

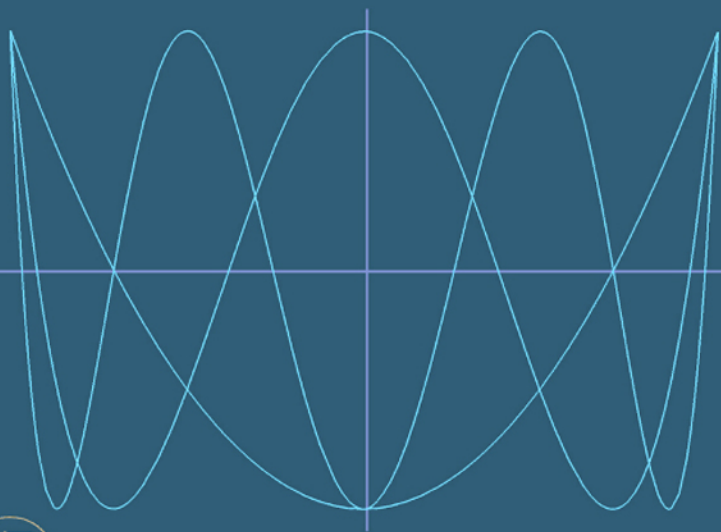
I. S. GRADSHTEYN  
I. M. RYZHIK



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# TABLE OF INTEGRALS, SERIES, AND PRODUCTS

SEVENTH EDITION



*Edited by Alan Jeffrey and Daniel Zwillinger*

# Table of Integrals, Series, and Products

*Seventh Edition*

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### 1.44–1.45 Trigonometric (Fourier) series

#### 1.441

$$1. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \quad [0 < x < 2\pi] \quad \text{FI III 539}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\frac{1}{2} \ln [2(1 - \cos x)] \quad [0 < x < 2\pi] \quad \text{FI III 530a, AD (6814)}$$

$$3. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2} \quad [-\pi < x < \pi] \quad \text{FI III 542}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln \left( 2 \cos \frac{x}{2} \right) \quad [-\pi < x < \pi] \quad \text{FI III 550}$$

#### 1.442

$$1.^{11} \quad \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\pi}{4} \operatorname{sign} x \quad [-\pi < x < \pi] \quad \text{FI III 541}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} = \frac{1}{2} \ln \cot \frac{x}{2} \quad [0 < x < \pi]$$

BR\* 168, JO (266), GI III(195)

$$3. \quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \quad \left[ -\frac{\pi}{2} < x < \frac{\pi}{2} \right] \quad \text{BR* 168, JO (268)a}$$

$$4.^{10} \quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k-1)x}{2k-1} = \frac{\pi}{4} \quad \left[ -\frac{\pi}{2} < x < \frac{\pi}{2} \right]$$

$$= -\frac{\pi}{4} \quad \left[ \frac{\pi}{2} < x < \frac{3\pi}{2} \right]$$

BR\* 168, JO (269)

#### 1.443

$$1.^8 \quad \sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k} \rho^k$$

$$= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \left( \frac{x}{2} \right)$$

$$\left[ 0 \leq x \leq 2, \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor \right] \quad \text{CE 340, GE 71}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1} \rho^k$$

$$= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1} \left( \frac{x}{2} \right)$$

$$\left[ 0 < x < 1; \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor \right] \quad \text{CE 340}$$

$$3. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad [0 \leq x \leq 2\pi] \quad \text{FI III 547}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \quad [-\pi \leq x \leq \pi] \quad \text{FI III 544}$$

$$5. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} \quad [0 \leq x \leq 2\pi]$$

$$6. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48} \quad [0 \leq x \leq 2\pi] \quad \text{AD (6617)}$$

$$7. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^5} = \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240} \quad [0 \leq x \leq 2\pi] \quad \text{AD (6818)}$$

