Obecné souřadnice tenzorů

souřadnice: x^m

báze v prostoru vektorů: $\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m}$ báze v prostoru 1-forem: $\mathbf{d}x^m$ platí: $\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m} \cdot \mathbf{d}x^n = \delta^n_m$

Souřadnice tenzorů:

souřadnice x^k

$$\boldsymbol{a} = a^k \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \qquad \boldsymbol{\alpha} = \alpha_k \, \mathbf{d}x^k$$

$$\boldsymbol{A} = A^{k\dots}_{l\dots} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \cdots \mathbf{d}x^l \cdots$$

$$a^k = a^{l'} \frac{\partial x^k}{\partial y^{l'}} \qquad \alpha_k = \alpha_{l'} \frac{\partial y^{l'}}{\partial x^k}$$

$$A^{m\dots}_{n\dots} = A^{k'}_{l'} \frac{\partial x^m}{\partial y^{k'}} \cdots \frac{\partial y^{l'}}{\partial x^n} \cdots$$

souřadnice $y^{k'}$

$$\boldsymbol{a} = a^{k'} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^{k'}} \qquad \boldsymbol{\alpha} = \alpha_{k'} \, \mathbf{d}y^{k'}$$

$$\boldsymbol{A} = A^{k'\dots}_{l'\dots} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^{k'}} \cdots \mathbf{d}y^{l'} \cdots$$

$$a^{k'} = a^{l} \frac{\partial y^{k'}}{\partial x^{l}} \qquad \alpha_{k'} = \alpha_{l} \frac{\partial x^{l}}{\partial y^{k'}}$$

$$A^{m'\dots}_{n'\dots} = A^{k}_{l} \frac{\partial y^{m'}}{\partial x^{k}} \cdots \frac{\partial x^{l}}{\partial y^{n'}} \cdots$$

Metrika, snižování a zvyšování indexů

metrika definuje skalární součin mezi vektory: $(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{q} \cdot \boldsymbol{b}$

$$\boldsymbol{q} = q_{mn} \, \mathbf{d}x^m \, \mathbf{d}x^n \qquad q_{kn} \, q^{ln} = \delta^l_k \qquad q_{kl} = q_{lk} \qquad q^{kl} = q^{lk}$$

metrika definuje korespondenci mezi vektory a 1-formami

$$a_k = q_{kl} \, a^l \qquad a^k = q^{kl} \, a_l$$

Levi-Civitův tenzor

Levi-Civitův tenzor $\boldsymbol{\epsilon}$ je totálně antisymetrický tenzor se zvolenou orientací, normalizovaný na metriku

$\boldsymbol{\epsilon} \in V^0_d$ kde $d = \dim V$

$\boldsymbol{\epsilon} = \mathcal{A}(\boldsymbol{\epsilon})$ tj. $\epsilon_{k_1 \dots k_d} = \epsilon_{[k_1 \dots k_d]}$ antisymetrie

$\epsilon_{k_1 \dots k_d} \, \epsilon_{k_1 \dots k_d} \, q^{k_1 l_1} \cdots q^{k_d l_d} = \pm d!$ normalizace

$\epsilon_{1 \dots d} > 0$ v positivně orientované bázi orientace

$\Rightarrow \quad \epsilon_{1 \dots d} = \pm \sqrt{|\det q_{kl}|}$

∇-operátor – kovariantní derivace

pro lineární souřadnice $\bar{x}^{\bar{k}}$ v afinním prostoru platí: $\nabla \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} \bar{x}^{\bar{k}}} = 0 \qquad \nabla \, \mathbf{d}\bar{x}^{\bar{k}} = 0$

v obecných souřadnicích x^k definujeme $\Gamma^{\,n}_{kl}$: $\nabla \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} = \Gamma^{\,n}_{mk} \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n} \, \mathbf{d}x^m \qquad \nabla \, \mathbf{d}x^k = -\Gamma^{\,k}_{mn} \, \mathbf{d}x^m \, \mathbf{d}x^n$

metrika je konstatní tenzor: $\nabla \boldsymbol{q} = 0$

souřadnice kovariantní derivace

$A^k_{l;m}$ označuje souřadnice tenzoru $\nabla \boldsymbol{A}$, $A^k_{l,m}$ označuje parciální derivace podle x^m

$$f_{;k} = f_{,k} \qquad a^k{}_{;m} = a^k{}_{,m} + \Gamma^{\,k}_{mn} a^n \qquad a_{n;m} = a_{n,m} - \Gamma^{\,k}_{mn} a_k$$

$$A^{k\dots}_{l\dots;m} = A^{k\dots}_{l\dots,m} + \Gamma^{\,k}_{mn} A^{n\dots}_{l\dots} + \dots - \Gamma^{\,n}_{ml} A^{k\dots}_{n\dots} - \cdots$$

$$\Gamma^{\,a}_{kl} = \frac{1}{2} \, q^{an} \big(q_{nk,l} + q_{nl,k} - q_{kl,n} \big) \qquad \Gamma^{\,a}_{kl} = \frac{\partial^2 \bar{x}^{\bar{n}}}{\partial x^k \partial x^l} \frac{\partial x^a}{\partial \bar{x}^{\bar{n}}} \qquad \Gamma^{\,a}_{kl} = -\frac{\partial \bar{x}^{\bar{b}}}{\partial x^k} \frac{\partial^2 x^a}{\partial \bar{x}^{\bar{b}} \partial \bar{x}^{\bar{c}}} \frac{\partial \bar{x}^{\bar{c}}}{\partial x^l}$$

Ortogonální souřadnice ($d = 3$)

ortogonální souřadnice x^m mají na sebe kolmé souřadnicové čáry, tj. q_{kl} je diagonální
v takovém případě zavádíme Lamého koeficienty: $h_k = \sqrt{q_{kk}}$
ve výrazech s Lamého koeficienty se nepoužívá sčítací konvence!

