I. S. GRADSHTEYN
I. M. RYZHIK

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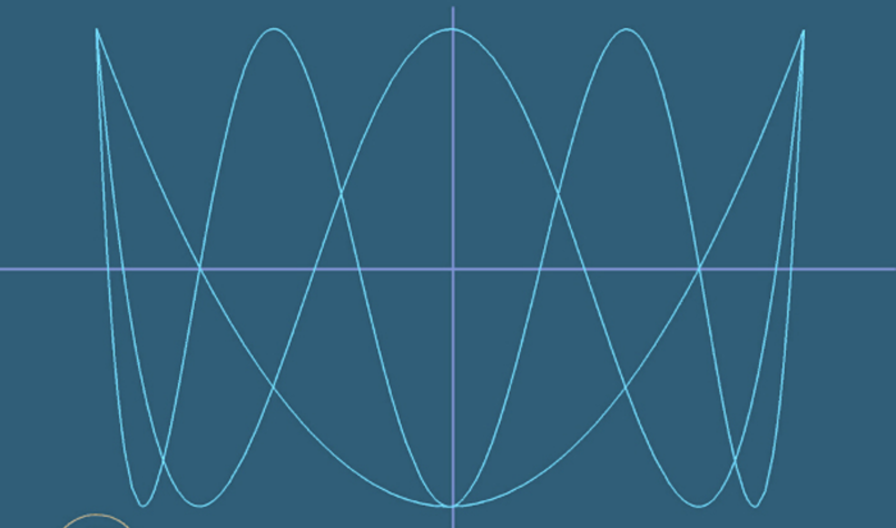
TABLE OF INTEGRALS, SERIES, AND PRODUCTS

SEVENTH EDITION



Edited by Alan Jeffrey and Daniel Zwillinger

Table of Integrals, Series, and Products

Seventh Edition

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1.44–1.45 Trigonometric (Fourier) series

1.441

1. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad [0 < x < 2\pi]$$ FI III 539

2. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\frac{1}{2} \ln\left[2\left(1 - \cos x\right)\right] \qquad [0 < x < 2\pi]$$ FI III 530a, AD (6814)

3. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2} \qquad [-\pi < x < \pi]$$ FI III 542

4. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln\left(2 \cos \frac{x}{2}\right) \qquad [-\pi < x < \pi]$$ FI III 550

1.442

1.[11] $$\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\pi}{4} \operatorname{sign} x \qquad [-\pi < x < \pi]$$ FI III 541

2. $$\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} = \frac{1}{2} \ln \cot \frac{x}{2} \qquad [0 < x < \pi]$$

BR* 168, JO (266), GI III(195)

3. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2} \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$ BR* 168, JO (268)a

4.[10] $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k-1)x}{2k-1} = \frac{\pi}{4} \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$
$$= -\frac{\pi}{4} \qquad \left[\frac{\pi}{2} < x < \frac{3\pi}{2}\right]$$

BR* 168, JO (269)

1.443

1.[8] $$\sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k} \rho^k$$
$$= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}\left(\frac{x}{2}\right)$$
$$\left[0 \le x \le 2, \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor\right]$$ CE 340, GE 71

2. $$\sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1} \rho^k$$
$$= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{x}{2}\right)$$
$$\left[0 < x < 1; \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor\right]$$ CE 340

3. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}$$ $[0 \le x \le 2\pi]$ FI III 547

4. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4}$$ $[-\pi \le x \le \pi]$ FI III 544

5. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12}$$ $[0 \le x \le 2\pi]$

6. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48}$$ $[0 \le x \le 2\pi]$ AD (6617)

7. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k^5} = \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240}$$ $[0 \le x \le 2\pi]$ AD (6818)

1.444

1. $$\sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} = \sin 2x - (\pi - 2x)\sin^2 x - \sin x \cos x \ln\left(4\sin^2 x\right)$$
$[0 \le x \le \pi]$ BR* 168, GI III (190)

2. $$\sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} = \cos 2x - \left(\frac{\pi}{2} - x\right)\sin 2x + \sin^2 x \ln\left(4\sin^2 x\right)$$
$[0 \le x \le \pi]$ BR* 168

3. $$\sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} = \sin x - \frac{x}{2}(1+\cos x) - \sin x \ln\left|2\cos\frac{x}{2}\right|$$ MO 213

