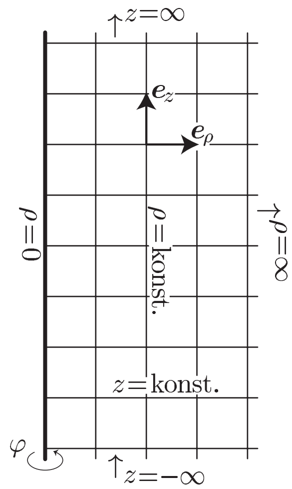
Cylindrické souřadnice $(\rho,\ \varphi,\ z)$



Vztah ke kartézským souřadnicím:

$$x = \rho\cos\varphi \qquad \rho = \sqrt{x^2+y^2}$$
$$y = \rho\sin\varphi \qquad \varphi = \arctan\frac{y}{x}$$
$$z = z \qquad z = z$$

Metrika a objemový element:

$$\boldsymbol{g} = \mathbf{d}\rho\,\mathbf{d}\rho + \rho^2\,\mathbf{d}\varphi\,\mathbf{d}\varphi + \mathbf{d}z\,\mathbf{d}z \qquad dV = \rho\,d\rho\,d\varphi\,dz$$

Laméovy koeficienty:

$$h_\rho = 1 \qquad h_\varphi = \rho \qquad h_z = 1$$

Triáda (normovaná báze):

$$\boldsymbol{e}_\rho = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} \qquad \boldsymbol{e}_\varphi = \frac{1}{\rho}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} \qquad \boldsymbol{e}_z = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z}$$
$$\boldsymbol{e}^\rho = \mathbf{d}\rho \qquad \boldsymbol{e}^\varphi = \rho\,\mathbf{d}\varphi \qquad \boldsymbol{e}^z = \mathbf{d}z$$

Vztah cylindrických a kartézských bází vektorů

$$\boldsymbol{e}_x = \cos\varphi\,\boldsymbol{e}_\rho - \sin\varphi\,\boldsymbol{e}_\varphi = \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_\rho - \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_\varphi$$
$$\boldsymbol{e}_y = \sin\varphi\,\boldsymbol{e}_\rho + \cos\varphi\,\boldsymbol{e}_\varphi = \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_\rho + \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_\varphi$$
$$\boldsymbol{e}_z = \boldsymbol{e}_z$$

$$\boldsymbol{e}_\rho = \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_x + \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_y = \cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y$$
$$\boldsymbol{e}_\varphi = -\frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_x + \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_y = -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y$$
$$\boldsymbol{e}_z = \boldsymbol{e}_z$$

Derivace bázových vektorů a forem

$$\nabla \boldsymbol{e}_\rho = \frac{1}{\rho}\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\varphi \qquad \nabla\cdot\boldsymbol{e}_\rho = \frac{1}{\rho} \qquad \nabla\times\boldsymbol{e}_\rho = 0$$

$$\nabla \boldsymbol{e}_\varphi = -\frac{1}{\rho}\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\rho \qquad \nabla\cdot\boldsymbol{e}_\varphi = 0 \qquad \nabla\times\boldsymbol{e}_\varphi = \frac{1}{\rho}\,\boldsymbol{e}_z$$

$$\nabla \boldsymbol{e}_z = 0 \qquad \nabla\cdot\boldsymbol{e}_z = 0 \qquad \nabla\times\boldsymbol{e}_z = 0$$

Operátory v cylindrických souřadnicích

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_\rho = f_{,\rho} \qquad a_\varphi = \frac{1}{\rho}\,f_{,\varphi} a^\varphi \qquad a_z = f_{,z}$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$f = \frac{1}{\rho}(\rho\,a^\rho)_{,\rho} + \frac{1}{\rho}\,a^\varphi{}_{,\varphi} + a^z{}_{,z}$$

