Black Hole Thermodynamics: Classical and Quantum Aspects of (Curved Space-Time) Quantum Field Theory

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19$^{\text{th}}$–26$^{\text{th}}$ March 2021

Introducing Me

- If you have questions & feedback: `sebastian.schuster@utf.mff.cuni.cz`
- Let me know if it's too fast—I can't see y'all, nor when you finish note taking
- **Warning!**

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- Much of this lecture had to be written half-blind—tell me if you see errors!

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Suggestions/Information for the "Seminar"—Work in Progress!

- At the end of the "Analogue" part of the lecture, we will do a poll
- There, we will decide the paper for the "seminar" part
- The one we choose, we will read before the "seminar"
- We will then have a guided discussion on it
- **Feel free to suggest papers you would like to discuss together**

Outline

Conventions/Warnings

- Signature: $-+++$
- Space-time indices: *Should* be Greek.[2]
- Spatial indices: $ijkl\ldots$
- In green, suggestions for exercises[3]

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- This list is probably incomplete and might grow during the next weeks.

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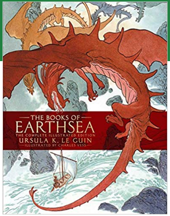

- **"Easiest":** Jacobson arXiv:1212.6821, arXiv:gr-qc/0308048 & these lectures; *Fabbri & Navarro-Salas—Modeling Black Hole Evaporation*; Frolov & Zelnikov—Intro. to [. . .]; Fewster, lecture notes; Traschen, arXiv:gr-qc/0010055

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- **Classics:** Hawking, DOI:10.1007/BF02345020; deWitt, DOI:10.1016/0370-1573(75)90051-4; Unruh, DOI:10.1103/PhysRevD.14.870; Unruh & Weiss, DOI:10.1103/PhysRevD.29.1656

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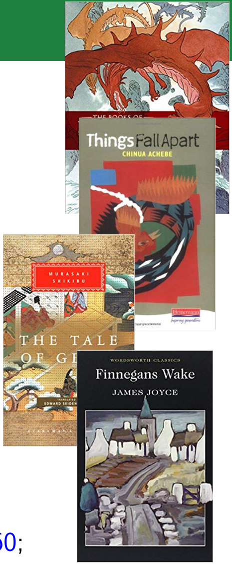

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- Here be dragons!: *Wald—QFT in CST and BH TD*; *Bär & Fredenhagen—QFT in CST*; Fewster, Pfeifer, Siemssen, arXiv:1709.01760; Brunetti, Dappiaggi, Fredenhagen, Yngvason—Advances in Algebraic QFT

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Finally: Shameless ~~Self~~ Promotion

Úvod do kvantové teorie pole na křivém pozadí

NTMF065, Pavel Krtouš

Pokročilé partie kvantové teorie pole na křivém pozadí

NTMF095, Andrei Zelnikov & Pavel Krtouš

Quantum Field Theory Gone Flat

Reminder: QFT in Minkowski Space

- Time and space completely on equal footing, unlike in non-relativistic QM
- Use simplest choice for space-time: $\mathbb{R}^{(3,1)}$, $(-+++)$
- Ladder operators: "Creation" and "annihilation" operators,

$$\hat{a}^\dagger_{\vec{k}} \left|\ldots; N_{\vec{k}}; \ldots\right\rangle = \sqrt{N_{\vec{k}} + 1} \left|\ldots; N_{\vec{k}} + 1; \ldots\right\rangle,$$
$$\hat{a}_{\vec{k}} \left|\ldots; N_{\vec{k}}; \ldots\right\rangle = \sqrt{N_{\vec{k}}} \left|\ldots; N_{\vec{k}} - 1; \ldots\right\rangle$$

- Vacuum $|0\rangle$ characterized as Lorentz-invariant state of "no particles", i.e. for ladder operators

$$\hat{a}_{\vec{k}} \left|0\right\rangle = 0$$

Summary

$\underbrace{\text{Successful}}_{[\ldots]}$ $\underbrace{\text{relativistic}}_{\textit{Special}\text{ Relativity}}$ $\underbrace{\text{many-particle quantum physics}}_{\text{Diff. Fock spaces for diff. particles}}$ $\underbrace{\text{in inertial frames}}_{\text{SR, i.e. Lorentz frames}}$

From Ladder Operators to Fock Spaces

- Assume you have a vacuum state.
- Define a one particle state $|\dots i \dots\rangle$ as

$$|\dots i \dots\rangle := \hat{a}_i^\dagger |0\rangle$$

- Inductively build possible states out of this
- The span of all such states is the corresponding **Fock space**
- 𝔚𝔞𝔯𝔫𝔦𝔫𝔤!: Mathematically there are issues with the whole concept, especially in interacting QFT.[5]

[5] See, for example, Strocchi—An Introduction to Non-Perturbative Foundations of Quantum Field Theory.

Particle Content of a State

- Expectation values of operators $\hat{A}$ for states $|\psi\rangle$ calculated as in QM:

$$\langle \hat{A} \rangle_\psi := \langle \psi | \hat{A} | \psi \rangle$$

- For us most important: How many particles of type i are there?

$$\langle \hat{N}_i \rangle_\psi = \langle \psi | \hat{a}_i^\dagger \hat{a}_i | \psi \rangle$$

- Remind yourselves how normalization of (multi-)particle states works; play with (different) ladder operators acting on states; what is a coherent state?