4. $$\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} = \cos x - \frac{x}{2}\sin x - (1+\cos x)\ln\left|2\cos\frac{x}{2}\right|$$ MO 213

5. $$\sum_{k=0}^{\infty} (-1)^k \frac{\sin(2k+1)x}{(2k+1)^2} = \frac{\pi}{4}x \qquad \left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$$
$$= \frac{\pi}{4}(\pi - x) \qquad \left[\frac{\pi}{2} \le x \le \frac{3}{2}\pi\right]$$
MO 213

6.[6] $$\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - |x|\right)$$ $[-\pi \le x \le \pi]$ FI III 546

7. $$\sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} = \frac{1}{2} - \frac{\pi}{4}\sin x$$ $\left[0 \le x \le \frac{\pi}{2}\right]$ JO (591)

1.445

1. $$\sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi}{2}\frac{\sinh \alpha(\pi - x)}{\sinh \alpha\pi}$$ $[0 < x < 2\pi]$ BR* 157, JO (411)

2. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2 + \alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh \alpha(\pi - x)}{\sinh \alpha\pi} - \frac{1}{2\alpha^2}$$ $[0 \le x \le 2\pi]$ BR* 257, JO (410)

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh \alpha x}{\sinh \alpha\pi} - \frac{1}{2\alpha^2}$$ $[-\pi \le x \le \pi]$ FI III 546

4. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \sin kx}{k^2+\alpha^2} = \frac{\pi}{2}\frac{\sinh \alpha x}{\sinh \alpha\pi}$$ $[-\pi < x < \pi]$ FI III, 546

5. $$\sum_{k=1}^{\infty} \frac{k \sin kx}{k^2-\alpha^2} = \pi \frac{\sin\left\{\alpha[(2m+1)\pi - x]\right\}}{2\sin\alpha\pi}$$ $\left[\text{if } x = 2m\pi, \text{ then } \sum \cdots = 0\right]$

$[2m\pi < x < (2m+2)\pi, \quad \alpha \text{ not an integer}]$ MO 213

6. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2}\frac{\cos\left[\alpha\left\{(2m+1)\pi - x\right\}\right]}{\alpha\sin\alpha\pi}$$

$[2m\pi \le x \le (2m+2)\pi, \quad \alpha \text{ not an integer}]$ MO 213

7. $$\sum_{k=1}^{\infty} (-1)^k \frac{k \sin kx}{k^2-\alpha^2} = \pi \frac{\sin[\alpha(2m\pi - x)]}{2\sin\alpha\pi}$$ $\left[\text{if } x = (2m+1)\pi, \text{ then } \sum \cdots = 0\right]$,

$[(2m-1)\pi < x < (2m+1)\pi, \alpha \text{ not an integer}]$ FI III 545a

8. $$\sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2}\frac{\cos[\alpha(2m\pi - x)]}{\alpha\sin\alpha\pi}$$

$[(2m-1)\pi \le x \le (2m+1)\pi, \alpha \text{ not an integer}]$ FI III 545a

9.* $$\sum_{n=-\infty}^{\infty} \frac{e^{in\alpha}}{(n-\beta)^2+\gamma^2} = \frac{\pi}{\gamma}\frac{e^{i\beta(\alpha-2\pi)}\sinh(\gamma\alpha) + e^{i\beta\alpha}\sinh\left[\gamma(2\pi-\alpha)\right]}{\cosh(2\pi\gamma) - \cos(2\pi\beta)}$$

$[0 \le \alpha \le 2\pi]$

1.446 $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\cos(2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8}\cos^2 x - \frac{1}{3}\cos x$$

$\left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$ BR* 256, GI III (189)

1.447

1. $$\sum_{k=1}^{\infty} p^k \sin kx = \frac{p\sin x}{1-2p\cos x+p^2}$$

$[|p| < 1]$ FI II 559

2. $$\sum_{k=0}^{\infty} p^k \cos kx = \frac{1-p\cos x}{1-2p\cos x+p^2}$$

$[|p| < 1]$ FI II 559

3. $$1 + 2\sum_{k=1}^{\infty} p^k \cos kx = \frac{1-p^2}{1-2p\cos x+p^2}$$

$[|p| < 1]$ FI II 559a, MO 213

1.448

1. $$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \arctan \frac{p \sin x}{1 - p \cos x}$$ $\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ FI II 559