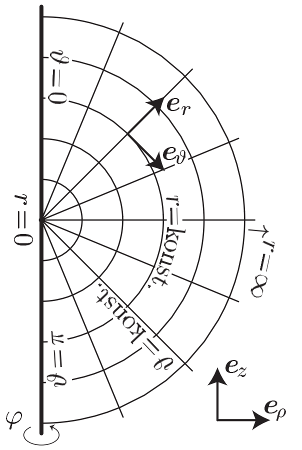
rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

$$b_\rho = \frac{1}{\rho}\,a_{z,\varphi} - a_{\varphi,z} \quad b_\varphi = a_{\rho,z} - a_{z,\rho} \quad b_z = \frac{1}{\rho}\,\left((\rho\,a_\varphi)_{,\rho} - a_{\rho,\varphi}\right)$$

laplace:

$$\nabla^2 f = \frac{1}{\rho}(\rho\,f_{,\rho})_{,\rho} + \frac{1}{\rho^2}f_{,\varphi\varphi} + f_{,zz} = f_{,\rho\rho} + \frac{1}{\rho^2}f_{,\varphi\varphi} + f_{,zz} + \frac{1}{\rho}f_{,\rho}$$

Sférické souřadnice $(r,\ \vartheta,\ \varphi)$



Vztah ke kartézským souřadnicím:

$$x = r \sin\vartheta\, \cos\varphi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r \sin\vartheta\, \sin\varphi \qquad \vartheta = \arctan \frac{\sqrt{x^2 + y^2}}{z}$$

$$z = r \cos\vartheta \qquad \varphi = \arctan \frac{y}{x}$$

$$\rho = r \sin\vartheta \qquad x = \rho\cos\varphi \qquad y = \rho\sin\varphi$$

Metrika a objemový element:

$$\boldsymbol{q} = \mathbf{d}r\, \mathbf{d}r + r^2\, \mathbf{d}\vartheta\, \mathbf{d}\vartheta + r^2 \sin^2\vartheta\ \mathbf{d}\varphi\, \mathbf{d}\varphi \qquad dV = r^2 \sin\vartheta\ dr\, d\vartheta\, d\varphi$$

Laméovy koeficienty:

$$h_r = 1 \qquad h_\vartheta = r \qquad h_\varphi = r\sin\vartheta$$

Triáda (normovaná báze):

$$\boldsymbol{e}_r = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} \qquad \boldsymbol{e}_\vartheta = \frac{1}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} \qquad \boldsymbol{e}_\varphi = \frac{1}{r\sin\vartheta}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\boldsymbol{e}^r = \mathbf{d}r \qquad \boldsymbol{e}^\vartheta = r\,\mathbf{d}\vartheta \qquad \boldsymbol{e}^\varphi = r\sin\vartheta\ \mathbf{d}\varphi$$

Vztah sférických a kartézských bází vektorů

$$\boldsymbol{e}_x = \sin\vartheta\cos\varphi\, \boldsymbol{e}_r + \cos\vartheta\cos\varphi\, \boldsymbol{e}_\vartheta - \sin\varphi\, \boldsymbol{e}_\varphi = \frac{x}{r}\, \boldsymbol{e}_r + \frac{z}{r}\frac{x}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_\vartheta - \frac{y}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_\varphi$$

$$\boldsymbol{e}_y = \sin\vartheta\sin\varphi\, \boldsymbol{e}_r + \cos\vartheta\sin\varphi\, \boldsymbol{e}_\vartheta + \cos\varphi\, \boldsymbol{e}_\varphi = \frac{y}{r}\, \boldsymbol{e}_r + \frac{z}{r}\frac{y}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_\vartheta + \frac{x}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_\varphi$$

$$\boldsymbol{e}_z = \cos\vartheta\, \boldsymbol{e}_r - \sin\vartheta\, \boldsymbol{e}_\vartheta = \frac{z}{r}\, \boldsymbol{e}_r - \frac{\sqrt{x^2+y^2}}{r}\, \boldsymbol{e}_\vartheta$$