The Crux

- Notice the importance of Lorentz invariance
- Depending on the formalism, *showing* Lorentz invariance already is hard work
- In its absence, the construction of a unique vacuum already is impossible
- Hint: Remember how Ana constructed new time-like KVF from a space-like and a time-like one

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- Pick your poison—welcome to research!
- Important: Core results are established now! Hawking and Unruh effect are here to stay!
- Experiments in the making; if not found at all, serious rethinking required.

Quantum Field Theory Gone Flat: Classical Interlude: Rindler Space-Time

Facts about Minkowski

- Born 22nd June 1864, Aleksotas (Russia) (now: Kaunas (Lithuania))
- 1880: Finishes highschool
- PhD 1885 (Königsberg)
- 1887: Teaches in Bonn
- 1894: Teaches in Königsberg
- 1896: Teaches in Zurich (Einstein among his students)
- 1897: Marries in Strasbourg
- 1902: Teaches in Göttingen
- Died 12th January 1909, Göttingen (Germany), due to appendicits

Source:

https://commons.wikimedia.org/wiki/File:

Facts about Minkowski

- Maximally symmetric: 10 KVFs to choose from!
- Globally hyperbolic
- Homogeneous
- Model space
- Flat
- No cosmological constant
- No gravity
- ...

Games with Time-Like KVFs in Minkowski

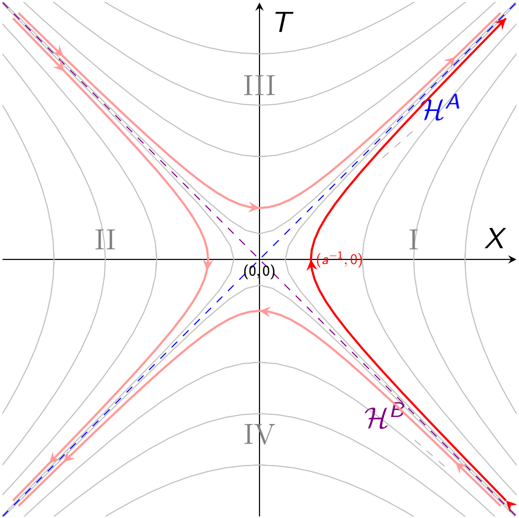
- Take global inertial coordinates on Minkowski space, say, (T, X, Y, Z)
- Translational symmetry in time and space:
  - Time translations generated by: $(\partial/\partial T)^\mu$
  - Space translations in X-direction generated by: $(\partial/\partial X)^\mu$
- Then

$$b^\mu := a\left[X\left(\frac{\partial}{\partial T}\right)^\mu + T\left(\frac{\partial}{\partial X}\right)^\mu\right], \tag{1}$$

  where a is a constant, is a KVF.
- Show this!
- Assuming $b^\mu b_\mu = -1$, show that a is the proper acceleration of this curve.
- Find the coordinate expression of the integral curves of this KVF!
- Find the Minkowski metric in terms of the coordinates suggested by these integral curves.

More Info and a Picture

- Cauchy surface Σ for Minkowski space, $\mathrm{d}\,s^2 = -\mathrm{d}\,T^2 + \mathrm{d}\,X^2 + \mathrm{d}\,Y^2 + \mathrm{d}\,Z^2$, is $X = 0$
- Cauchy surface for I is $\Sigma_{\mathrm{I}} := \Sigma \cap \mathrm{I}$
- Cauchy surface for II is $\Sigma_{\mathrm{II}} := \Sigma \cap \mathrm{II}$
- Note: $\Sigma = \Sigma_{\mathrm{I}} \cup \Sigma_{\mathrm{II}}$
- Bifurcation two-surface S of the KVF at $(0, 0)$ in picture
- $\mathcal{H}^A \cup \mathcal{H}^B$ is a bifurcate Killing horizon



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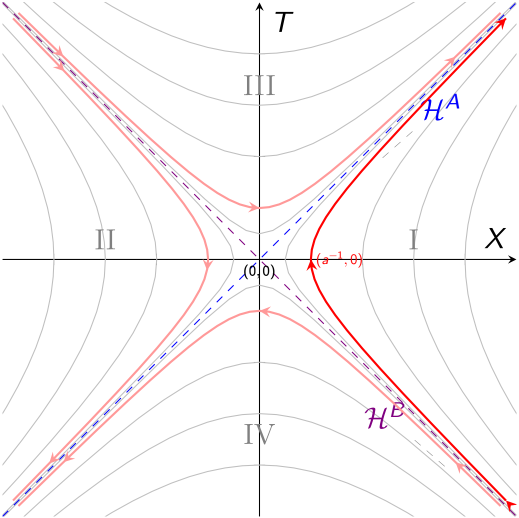
- Yes, the right column allows for LaTeX bragging rights.😎 😋 😃



Quantum Field Theory Gone Flat:
The Unruh Effect

We will consider:

- One scalar field
- non-interacting
- described by the Klein–Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\psi = 0$$

- Appropriately fill in boundary/IV conditions

Warning! We will skip **a lot** of steps, and very roughly follow Wald's book.[6]

[6] R. M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.* ISBN: 978-0-226-87027-4 (University of Chicago Press, 1994).

Nothing to Consider

- Let's glibly say: A Hilbert space $\mathfrak{H}$ will have its vacuum $|0\rangle_{\mathfrak{H}}$ (and associated Fock space)
- Wave your hands: Hilbert spaces need a notion of a Cauchy surface to go with ($3+1$ split!)

