamma\rangle + |\alpha\alpha\gamma\rangle\right] + (\quad) + (\quad)$$

$$2\,|\alpha\alpha\gamma\rangle \qquad N!/N_1! \cdot N!$$

$$\hat{S}\,|m_1 \cdots m_N\rangle = \frac{1}{N!}\left(\frac{N!}{N_1!\,N_2!\cdots}\ \text{různých členů}\right)(N_1!\,N_2!\cdots)$$

správný norm. faktor $\boxed{\sqrt{\frac{N_1!\cdots N_2!}{N!}}}$

def. báze obsaz. čísel

$$\boxed{|N_1 N_2 \cdots\rangle_N = \sqrt{\frac{N!}{N_1!\,N_2!\cdots}}\ \hat{S}\,|m_1 m_2 \cdots m_N\rangle_m}$$

Rozklad $\hat{I}$ na $\mathcal{H}^{(S)}$ .. $\hat{I}_S \equiv \hat{S} = \sum_{\substack{N_1=0 \\ \sum N_k = N}}^{N} \sum_{N_2=0}^{N} \cdots |N_1 N_2 \cdots\rangle_N \langle N_1 N_2 \cdots|_N$

FERMIONY: .. stejně $\mathcal{H}^{(S)} \to \mathcal{H}^{(A)}$ $\quad \hat{S} \to \hat{A}$

$N_k \in \{0,1\}$ protože $\hat{A}\,|m_1 m_1\rangle = \frac{1}{2}\left(|m_1 m_1\rangle - |m_1 m_1\rangle\right) = 0$

$$\boxed{|N_1 N_2 \cdots\rangle_N = \sqrt{N!}\ \hat{A}\,|m_1 m_2 \cdots m_N\rangle_m}$$

pozn.: na $\mathcal{H}^{(N)}$ $\quad (\cdot)$ $\quad \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} |m_1 m_2 \cdots m_N\rangle\langle m_1 m_2 \cdots m_N| = I_{\mathcal{H}^{(N)}}$

$\hat{A}\,(\cdot)\,\hat{A}$

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} \underbrace{A\,|m_1 m_2 \cdots m_N\rangle}_{\frac{1}{\sqrt{N!}}|N_1 N_2 \cdots\rangle}\langle m_1 m_2 \cdots m_N|\,A = A^2 = A$$

$$N! \sum_{\substack{N_1=0 \\ N = \sum_k N_k}} \sum_{N_2=0} \cdots \ \frac{1}{\sqrt{N!}}\,|N_1 N_2 \cdots\rangle\langle N_1 N_2 \cdots|\,\frac{1}{\sqrt{N!}}$$

$$\cdots \quad \hat{I}_{\mathcal{H}^{(A)}} = \hat{A} = \sum_{\substack{\{N_1, N_2, N_3 \cdots\} \\ \sum_k N_k = N}} |N_1 N_2 \cdots\rangle\langle N_1 N_2 \cdots|$$

• Operátory ve Fockově prostoru — Fock space

Фок

opakování:

direkt. součet: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ — $|\psi\rangle \in \mathcal{H}$ — $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

$|\psi\rangle = (|\psi_1\rangle, |\psi_2\rangle)$ — $|\psi_1\rangle \in \mathcal{H}_1$, $|\psi_2\rangle \in \mathcal{H}_2$

$(|\psi_1\rangle, 0) \in \mathcal{H}_1 \subset \mathcal{H}$

Def: $\mathcal{H}_F = \bigoplus_{N=0}^{\infty} \mathcal{H}_S^{(N)}$ bosony $= \bigoplus_{N=0}^{\infty} \mathcal{H}_A^{(N)}$ .. fermiony — $\boxed{\text{Fock space}}$

$= \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}_S^{(2)} \oplus \mathcal{H}_S^{(3)} \oplus \dots$

$\boxed{\text{① BOSONY}}$

$\mathcal{H}^{(0)}$ ... $|0\rangle$ vakuum ~ žádná částice

$\mathcal{H}_S^{(N)}$ ~ báze $|N_1 N_2 \dots\rangle$ — $N_1 + N_2 + \dots = N$, $N_k = 0, 1, 2, \dots$

Def: kreační operátor $\hat{a}_m^+$ působí $|\psi\rangle \in \mathcal{H}_S^{(N)}$

$\hat{a}_m^+|0\rangle = |m\rangle$ přidá částici do stavu $|m\rangle$ — $\hat{a}_m^+|\psi\rangle \in \mathcal{H}_S^{(N+1)}$