## 1.444

$$1. \quad \sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} = \sin 2x - (\pi - 2x) \sin^2 x - \sin x \cos x \ln(4 \sin^2 x) \quad [0 \leq x \leq \pi] \quad \text{BR* 168, GI III (190)}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} = \cos 2x - \left(\frac{\pi}{2} - x\right) \sin 2x + \sin^2 x \ln(4 \sin^2 x) \quad [0 \leq x \leq \pi] \quad \text{BR* 168}$$

$$3. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} = \sin x - \frac{x}{2} (1 + \cos x) - \sin x \ln \left| 2 \cos \frac{x}{2} \right| \quad \text{MO 213}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} = \cos x - \frac{x}{2} \sin x - (1 + \cos x) \ln \left| 2 \cos \frac{x}{2} \right| \quad \text{MO 213}$$

$$5. \quad \sum_{k=0}^{\infty} (-1)^k \frac{\sin(2k+1)x}{(2k+1)^2} = \frac{\pi}{4} x \quad \left[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right] \\ = \frac{\pi}{4} (\pi - x) \quad \left[ \frac{\pi}{2} \leq x \leq \frac{3}{2}\pi \right] \quad \text{MO 213}$$

$$6.^6 \quad \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\pi}{4} \left( \frac{\pi}{2} - |x| \right) \quad [-\pi \leq x \leq \pi] \quad \text{FI III 546}$$

$$7. \quad \sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} = \frac{1}{2} - \frac{\pi}{4} \sin x \quad \left[ 0 \leq x \leq \frac{\pi}{2} \right] \quad \text{JO (591)}$$

## 1.445

$$1. \quad \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi \sinh \alpha(\pi - x)}{2 \sinh \alpha\pi} \quad [0 < x < 2\pi] \quad \text{BR* 157, JO (411)}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^2 + \alpha^2} = \frac{\pi \cosh \alpha(\pi - x)}{2\alpha \sinh \alpha\pi} - \frac{1}{2\alpha^2} \quad [0 \leq x \leq 2\pi] \quad \text{BR* 257, JO (410)}$$

3. 
$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2 + \alpha^2} = \frac{\pi \cosh \alpha x}{2\alpha \sinh \alpha \pi} - \frac{1}{2\alpha^2} \quad [-\pi \leq x \leq \pi] \quad \text{FI III 546}$$
4. 
$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi \sinh \alpha x}{2 \sinh \alpha \pi} \quad [-\pi < x < \pi] \quad \text{FI III, 546}$$
5. 
$$\sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin \{\alpha[(2m+1)\pi - x]\}}{2 \sin \alpha \pi} \quad \left[ \text{if } x = 2m\pi, \text{ then } \sum \dots = 0 \right]$$

$$[2m\pi < x < (2m+2)\pi, \quad \alpha \text{ not an integer}] \quad \text{MO 213}$$
6. 
$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi \cos [\alpha \{(2m+1)\pi - x\}]}{2 \alpha \sin \alpha \pi}$$

$$[2m\pi \leq x \leq (2m+2)\pi, \quad \alpha \text{ not an integer}] \quad \text{MO 213}$$
7. 
$$\sum_{k=1}^{\infty} (-1)^k \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin[\alpha(2m\pi - x)]}{2 \sin \alpha \pi} \quad \left[ \text{if } x = (2m+1)\pi, \text{ then } \sum \dots = 0 \right],$$

$$[(2m-1)\pi < x < (2m+1)\pi, \alpha \text{ not an integer}] \quad \text{FI III 545a}$$
8. 
$$\sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi \cos[\alpha(2m\pi - x)]}{2 \alpha \sin \alpha \pi}$$

$$[(2m-1)\pi \leq x \leq (2m+1)\pi, \alpha \text{ not an integer}] \quad \text{FI III 545a}$$
- 9.\* 
$$\sum_{n=-\infty}^{\infty} \frac{e^{in\alpha}}{(n-\beta)^2 + \gamma^2} = \frac{\pi e^{i\beta(\alpha-2\pi)} \sinh(\gamma\alpha) + e^{i\beta\alpha} \sinh[\gamma(2\pi-\alpha)]}{\gamma \cosh(2\pi\gamma) - \cos(2\pi\beta)}$$

$$[0 \leq \alpha \leq 2\pi]$$
- 1.446 
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8} \cos^2 x - \frac{1}{3} \cos x$$

$$\left[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right] \quad \text{BR* 256, GI III (189)}$$
- 1.447
1. 
$$\sum_{k=1}^{\infty} p^k \sin kx = \frac{p \sin x}{1 - 2p \cos x + p^2}$$

$$[|p| < 1] \quad \text{FI II 559}$$
2. 
$$\sum_{k=0}^{\infty} p^k \cos kx = \frac{1 - p \cos x}{1 - 2p \cos x + p^2}$$