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- Here, we've got 3:
  - For inertial observers in Minkowski space, the Cauchy surface is Σ
  - For future-pointing b^{μ}, the Cauchy surface is Σ_{I}
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- Each region will have its natural Hilbert space/vacuum/Fock space/particles, linked to initial data for it on its corresponding Cauchy surface
- This gives us 3 obvious Hilbert spaces[7]:
  - For Σ we have $\mathfrak{H}_{\mathrm{M}}$ (well-known)
  - For Σ_{I} we have $\mathfrak{H}_{\mathrm{I}}$
  - For Σ_{II} we have $\mathfrak{H}_{\mathrm{II}}$

[7] We assume it is one. We assume this is this simple. We ignore any indication to the contrary.

Nothing to Consider

- $|0\rangle_{\mathsf{M}} \longleftrightarrow \Sigma$ KVF & $\xi^\mu := \left(\frac{\partial}{\partial T}\right)^\mu$—they define positive and negative frequency
- What about observers following the (time-like) trajectories of b^μ in region I or II?

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- Now construct Hilbert space[8] $\mathfrak{H}_{\mathsf{M}'} = \mathfrak{H}_{\mathrm{I}} \oplus \mathfrak{H}_{\mathrm{II}}$

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Preparing the Tools

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- Good to know, part II: A solution is uniquely specified by boundary values on $\mathcal{H}^A \cup \mathcal{H}^B$
- Good to know, part III: Check the following!
  - On $\mathcal{H}^A$, inertial time V and a Killing parameter time v can be related by

$$v = \frac{1}{a} \ln |V|$$

  - On $\mathcal{H}^B$, inertial time U and a Killing parameter time u can be related by

$$u = -\frac{1}{a} \ln |U|$$

Frequent Encounters with Fourier

- Construct a solution ψ_{I} that vanishes in II and oscillates with $\omega > 0$ (by v) as seen by b^μ in I
- Restrict this to ψ_{I}^A on $\mathcal{H}^A$:

$$\psi_{\mathrm{I}}^A(V, Y, Z) = \begin{cases} \mathrm{fudge}(Y, Z) \exp(-i\omega v(V)) & V > 0 \\ 0 & V < 0 \end{cases}$$

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- This means, we need to know:

$$\tilde{\psi}_{\mathrm{I}}^{A}(\omega_{\mathrm{M}}, Y, Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(i\omega_{\mathrm{M}} V) \psi_{\mathrm{I}}^{A}(V, Y, Z) \, \mathrm{d}\, V$$

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- Warning! While important to know, this is a statement about the "basis" of our *classical* solution space
- Such states are then created by the associated ladder operators
- This distinction is important for the Hawking effect!

Connect(ing) the Dots

- After a quick calculation:

$$\tilde{\psi}_{\mathrm{I}}^{A}(\omega_{\mathrm{M}}, Y, Z) = \frac{1}{\sqrt{2\pi}} \mathrm{fudge}(Y, Z) \int_0^\infty \exp(i\omega_{\mathrm{M}} V) \exp(-\frac{i\omega}{a} \ln V) \, \mathrm{d} V$$

- Warning! Convergence of integral subtle and complicated

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- After complex analysis magic we get for positive $\omega_{\mathrm{M}} > 0$ that

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- Rinse and repeat by appropriate swaps to get something for $\tilde{\psi}_{\mathrm{II}}^{A}(\omega_{\mathrm{M}}, Y, Z)$
- Relates (in a simple way) frequencies of I and II

Connect(ing) the Dots

- After a quick calculation:

$$\tilde{\psi}^A_{\mathrm{I}}(\omega_{\mathrm{M}}, Y, Z) = \frac{1}{\sqrt{2\pi}}\mathrm{fudge}(Y, Z) \int_0^\infty \exp(i\omega_{\mathrm{M}} V) \exp(-\frac{i\omega}{a} \ln V) \, \mathrm{d}\, V$$

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- Rinse and repeat by appropriate swaps to get something for $\tilde{\psi}^A_{\mathrm{II}}(\omega_{\mathrm{M}}, Y, Z)$
- Relates (in a simple way) frequencies of I and II
- Then on $\mathcal{H}^A$

$$\Psi_M = \psi^A_{\mathrm{I}} + \exp(-\pi\omega/a)\psi^A_{\mathrm{II}}$$

will have only positive frequency w.r.t. $T/V/U$

Connect(ing) the Dots

- After a quick calculation:

$$\tilde{\psi}^A_{\mathrm{I}}(\omega_{\mathrm{M}}, Y, Z) = \frac{1}{\sqrt{2\pi}} \mathrm{fudge}(Y, Z) \int_0^\infty \exp(i\omega_{\mathrm{M}} V) \exp(-\frac{i\omega}{a} \ln V) \,\mathrm{d}\, V$$

- Warning! Convergence of integral subtle and complicated
- After complex analysis magic we get for positive $\omega_{\mathrm{M}} > 0$ that

$$\tilde{\psi}^A_{\mathrm{I}}(-\omega_{\mathrm{M}}, Y, Z) = -\exp(-\pi\omega/a)\tilde{\psi}^A_{\mathrm{I}}(\omega_{\mathrm{M}}, Y, Z)$$