$$\boldsymbol{e}_r = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\, \boldsymbol{e}_x + y\, \boldsymbol{e}_y + z\, \boldsymbol{e}_z\right) = \sin\vartheta\cos\varphi\, \boldsymbol{e}_x + \sin\vartheta\sin\varphi\, \boldsymbol{e}_y + \cos\vartheta\, \boldsymbol{e}_z$$

$$\boldsymbol{e}_\vartheta = \frac{z}{\sqrt{x^2+y^2+z^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_x + \frac{y}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_y - \frac{\sqrt{x^2+y^2}}{z}\, \boldsymbol{e}_z\right)$$

$$= \cos\vartheta\cos\varphi\, \boldsymbol{e}_x + \cos\vartheta\sin\varphi\, \boldsymbol{e}_y - \sin\vartheta\, \boldsymbol{e}_z$$

$$\boldsymbol{e}_\varphi = -\frac{y}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_x + \frac{x}{\sqrt{x^2+y^2}}\, \boldsymbol{e}_y = -\sin\varphi\, \boldsymbol{e}_x + \cos\varphi\, \boldsymbol{e}_y$$

Derivace bázových vektorů a forem

$$\nabla \boldsymbol{e}_r = \frac{1}{r}\,\boldsymbol{e}^\vartheta\,\boldsymbol{e}_\vartheta + \frac{1}{r}\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\varphi \qquad \nabla\cdot\boldsymbol{e}_r = \frac{2}{r} \qquad \nabla\times\boldsymbol{e}_r = 0$$

$$\nabla \boldsymbol{e}_\vartheta = -\frac{1}{r}\,\boldsymbol{e}^\vartheta\,\boldsymbol{e}_r + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\varphi \qquad \nabla\cdot\boldsymbol{e}_\vartheta = \frac{1}{r}\cot\vartheta \qquad \nabla\times\boldsymbol{e}_\vartheta = \frac{1}{r}\,\boldsymbol{e}_\varphi$$

$$\nabla \boldsymbol{e}_\varphi = -\frac{1}{r}\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_r - \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\vartheta \qquad \nabla\cdot\boldsymbol{e}_\varphi = 0 \qquad \nabla\times\boldsymbol{e}_\varphi = -\frac{1}{r}\,\boldsymbol{e}_\vartheta + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}_r$$

Operátory ve sférických souřadnicích

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_r = f_{,r} \qquad a_\vartheta = \frac{1}{r}\,f_{,\vartheta} \qquad a_\varphi = \frac{1}{r\sin\vartheta}\,f_{,\varphi}$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$f = \frac{1}{r^2}(r^2\,a^r)_{,r} + \frac{1}{r\sin\vartheta}\,(\sin\vartheta\,a^\vartheta)_{,\vartheta} + \frac{1}{r\sin\vartheta}\,a^\varphi{}_{,\varphi}$$

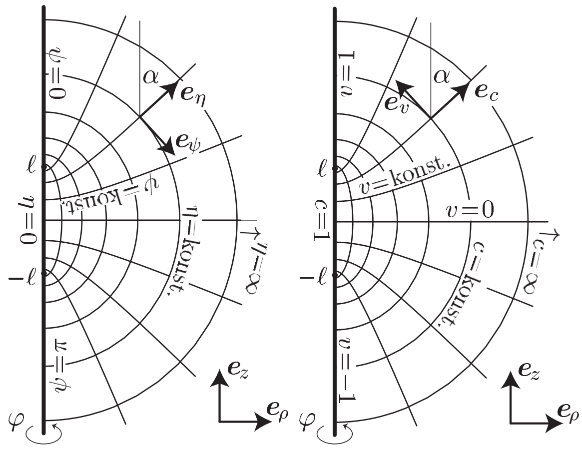
rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

$$b_r = \frac{1}{r\sin\vartheta}\,(\sin\vartheta\,a_\varphi)_{,\vartheta} - a_{\vartheta,\varphi} \qquad b_\vartheta = \frac{1}{r}\left(\frac{1}{\sin\vartheta}\,a_{r,\varphi} - (r\,a_\varphi)_{,r}\right) \qquad b_\varphi = \frac{1}{r}\,\left((r\,a_\vartheta)_{,r} - a_{r,\vartheta}\right)$$

laplace:

