Prostor antisymetrických tenzorů

definice:

$V^{[k]} \subset V^k$ je prostor antisymetrických tenzorů k-tého stupně, $k = 0, \dots, d$; $\dim V^{[k]} = \binom{d}{k}$

$$A \in V^{[k]} \quad \equiv \quad \forall\, \sigma - \text{permutace } [1,\dots,k] \ : \quad A^{a_1 \dots a_k} = \operatorname{sign} \sigma \ A^{a_{\sigma_1} \dots a_{\sigma_k}}$$

antisymetrizace:

$$A = \mathcal{A} B \qquad A^{a_1 \dots a_k} = B^{[a_1 \dots a_k]} = \frac{1}{k!} \sum_\sigma \operatorname{sign} \sigma \ B^{a_{\sigma_1} \dots a_{\sigma_k}}$$

$$A \in V^{[k]} \quad \Leftrightarrow \quad A^{a_1 \dots a_k} = A^{[a_1 \dots a_k]}$$

projektor na $V^{[k]}$:

$${}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k} = \delta^{[a_1}_{b_1} \dots \delta^{a_k]}_{b_k} = \delta^{a_1}_{[b_1} \dots \delta^{a_k}_{b_k]} \qquad , \qquad {}^{[k]}\delta \in V^{[k]}_{[k]}$$

vlastnosti projektoru:

$${}^{[k]}\delta^{a_1 \dots a_k}_{r_1 \dots r_k}\, {}^{[k]}\delta^{r_1 \dots r_k}_{b_1 \dots b_k} = {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k} \qquad , \qquad A^{[a_1 \dots a_k]} = {}^{[k]}\delta^{a_1 \dots a_k}_{r_1 \dots r_k} A^{r_1 \dots r_k}$$

$${}^{[k]}\delta^{a_1 \dots a_{k-l} a_{k-l+1} \dots a_k}_{b_1 \dots b_{k-l} r_1 \ \dots \ r_l}\, {}^{[l]}\delta^{r_1 \ \dots \ r_l}_{b_{k-l+1} \dots b_k} = {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k}$$

$${}^{[k]}\delta^{a_1 \dots a_l r_1 \dots r_{k-l}}_{b_1 \dots b_l r_1 \dots r_{k-l}} = \frac{(d-l)!\, l!}{(d-k)!\, k!}\, {}^{[l]}\delta^{a_1 \dots a_l}_{b_1 \dots b_l} \qquad , \qquad {}^{[k]}\delta^{r_1 \dots r_k}_{r_1 \dots r_k} = \dim V^{[k]}$$

$${}^{[k]}\delta^{a_{\sigma_1} \dots a_{\sigma_k}}_{b_{\sigma_1} \dots b_{\sigma_k}} = {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k} \qquad \sigma \text{ je permutace } [1,\dots,k]$$

souřadnice:

$$A = A^{a_1 \dots a_k}\ \vec{e}_{a_1} \dots \vec{e}_{a_k} = \sum_{a_1 < \dots < a_k} A^{a_1 \dots a_k}\ k!\ \mathcal{A}(\vec{e}_{a_1} \dots \vec{e}_{a_k})$$

totálně antisymetrické formy a tenzory:

prostory $V_{[d]}$ a $V^{[d]}$, kde d je dimenze prostoru V; $\dim V_{[d]} = \dim V^{[d]} = 1$

souřadnice ($\alpha \in V_{[d]}$):

$$\alpha = \alpha_{a_1 \dots a_d}\ \underset{\rightarrow}{e}^{a_1} \dots \underset{\rightarrow}{e}^{a_d} = \alpha_{1 \dots d} \sum_\sigma \operatorname{sign}\sigma\ \underset{\rightarrow}{e}^{\sigma_1} \dots \underset{\rightarrow}{e}^{\sigma_d} = \alpha_{1 \dots d}\ d!\ \mathcal{A}(\underset{\rightarrow}{e}^{1} \dots \underset{\rightarrow}{e}^{d})$$

inverze:

$${}^{-1}\ :\ V_{[d]} \leftrightarrow V^{[d]}\ , \quad \alpha \rightarrow \alpha^{-1}\ , \quad (\alpha^{-1})^{-1} = \alpha\ , \quad \alpha_{r_1 \dots r_d}\, \alpha^{-1\, r_1 \dots r_d} = d!$$

vlastnosti inverze:

$$\alpha_{b_1 \dots b_k r_1 \dots r_{d-k}}\, \alpha^{-1\, a_1 \dots a_k r_1 \dots r_{d-k}} = (d-k)!\, k!\ {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k}$$

$$\alpha_{b_1 \dots b_d}\, \alpha^{-1\, a_1 \dots a_d} = d!\ {}^{[d]}\delta^{a_1 \dots a_d}_{b_1 \dots b_d} \qquad , \qquad \alpha_{r_1 \dots r_d}\, \alpha^{-1\, r_1 \dots r_d} = d!$$

$$\alpha^{-1\, 1 \dots d} = (\alpha_{1 \dots d})^{-1}$$

determinant:

$$\det A = {}^{[d]}\delta^{a_1 \dots a_d}_{b_1 \dots b_d}\ A^{b_1}_{a_1} \dots A^{b_d}_{a_d} = \sum_\sigma \operatorname{sign}\sigma\ A^{\sigma_1}_1 \dots A^{\sigma_d}_d \qquad , \qquad A \in V^1_1$$

Prostor symetrických tenzorů

definice:

$V^{(k)} \subset V^k$ je prostor symetrických tenzorů k-tého stupně, $k \in \mathbb{N}_0$; $\quad \dim V^{(k)} = \binom{k+d-1}{k}$

$$A \in V^{(k)} \quad \equiv \quad \forall\, \sigma - \text{permutace } [1, \ldots, k] \ : \quad A^{a_1 \ldots a_k} = A^{a_{\sigma_1} \ldots a_{\sigma_k}}$$

symetrizace:

$$A = \mathcal{S}B \qquad A^{a_1 \ldots a_k} = B^{(a_1 \ldots a_k)} = \frac{1}{k!} \sum_{\sigma} B^{a_{\sigma_1} \ldots a_{\sigma_k}}$$

$$A \in V^{(k)} \quad \Leftrightarrow \quad A^{a_1 \ldots a_k} = A^{(a_1 \ldots a_k)}$$

projektor na $V^{(k)}$:

$${}^{(k)}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k} = \delta^{(a_1}_{b_1} \ldots \delta^{a_k)}_{b_k} = \delta^{a_1}_{(b_1} \ldots \delta^{a_k}_{b_k)} \qquad , \qquad {}^{(k)}\delta \in V^{(k)}_{(k)}$$

vlastnosti projektoru:

$${}^{(k)}\delta^{a_1 \ldots a_k}_{r_1 \ldots r_k} \, {}^{(k)}\delta^{r_1 \ldots r_k}_{b_1 \ldots b_k} = {}^{(k)}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k} \qquad , \qquad A^{(a_1 \ldots a_k)} = {}^{(k)}\delta^{a_1 \ldots a_k}_{r_1 \ldots r_k} A^{r_1 \ldots r_k}$$

$${}^{(k)}\delta^{a_1 \ldots a_{k-l} a_{k-l+1} \ldots a_k}_{b_1 \ldots b_{k-l} r_1 \quad \ldots \quad r_l} \, {}^{(l)}\delta^{r_1 \quad \ldots \quad r_l}_{b_{k-l+1} \ldots b_k} = {}^{(k)}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k}$$

$${}^{(k)}\delta^{a_1 \ldots a_l r_1 \ldots r_{k-l}}_{b_1 \ldots b_l r_1 \ldots r_{k-l}} = \frac{(k+d-1)!\, l!}{(l+d-1)!\, k!} \, {}^{(l)}\delta^{a_1 \ldots a_l}_{b_1 \ldots b_l} \qquad , \qquad {}^{(k)}\delta^{r_1 \ldots r_k}_{r_1 \ldots r_k} = \dim V^{(k)}$$

$${}^{(k)}\delta^{a_{\sigma_1} \ldots a_{\sigma_k}}_{b_{\sigma_1} \ldots b_{\sigma_k}} = {}^{(k)}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k} \qquad \sigma \text{ je permutace } [1, \ldots, k]$$

souřadnice:

$$A = A^{a_1 \ldots a_k} \, \vec{e}_{a_1} \ldots \vec{e}_{a_k} = \sum_{a_1 \leq \cdots \leq a_k} A^{a_1 \ldots a_k} \; n(a_1, \ldots, a_k)\, \mathcal{S}(\vec{e}_{a_1} \ldots \vec{e}_{a_k})$$

$n(a_1, \ldots, a_k)$ je počet vzájemně odlišných permutací indexů $a_1 \ldots a_k$