- Rinse and repeat by appropriate swaps to get something for $\tilde{\psi}^A_{\mathrm{II}}(\omega_{\mathrm{M}}, Y, Z)$
- Relates (in a simple way) frequencies of I and II
- Then on $\mathcal{H}^A$

$$\Psi_M = \psi^A_{\mathrm{I}} + \exp(-\pi\omega/a)\psi^A_{\mathrm{II}}$$

  will have only positive frequency w.r.t. $T/V/U$
- Repeat all of this for $\mathcal{H}^B$

Add Quantum

- This was only providing us with the modes our creation/annihilation operators create/annihilate
- Connecting this with the ladder operators (finally) gives us a way to write the Minkowski vacuum as

$$U\left|0\right\rangle_{\mathrm{M}} = \prod_i \left(\sum_{n=0}^{\infty} \exp(-n\pi\omega_i/a) \left|n_{i,\mathrm{I}}\right\rangle \otimes \left|n_{i,\mathrm{II}}\right\rangle \right)$$

- Here
  - i goes through our modes (with frequency ω_i spanning $\mathfrak{H}_{\mathrm{I}}$ and $\mathfrak{H}_{\mathrm{II}}$
  - U unitarily maps the Minkowski vacuum $\left|0\right\rangle_{\mathrm{M}} \in \mathfrak{H}_{\mathrm{M}}$ into $\mathfrak{H}_{\mathrm{M}'}$
  - n_i says how many particle in mode i there are

Get Effect

$$U\,|0\rangle_{\mathrm{M}} = \prod_i \left(\sum_{n=0}^{\infty} \exp(-n\pi\omega_i/a)\,|n_{i,\mathrm{I}}\rangle \otimes |n_{i,\mathrm{II}}\rangle \right)$$

- Observer b^μ now only sees parts of this state (Killing horizon!)
- Trace out the unseen bit in $\mathfrak{H}_{\mathrm{II}}$, get density matrix

$$\rho = \prod_i \left(\sum_{n=0}^{\infty} \exp(-2\pi n\omega_i/a)\,|n_{i,\mathrm{I}}\rangle \langle n_{i,\mathrm{I}}| \right)$$

- This has the form of a thermal density matrix with temperature

$$T_{\mathrm{Unruh}} = \frac{a}{2\pi} \frac{\hbar}{ck_{\mathrm{B}}}$$

Comments

- With less qualms about distributions, this can be calculated using Bogoliubov coefficients[9]
- One can also look at the "surface gravity" of the Killing horizon of Rindler space-time
- If we then connect this to a surface gravity[10] κ, we get the delicate problem

$$\kappa = \lim_{\to \mathcal{H}^A} \frac{a}{\sqrt{-b_\mu b^\mu}} \to \infty$$

- Vigorous hand-waving can connect this to black holes for $M \to 0$ 😇

---

[9] V. Mukhanov & S. Winitzki. *Introduction to Quantum Effects in Gravity.* ISBN: 9780521868341 (Cambridge University Press, 2010), Chapter 8.

[10] T. Jacobson & G. Kang. Conformal Invariance of Black Hole Temperature. *Classical and Quantum Gravity* **10,** L201–L206. doi:10.1088/0264-9381/10/11/002. arXiv: gr-qc/9307002 (1993).

Curved Space-Time Quantum Field Theory Done Quick

Hawking Effect: A 1-Slide Summary

Black Hole Radiation

Thermal radiation arising from the observer dependence of quantum vacua in curved space-times with apparent horizons.[11]

- Plenty of different approaches to its derivation.
- Spectrum of radiation akin to Planckian black body radiation:

$$\mathrm{d}\,\Gamma = \frac{g}{(2\pi)^3}\,\frac{c\;T_{\text{grey}}\,(\hat{k}\cdot\hat{n})}{\exp\left((\epsilon-\mu)/k_{\text{B}}T_{\text{H}}\right)+s}\,\mathrm{d}^3\vec{k}\;\mathrm{d}\,A, \qquad s\in\{-1,0,1\}$$

- Usually: Neglect T_{grey}, treat as black body

[11] However, note arXiv:gr-qc/0607008.

Goals for the Lectures on the Hawking effect

- The traditional derivation—a sketch
- Quick & dirty: Parikh–Wilczek
- Quickest & dirtiest: The equivalence principle
- What Else Is There?
    - Algebraic approaches
    - Euclidean space-“times” and periodicity
    - ...

To be Ignored 😇

- Interacting fields
- Renormalization
- Cosmology/Inflation/Yaddayaddayadda
- Particle creation in/by GWs
- Fascinating recent stuff regarding CPT or neutrino oscillations[12]
- Graybody factos, special **fun**ctions
- ...

[12] I'm Jon Snow. I know nothing.☹

Important Reminder of the Ingredients from the Unruh Effect

- We want to expand quantum fields $\hat{\psi}$ for different observers
- For this, we work with something of the sort

$$\hat{\psi} = \sum_i c_i \hat{a}_i^\dagger f_i$$

where

i Collective dummy variable: Momentum, spin, quantum numbers, charges ...

c_i Expansion coefficients

f_i Classical solution describing the coordinate dependence for input i

$\hat{a}_i^\dagger$ Stand in for the ladder operator creating a particle of type i in a given Fock space

- **Warning!** Different formalisms (*e.g.*, the algebraic one) place the emphasis elsewhere, and this may not turn up—for good reasons!

Note on How to Build Black Hole Evaporation

- Black hole evaporation is kinematic—not dynamic!
- You only need:

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[14] In the sense of: Hyperbolic, partial differential equation with "enough geometry" [sic!]