$$[|p| < 1] \quad \text{FI II 559}$$
3. 
$$1 + 2 \sum_{k=1}^{\infty} p^k \cos kx = \frac{1 - p^2}{1 - 2p \cos x + p^2}$$

$$[|p| < 1] \quad \text{FI II 559a, MO 213}$$

## 1.448

1. 
$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \arctan \frac{p \sin x}{1 - p \cos x}$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{FI II 559}$$
2. 
$$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln(1 - 2p \cos x + p^2)$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{FI II 559}$$
3. 
$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \sin x}{1 - p^2}$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{JO (594)}$$
4. 
$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2}$$

$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{JO (259)}$$
5. 
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \sin x + p^2}{1 - 2p \sin x + p^2}$$

$$[0 < x < \pi, \quad p^2 \leq 1] \quad \text{JO (261)}$$
6. 
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \cos x}{1 - p^2}$$

$$[0 < x < \pi, \quad p^2 \leq 1] \quad \text{JO (597)}$$

## 1.449

1. 
$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p \cos x} \sin(p \sin x)$$

$$[p^2 \leq 1] \quad \text{JO (486)}$$
2. 
$$\sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p \cos x} \cos(p \sin x)$$

$$[p^2 \leq 1] \quad \text{JO (485)}$$

Let  $S(x) = -\frac{1}{x} \cos x + \frac{1}{x}$  and  $C(x) = \frac{1}{x} \sin x$ .

- 3.\* 
$$\sum_{n=1}^{\infty} \frac{n}{n^2 - a^2} S(nx) = \frac{\pi}{2} [C(ax) - \cot(\pi a) S(ax)]$$

$$[0 < x < 2\pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$
- 4.\* 
$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} C(nx) = \frac{1}{2a^2} - \frac{\pi}{2a} [S(ax) - \cot(\pi a) C(ax)]$$

$$[0 \leq x \leq 2\pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$
- 5.\* 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \operatorname{cosec}(\pi a) S(ax)$$

$$[-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$6.* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - a^2} C(nx) = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cosec}(\pi a) C(ax) \quad [-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$7.* \quad \sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)^2 - a^2} S(nx) = \frac{\pi}{4} \left[ C(ax) + \tan\left(\frac{\pi a}{2}\right) S(ax) \right] \\ [0 < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$8.* \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - a^2} C(nx) = -\frac{\pi}{4a} \left[ S(ax) - \tan\left(\frac{\pi a}{2}\right) C(ax) \right] \\ [0 \leq x \leq \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$9.* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2 - a^2} S(nx) = \frac{\pi}{4a} \sec\left(\frac{\pi a}{2}\right) S(ax) \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \dots\right]$$

$$10.* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2 - a^2} C(nx) = \frac{\pi}{4} \sec\left(\frac{\pi a}{2}\right) C(ax) \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \dots\right]$$

### Fourier expansions of hyperbolic functions

#### 1.451

$$1. \quad \sinh x = \cos x \sum_{k=0}^{\infty} \frac{(1^2 + 0^2)(1^2 + 2^2) \dots [1^2 + (2k)^2]}{(2k+1)!} \sin^{2k+1} x \quad \text{JO (504)}$$

$$2. \quad \cosh x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{(1^2 + 1^2)(1^2 + 3^2) \dots [1^2 + (2k-1)^2]}{(2k)!} \sin^{2k} x \quad \text{JO (503)}$$

#### 1.452

$$1. \quad \sinh(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \cos(2k+1)\theta}{(2k+1)!} \\ [x^2 < 1] \quad \text{JO (391)}$$

$$2. \quad \cosh(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k} \cos 2k\theta}{(2k)!} \\ [x^2 < 1] \quad \text{JO (390)}$$

$$3. \quad \sinh(x \cos \theta) = \operatorname{cosec}(x \sin \theta) \sum_{k=1}^{\infty} \frac{x^{2k} \sin 2k\theta}{(2k)!} \\ [x^2 < 1, \quad x \sin \theta \neq 0] \quad \text{JO (393)}$$

$$4. \quad \cosh(x \cos \theta) = \operatorname{cosec}(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \sin(2k+1)\theta}{(2k+1)!} \\ [x^2 < 1, \quad x \sin \theta \neq 0] \quad \text{JO (392)}$$