2. $$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln\left(1 - 2p\cos x + p^2\right)$$ $\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ FI II 559

3. $$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \sin x}{1 - p^2}$$ $\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ JO (594)

4. $$\sum_{k=1}^{\infty} \frac{p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p\cos x + p^2}{1 - 2p\cos x + p^2}$$ $\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ JO (259)

5. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p\sin x + p^2}{1 - 2p\sin x + p^2}$$ $\left[0 < x < \pi, \quad p^2 \le 1\right]$ JO (261)

6. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \cos x}{1 - p^2}$$ $\left[0 < x < \pi, \quad p^2 \le 1\right]$ JO (597)

1.449

1. $$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p\cos x} \sin\left(p \sin x\right)$$ $\left[p^2 \le 1\right]$ JO (486)

2. $$\sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p\cos x} \cos\left(p \sin x\right)$$ $\left[p^2 \le 1\right]$ JO (485)

Let $S(x) = -\frac{1}{x}\cos x + \frac{1}{x}$ and $C(x) = \frac{1}{x}\sin x$.

3.* $$\sum_{n=1}^{\infty} \frac{n}{n^2 - a^2} S(nx) = \frac{\pi}{2}\left[C(ax) - \cot(\pi a) S(ax)\right]$$ $[0 < x < 2\pi, \quad a \ne 0, \pm 1, \pm 2, \ldots]$

4.* $$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} C(nx) = \frac{1}{2a^2} - \frac{\pi}{2a}\left[S(ax) - \cot(\pi a) C(ax)\right]$$ $[0 \le x \le 2\pi, \quad a \ne 0, \pm 1, \pm 2, \ldots]$

5.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \operatorname{cosec}(\pi a) S(ax)$$ $[-\pi < x < \pi, \quad a \ne 0, \pm 1, \pm 2, \ldots]$

6.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-a^2} C(nx) = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cosec}(\pi a) C(ax) \qquad [-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$$

7.* $$\sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)^2-a^2} S(nx) = \frac{\pi}{4} \left[C(ax) + \tan\left(\frac{\pi a}{2}\right) S(ax) \right]$$

$$[0 < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$$

8.* $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2-a^2} C(nx) = -\frac{\pi}{4a} \left[S(ax) - \tan\left(\frac{\pi a}{2}\right) C(ax) \right]$$

$$[0 \leq x \leq \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$$

9.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2-a^2} S(nx) = \frac{\pi}{4a} \sec\left(\frac{\pi a}{2}\right) S(ax) \qquad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \ldots\right]$$

10.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2-a^2} C(nx) = \frac{\pi}{4} \sec\left(\frac{\pi a}{2}\right) C(ax) \qquad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \ldots\right]$$

Fourier expansions of hyperbolic functions

1.451

1. $$\sinh x = \cos x \sum_{k=0}^{\infty} \frac{\left(1^2+0^2\right)\left(1^2+2^2\right)\ldots\left[1^2+(2k)^2\right]}{(2k+1)!} \sin^{2k+1} x$$ JO (504)

2. $$\cosh x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{\left(1^2+1^2\right)\left(1^2+3^2\right)\ldots\left[1^2+(2k-1)^2\right]}{(2k)!} \sin^{2k} x$$ JO (503)

1.452

1. $$\sinh(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k+1}\cos(2k+1)\theta}{(2k+1)!}$$

$$\left[x^2 < 1\right]$$ JO (391)

2. $$\cosh(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k}\cos 2k\theta}{(2k)!}$$

$$\left[x^2 < 1\right]$$ JO (390)

3. $$\sinh(x\cos\theta) = \operatorname{cosec}(x\sin\theta) \sum_{k=1}^{\infty} \frac{x^{2k}\sin 2k\theta}{(2k)!}$$

$$\left[x^2 < 1, \quad x\sin\theta \neq 0\right]$$ JO (393)

4. $$\cosh(x\cos\theta) = \operatorname{cosec}(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k+1}\sin(2k+1)\theta}{(2k+1)!}$$

$$\left[x^2 < 1, \quad x\sin\theta \neq 0\right]$$ JO (392)