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  - The Einstein equations[15]
  - The precise Hamiltonian/Lagrangian for the quantum part is not needed, either; its existence is enough

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Curved Space-Time Quantum Field Theory Done Quick:
The Hawking Effect

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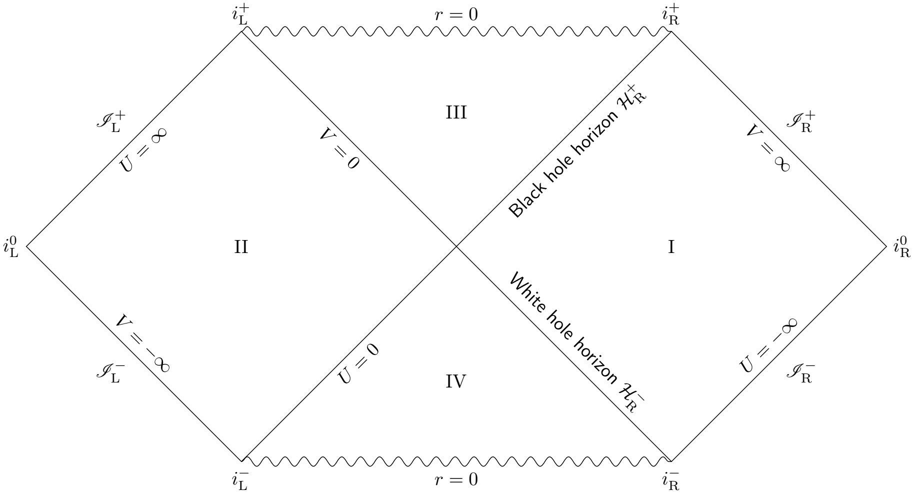
- Let's do better than that! ☺

Our Space-Time from Now: Eternal Schwarzschild



Slower and Less Dirty: Parikh & Wilczek

The idea:

- Make maximal use of spherical symmetry
- Make use of the WKB method
- Choose appropriate coordinates to define positive frequency
- Get an s-wave Hawking effect

Kraus & Wilczek (1994), arXiv:gr-qc/9406042; Kraus & Wilczek (1995), arXiv:gr-qc/9408003; Parikh & Wilczek (2000), arXiv:hep-th/9907001

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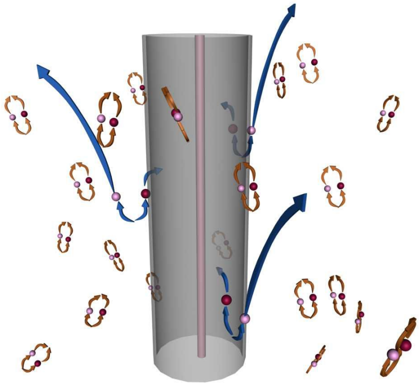

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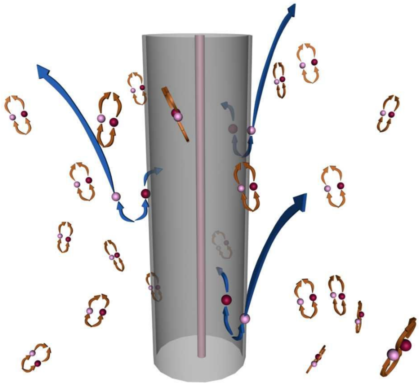
- **Warning!** The picture on the right is *not* for *s*-waves



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The Coordinates

- Take Painlevé–Gullstrand coordinate, where

$$t_{\mathrm{PG}} = t_{\mathrm{S}} + 2M\sqrt{2Mr} + 2M \ln\left(\frac{\left|\sqrt{r} - \sqrt{2M}\right|}{\sqrt{r} + \sqrt{2M}}\right)$$

- Get metric in form:

$$\mathrm{d}\,s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}\,t^2 + 2\sqrt{\frac{2M}{r}}\,\mathrm{d}\,t\,\mathrm{d}\,r + \mathrm{d}\,r^2 + r^2\,\mathrm{d}\,\Omega^2$$

- Now assume your wave equation to be separable, and focus on the r-coordinate in a Hamiltonian formulation

- Focus on a (massless) particle pair[16] created just about at the horizon
- One moves inside to r_{in}, the other barely escapes to r_{out}
- They will follow geodesics:

$$\frac{\mathrm{d}\, r}{\mathrm{d}\, t} = \pm 1 - \sqrt{\frac{2M}{r}} = \frac{\mathrm{d}\, H}{\mathrm{d}\, p_r} \tag{2}$$

- Finally, calculate the transition amplitude Γ for this process using the WKB-method:

$$\Gamma \sim \exp \mathrm{Im} S$$

[16] *Of course*, it's an uncharged scalar. The standard model has so many to play with.

$$\mathrm{Im} S = \mathrm{Im}\left(\int_{r_{\text{in}}}^{r_{\text{out}}} p_r \,\mathrm{d}\, r\right)$$

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Funky Stuff with Hamiltonians and Intelligent Ones

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Remember: $\dfrac{\mathrm{d}\, r}{\mathrm{d}\, t} = \dfrac{\mathrm{d}\, H}{\mathrm{d}\, p_r} \iff \color{red}{1} = \left(\dfrac{\mathrm{d}\, r}{\mathrm{d}\, t}\right)^{-1} \dfrac{\mathrm{d}\, H}{\mathrm{d}\, p_r}$