$\hat{a}_m^+ |N_1 \dots N_m \dots\rangle = \sqrt{N_m+1}\,|N_1 \dots (N_m+1) \dots\rangle$ .. → $\begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & 0 & \sqrt{3} \\ & & & \ddots \end{pmatrix}$ — $(a^+)^+$

• anihilační operátor $\hat{a}_m = (\hat{a}_m^+)^+$

$\hat{a}_m |N_1 \dots N_m \dots\rangle = \sqrt{N_m}\,|N_1 \dots (N_m-1) \dots\rangle$

pokud $N_m = 0$ $\to = 0$ — $\hat{a}_m|N_1 \dots 0 \dots\rangle = 0 \quad (\neq |0\rangle)$

• vyjádření bát. vektoru: $\boxed{|N_1 N_2 \dots\rangle = \frac{1}{\sqrt{N_1!}}(\hat{a}_1^+)^{N_1} \frac{1}{\sqrt{N_2!}}(\hat{a}_2^+)^{N_2} \dots |0\rangle}$

$N = \sum_k N_k$

alternativa $|N_1 N_2 \dots\rangle = \sqrt{\frac{N!}{N_1! N_2! \dots}}\,\hat{S}|m_1 m_2 \dots m_N\rangle$

$$\boxed{\hat{S}|m_1\rangle|m_2\rangle \dots |m_N\rangle = \frac{1}{\sqrt{N!}}\, a_{m_1}^+ a_{m_2}^+ \dots \hat{a}_{m_N}^+ |0\rangle}$$

• komutační relace: $[\hat{a}_i^+, \hat{a}_j^+] = [\hat{a}_i, \hat{a}_j] = 0$. $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}\hat{I}$.

Dk: $i \neq j$ — $a_i a_j^+ |\dots N_i \dots N_j \dots\rangle = \sqrt{N_i}\sqrt{N_j+1}\,|\dots N_i-1 \dots N_j+1 \dots\rangle$

$i = j$ — N_i+1 — $a_i a_i^+ |N_1 \dots N_i \dots\rangle = \sqrt{N_i+1}\sqrt{N_i+1}\,|N_1 \dots N_i \dots\rangle$

→ N_i — $a_i^+ a_i |N_1 \dots N_i \dots\rangle = \sqrt{N_i-1+1}\sqrt{N_i}\,|N_1 \dots N_i \dots\rangle$

- Operátor počtu částic: $\hat{N}_k = \hat{a}_k^+ \hat{a}_k$ počet částic ve stavu k

  → celk. počet částic $\hat{N} = \sum_k \hat{a}_k^+ \hat{a}_k$

$\Rightarrow \quad [\hat{N}_k, \hat{a}_l^+] = \delta_{kl} \hat{a}_l^+ \qquad [\hat{N}_k, \hat{a}_l] = -\delta_{kl} \hat{a}_l$

ZÁMĚNA BÁZE: vycházíme $\hat{C}|n\rangle = c_n|n\rangle$ ... $\hat{B}|b:n\rangle = b_n|b:n\rangle$

v $\mathcal{H}^{(1)}$

$|c:n\rangle = \hat{a}_n^+(c)|0\rangle$ ... $|b:n\rangle = \hat{a}_n^+(b)|0\rangle$

$\rightarrow \quad |b:n\rangle = \sum_m |c:m\rangle\langle c:m|b:n\rangle \qquad \sum_n \langle c:m|b:n\rangle\langle b:n|c:m'\rangle = \delta_{mm'}$

$\hat{a}_n^+(b) = \sum_m \hat{a}_m^+(c)\langle c:m|b:n\rangle \leftarrow c \leftrightarrow b$

$\hookrightarrow \quad a_n^+(c) = \sum_m \hat{a}_m^+(b)\langle b:m|c:n\rangle$

+ anihilační $\quad \hat{a}_n(b) = \sum_m a_m(c)\langle b:n|c:m\rangle$

$\Rightarrow \quad [\hat{a}_m(b), \hat{a}_n^+(b)] = \delta_{mn}$

Spojitá báze: $\langle x|c:n\rangle \equiv \phi_n(x) \qquad \hat{B} \rightarrow \hat{X}$

$\hat{a}_x^+ \longrightarrow \hat{\psi}^+(x)$

kreuje 1 částice do místa x ... $\hat{\psi}^+(x)|0\rangle = |x\rangle$

$$\boxed{\hat{\psi}^+(x) = \sum_n \hat{a}_n^+\langle n|x\rangle = \sum_n \phi_n(x)^* \hat{a}_n^+}$$

$$\boxed{\hat{\psi}(x) = \sum_n \phi_n(x)\,\hat{a}_n}$$

Field operators