Normalizovaná báze vektorů a 1-forem

$$\boldsymbol{e}_{\hat{k}} = \frac{1}{h_k} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \qquad |\boldsymbol{e}_{\hat{k}}| = 1 \qquad \left| \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \right| = h_k$$

$$\boldsymbol{e}^{\hat{k}} = h_k \, \mathbf{d}x^k \qquad |\boldsymbol{e}^{\hat{k}}| = 1 \qquad \left| \mathbf{d}x^k \right| = \frac{1}{h_k}$$

Metrika a objemový element

$$\boldsymbol{q} = h_1^2 \boldsymbol{e}^{\hat{1}} \boldsymbol{e}^{\hat{1}} + h_2^2 \boldsymbol{e}^{\hat{2}} \boldsymbol{e}^{\hat{2}} + h_3^2 \boldsymbol{e}^{\hat{3}} \boldsymbol{e}^{\hat{3}} \qquad dV = h_1 h_2 h_3 \, dx^1 \, dx^2 \, dx^3$$

Vztah souřadnic s různým typem indexů

$$\boldsymbol{a} = a^n \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n} = a^{\hat{n}} \, \boldsymbol{e}_{\hat{n}} \qquad \boldsymbol{a} = a_n \, \mathbf{d}x^n = a_{\hat{n}} \, \boldsymbol{e}^{\hat{n}}$$

$$h_n \, a^n = a^{\hat{n}} = a_{\hat{n}} = h_n^{-1} \, a_n \qquad \text{nesčítá se!}$$

Levi-Civitův tenzor a vektorové násobení

$$\boldsymbol{\epsilon} = h_1 h_2 h_3 \; \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 = \boldsymbol{e}^{\hat{1}} \wedge \boldsymbol{e}^{\hat{2}} \wedge \boldsymbol{e}^{\hat{3}}$$

$$\epsilon_{123} = h_1 h_2 h_3 \qquad \epsilon_{\hat{1}\hat{2}\hat{3}} = 1$$

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \quad \Rightarrow$$

$$c^1 = \frac{h_2 h_3}{h_1} \left(a^2 b^3 - a^3 b^2 \right) \qquad c^{\hat{1}} = \left(a^{\hat{2}} b^{\hat{3}} - a^{\hat{3}} b^{\hat{2}} \right) \qquad \text{a cyklické záměny}$$

Γ - koeficienty

$$\Gamma^{\ k}_{kn} = \Gamma^{\ k}_{nk} = (\log h_k)_{,n} = h_k^{-1} \, h_{k,n}$$

$$\Gamma^{\ k}_{nn} = -\frac{1}{2} \, h_k^{-2} \, (h_n^2)_{,k} \qquad k \neq n$$

$$\Gamma^{\ k}_{mn} = 0 \qquad k,\, m,\, n \text{ různé}$$

Operátory v ortogonálních souřadnicích

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_k = f_{,k} \qquad a^{\hat{k}} = a_{\hat{k}} = \frac{1}{h_k}\, f_{,k} \qquad a^k = \frac{1}{h_k^2}\, f_{,k}$$

divergence: $f = \nabla \cdot \boldsymbol{a}$

$$\begin{aligned} f &= \frac{1}{h_1 h_2 h_3}\Big(\big(h_1 h_2 h_3\; a^1\big)_{,1} + \big(h_1 h_2 h_3\; a^2\big)_{,2} + \big(h_1 h_2 h_3\; a^3\big)_{,3}\Big) \\ &= \frac{1}{h_1 h_2 h_3}\left(\big(h_2 h_3\; a^{\hat{1}}\big)_{,1} + \big(h_1 h_3\; a^{\hat{2}}\big)_{,2} + \big(h_1 h_2\; a^{\hat{3}}\big)_{,3}\right) \\ &= \frac{1}{h_1 h_2 h_3}\left(\left(\frac{h_2 h_3}{h_1}\, a_1\right)_{,1} + \left(\frac{h_1 h_3}{h_2}\, a_2\right)_{,2} + \left(\frac{h_1 h_2}{h_3}\, a_3\right)_{,3}\right) \end{aligned}$$

rotace: $\boldsymbol{b} = \nabla \times \boldsymbol{a}$

$$\begin{aligned} b^1 &= \frac{1}{h_1 h_2 h_3}\Big(\big(h_3^2\, a^3\big)_{,2} - \big(h_2^2\, a^2\big)_{,3}\Big) && \text{a cyklické záměny} \\ b^{\hat{1}} &= \frac{1}{h_2 h_3}\Big(\big(h_3\, a^{\hat{3}}\big)_{,2} - \big(h_2\, a^{\hat{2}}\big)_{,3}\Big) && \text{a cyklické záměny} \\ b_1 &= \frac{h_1}{h_2 h_3}\big(a_{3,2} - a_{2,3}\big) && \text{a cyklické záměny} \end{aligned}$$

laplace:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left(\left(\frac{h_2 h_3}{h_1}\, f_{,1}\right)_{,1} + \left(\frac{h_1 h_3}{h_2}\, f_{,2}\right)_{,2} + \left(\frac{h_1 h_2}{h_3}\, f_{,3}\right)_{,3}\right)$$