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$$\overset{\text{outgoing}}{=} \mathrm{Im}\left(\int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} \left(+1 - \sqrt{\frac{2M}{r}}\right)^{-1} \frac{\mathrm{d}\, H}{\mathrm{d}\, p_r'} \,\mathrm{d}\, p_r' \,\mathrm{d}\, r\right)$$

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Funky Stuff with Hamiltonians and Intelligent Ones

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&= -\mathrm{Im}\left(\int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{\omega} \left(+1 - \sqrt{\frac{2(M-\omega')}{r}}\right)^{-1} \mathrm{d}\, \omega' \,\mathrm{d}\, r\right)
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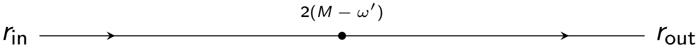
A Lazy Slide: Handwaving Complex Analysis, Part I

- Do the r-integral in the complex plane by looking at $\lim\limits_{\varepsilon,\eta\to 0} \omega' - i\varepsilon$:



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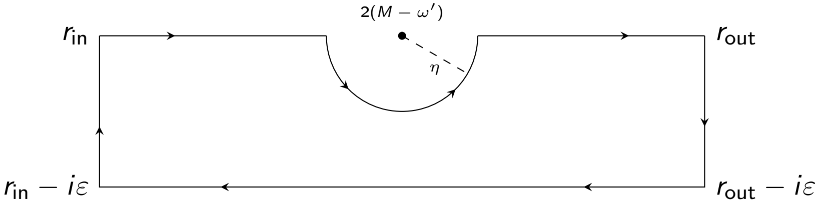

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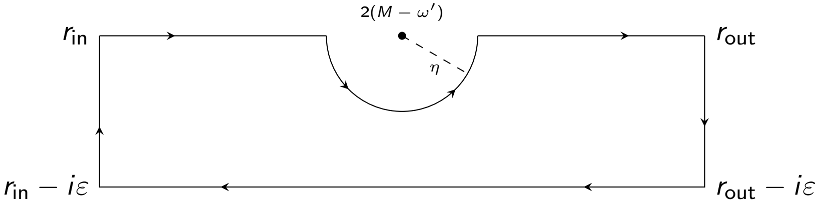

- Find that

$$\mathrm{Res}_{r=2(M-\omega')}\left(+1-\sqrt{\frac{2(M-\omega')}{r}}\right)^{-1} = 4(M-\omega')$$

$$\implies \qquad \mathrm{Im}S = 4\pi\omega(M-\frac{\omega}{2})$$

Another Lazy Slide: Handwaving Complex Analysis, Part II—A Result!

- Repeat without swapping integrals:

$$
\begin{aligned}
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- The **Hawking effect**, people!

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- Based on how long a particle pair can live as a vacuum fluctuation, this needs to happen **close** to the horizon
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- *I* would say that the horizon is a distraction

The Hawking Effect: The Traditional Way[17]

[17] S. W. Hawking. Particle creation by black holes. *Communications in Mathematical Physics* **43**, 199–220. doi:10.1007/BF02345020 (Aug. 1975). Erratum ibid. 46 (1976) 206, R. M. Wald. On Particle Creation by Black Holes. *Communications in Mathematical Physics* **45**, 9–34. doi:10.1007/BF01609863 (Feb. 1975).

Bogoliubov Coefficients/Bogoliubov(–Valatin) Transformations

- Suppose you have a Hilbert space $\mathfrak{H}$ and (anti-)commutation relations on it
- The automorphisms $\mathfrak{H} \to \mathfrak{H}$ which retain these (anti-)commutation relations are the **Bogoliubov transformations**
- The simple way to write this is in terms of annihilators $\hat{a}_i$ for a vacuum $|0\rangle$ as

$$\hat{a}'_i = \sum_j \alpha_{ji} \hat{a}_j + \beta^*_{ji} \hat{a}^\dagger_j \qquad \Longleftrightarrow \qquad \hat{a}_i = \sum_j \alpha^*_{ij} \hat{a}'_j - \beta^*_{ij} \hat{a}'^\dagger_j$$

- This means:

$$\hat{a}_i |0\rangle = 0 = \hat{a}'_i |0'\rangle$$

but $\hat{a}'_i |0\rangle$ and $\hat{a}_i |0'\rangle$ don't necessarily vanish

Quick Facts about Bogoliubov Coefficients

All of this is good exercise...

- In terms of orthonormal modes u_i and u'_i for the corresponding vacua, this means:

$$\alpha_{ij} = (u_i, u'_j), \qquad \beta_{ij} = -(u_i, u'^*_j)$$

- Two simple identities:

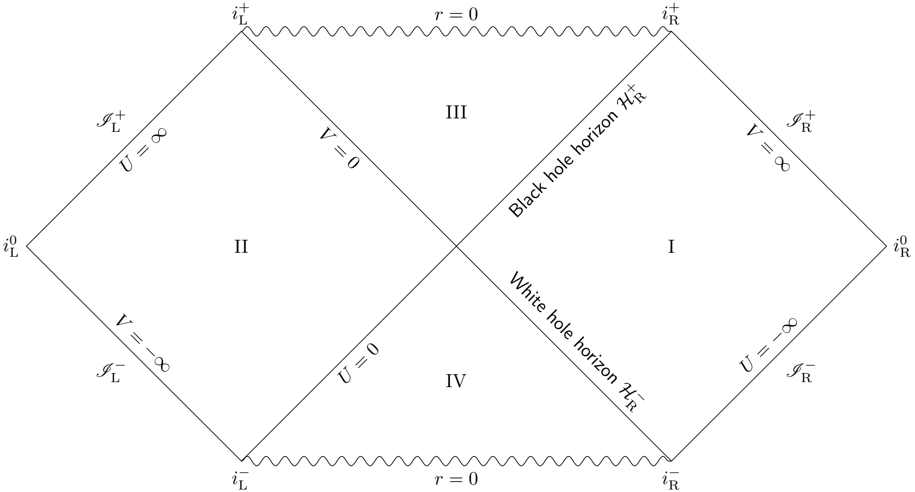
$$\sum_k \left(\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}\right) = \mathbb{1}, \qquad \sum_k \left(\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}\right) = 0$$

- For us of extreme importance: $\langle 0'|\hat{N}_i|0'\rangle = \sum_j |\beta_{ji}|^2$

- For the vacua, one has:

$$|0'\rangle = \langle 0|0'\rangle \exp\left\{-\frac{1}{2}\sum_{ij}\left[\sum_k \beta^*_{ik}\alpha^{-1}_{kj}\right]\hat{a}^\dagger_i \hat{a}^\dagger_j\right\}|0\rangle$$

Important Reminder



How You Traditionally Calculate It

- Take two Cauchy surface
  - Past: $\mathscr{I}^- \cup \mathcal{H}^-$
  - Future: $\mathscr{I}^+ \cup \mathcal{H}^+$

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- Number operator, BAM!, Hawking radiation!
- Technical aside: Modes on the horizons are tricky due to a lack of natural time; fortuitously, the main results are independent of this choice

More Comments on Literature

- Reasonably explicit textbook calculations: A. Fabbri & J. Navarro-Salas. *Modeling Black Hole Evaporation.* ISBN: 1-86094-527-9 (Imperial College Press, 2005), B. S. DeWitt. *The Global Approach to Quantum Field Theory, Volume 2.* ISBN: 978-0-19-871287-9 (Oxford University Press, 2014)

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- Also, many textbook treatments of collapse: R. M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.* ISBN: 978-0-226-87027-4 (University of Chicago Press, 1994), L. E. Parker & D. J. Toms. *Quantum Field Theory in Curved Spacetime.* ISBN: 978-0-521-87787-9 (Cambridge University Press, 2009), A. Fabbri & J. Navarro-Salas. *Modeling Black Hole Evaporation.* ISBN: 1-86094-527-9 (Imperial College Press, 2005)

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- There is more. Oh-so-much-more.

Back-scattering and Back-reaction

- Back-scattering:
  - Curvature can redirect waves from $\mathscr{I}^-(\mathcal{H}^-)$ to $\mathscr{I}^+(\mathcal{H}^+)$
  - This is very different from flat space-times
  - This complicates the maths *a lot*
  - This regularizes the luminosity through graybody factors $\Gamma_\ell(\omega)$
- Back-reaction:
  - A radiating black hole necessarily loses mass, hence, the metric is not static anymore
  - This is very different from flat space-times
  - This complicates the maths ~~a lot~~*impossibly much*
  - This regularizes the emitted energy

Lifetime

- Stephan's Law: $\frac{\mathrm{d}\,E}{\mathrm{d}\,t} \simeq -\sigma A_{\mathrm{H}} T^4_{\mathrm{Hawking}}$, where Stefan–Boltzmann constant $\sigma = \frac{\pi^2 k_{\mathrm{B}}^4}{60 c^2 \hbar^3}$

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$$\frac{\mathrm{d}\,M}{\mathrm{d}\,t} \sim \frac{\hbar c^4}{G^2 M^2}$$

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$$\frac{\mathrm{d}\,M}{\mathrm{d}\,t} \sim \frac{\hbar c^4}{G^2M^2}$$

- Solve:

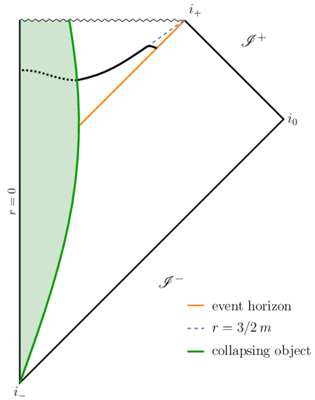
$$\tau_\mathrm{lifetime} \sim \frac{G^2}{\hbar c^4}M^3$$

Some "Numbers"

- Temperature:
$$T_{\mathrm{Hawking}} \approx 6 \times 10^{-8}\,\mathrm{K} \left(\frac{M_\odot}{M}\right)$$
- Lifetime:
$$\tau_{\mathrm{lifetime}} \approx 2.1 \times 10^{67}\,\mathrm{a} \left(\frac{M}{M_\odot}\right)^3$$
- A calculator on the web; **Warning!** I cannot say much about the source besides this calculator.

What Then?

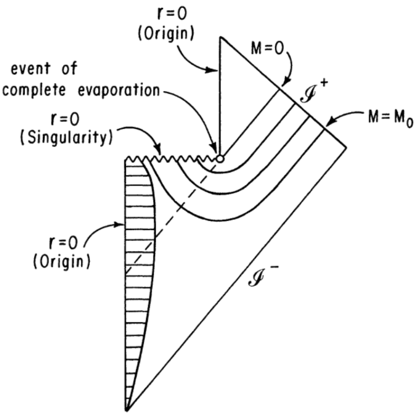
Purely classical, no CSTQFT



Source: M. Christodoulou & T. De Lorenzo. Volume inside old black holes. *Physical Review D* **94**, 104002. doi:10.1103/PhysRevD.94.104002. arXiv: 1604.07222 [gr-qc] (Nov. 2016)

What Then?

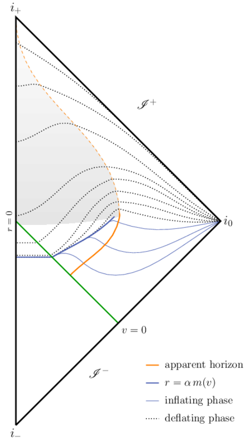
Formation of Cauchy horizon



Source: Wald—QFT in CST and BHTD, p.178

What Then?

Only apparent horizons



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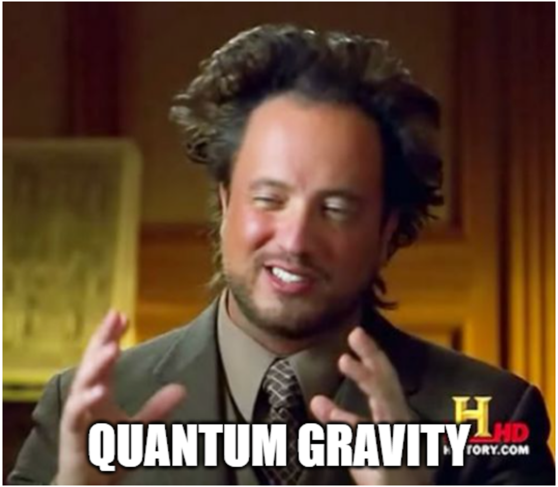
mumblesincoherently



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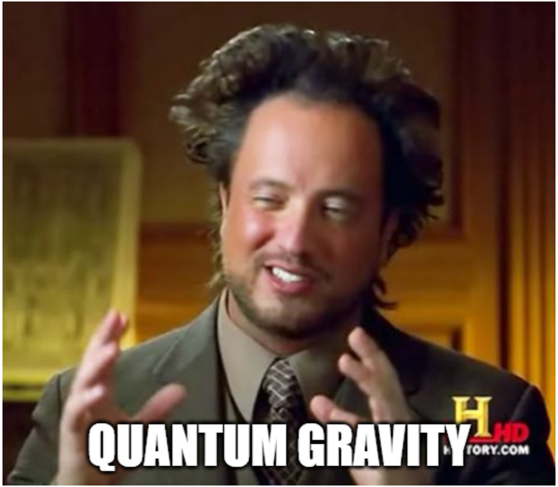

More later in Ana's lectures! ☺

Curved Space-Time Quantum Field Theory Done Quick: Detectors

More Subtleties

- As you noticed (with your questions ☺)—there sometimes is a bit of an issue how to define particle
- We are often relying on coordinate times
- Observers corresponding to these times would be at very different places (usually)
- So—what is it good for?

Simple Detector Models: The Unruh–DeWitt Detector

- Wave your hands a bit, and introduce a local interaction Hamiltonian between vacuum and a "detector"
- Unruh–DeWitt: 2-level quantum mechanical system in adiabatic perturbation theory for scalar field Lagrangian
$$\mathcal{L} = cm(\tau)\phi[\underline{x}(\tau)]$$
- m: monopole moment of detector, τ: proper time of detector, $\underline{x}$: detector's space-time trajectory
- For the sake of argument, let's go back to Minkowski
- We'll follow N. D. Birrell & P. C. Davies. *Quantum fields in curved space.* ISBN: 978-0-521-27858-4 (Cambridge University Press, 1984), p.48ff

Transition Amplitudes

- Ground state energy: E_0, excited state ψ's energy: E
- Transition amplitude to first order:

$$\Gamma = ic \langle E, \psi \rangle \int_{-\infty}^{\infty} m(\tau)\phi[\underline{x}(\tau)] \,\mathrm{d}\,\tau \,|0_M, E_0\rangle$$

- After some fiddling and summing over possible energies E

$$\Gamma \sim c^2 \sum_{E,\psi} \underbrace{|\langle E|m(0)|E_0\rangle|^2}_{\text{“selectivity”}} \underbrace{\int_{-\infty}^{\infty} \mathrm{d}\,\tau \int_{-\infty}^{\infty} \mathrm{d}\,\tau' e^{-iE(\tau-\tau')} G^+(\underline{x}(\tau), \underline{x}(\tau'))}_{\mathcal{F}(E),\ \text{“detector response function”}}$$

- Selectivity depends on detectors internal structure
- Detector response function encoding the “particle bath” of a given state that the detector feels

More Caveats

- Interacting: Non-linear. Green's function? Inherently linear. Dyson series approaches may or may not work.
- "Adiabatic" in CSTQFT is . . . *complicated*[18]
- This introduces *yet another* $3+1$-split; Wald calls this cheekily "the internal Hamiltonian"
- This is scalar. If free fields are hard for curved space-times, you ain't seen nothing yet. Detectors/interactions are worse.
- This is also of fundamental importance (and ever more so) in the field of "relativistic quantum information"

[18] For the Unruh effect there's still work on this—see arXiv:1605.01316

Some Last Words on These Lectures

- There is a lot I didn't even mention—even though I love it
- For example, curvature can appear non-trivially even in free field equations!
- Another example, curved space-times introduce all sorts of doors for anomalies
- Lastly, see the list of skipped topics—or any index/list of contents of a decent sized book
- I hope this still gives you a rough idea.☺

Thank you! Questions?