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2}\,(r^2\,f_{,r})_{,r} + \frac{1}{r^2\sin\vartheta}\,(\sin\vartheta\,f_{,\vartheta})_{,\vartheta} + \frac{1}{r^2\sin^2\vartheta}\,f_{,\varphi\varphi}\\ &= f_{,rr} + \frac{1}{r^2}f_{,\vartheta\vartheta} + \frac{1}{r^2\sin^2\vartheta}f_{,\varphi\varphi} + \frac{2}{r}f_{,r} + \frac{\cot\vartheta}{r^2}f_{,\vartheta}\end{aligned}$$

Protáhlé elipsoidální souřadnice



Hodnoty souřadnic:

$$\eta \in \mathbb{R}^+ \quad \psi \in \big(0,\pi\big) \qquad\qquad c \in (1,\infty) \quad v \in (-1,1)$$

Vztah ke kartézským souřadnicím:

$$x = \ell \sinh\eta \,\sin\psi\, \cos\varphi \;= \sqrt{c^2-\ell^2}\,\sqrt{1-v^2}\,\cos\varphi$$
$$y = \ell \sinh\eta \,\sin\psi\, \sin\varphi \;= \sqrt{c^2-\ell^2}\,\sqrt{1-v^2}\,\sin\varphi$$
$$z = \ell \cosh\eta \,\cos\psi \;= c\,v$$

$$\cosh\eta = \frac{c}{\ell} = \frac{\sqrt{\sigma^2+r^2+\ell^2}}{\sqrt{2}\,\ell} = \frac{\sqrt{\rho^2+(z+\ell)^2}+\sqrt{\rho^2+(z-\ell)^2}}{2\,\ell} \qquad \sinh\eta = \frac{\sqrt{c^2-\ell^2}}{\ell} = \frac{\sqrt{\sigma^2+r^2-\ell^2}}{\sqrt{2}\,\ell}$$

$$\cos\psi = v = \frac{\sqrt{2}\,z}{\sqrt{\sigma^2+r^2+\ell^2}} = \frac{\sqrt{\rho^2+(z+\ell)^2}-\sqrt{\rho^2+(z-\ell)^2}}{2\,\ell} \qquad \sin\psi = \sqrt{1-v^2} = \frac{\sqrt{\sigma^2-r^2+\ell^2}}{\sqrt{2}\,\ell}$$

$$\tan\varphi = \frac{y}{x} \qquad\qquad x = \rho\cos\varphi \qquad y = \rho\sin\varphi$$

pomocné funkce:

$$\rho^2 = x^2+y^2 = \ell^2\sinh^2\eta\,\sin^2\psi = (c^2-\ell^2)\,(1-v^2)$$
$$r^2 = \rho^2+z^2 = \ell^2\big(\cosh^2\eta-\sin^2\psi\big) = \ell^2\big(\sinh^2\eta+\cos^2\psi\big) = c^2+\ell^2v^2-\ell^2$$
$$\sigma^2 = \sqrt{\rho^2+(z+\ell)^2}\,\sqrt{\rho^2+(z-\ell)^2} = \sqrt{(r^2-\ell^2)^2+4\ell^2\rho^2} = \sqrt{(r^2+\ell^2)^2-4\ell^2z^2}$$
$$= \ell^2\big(\cosh^2\eta-\cos^2\psi\big) = \ell^2\big(\sinh^2\eta+\sin^2\psi\big) = c^2-\ell^2v^2$$

Metrika a objemový element:

$$\boldsymbol{q} = \sigma^2\ \mathbf{d}\eta\,\mathbf{d}\eta + \sigma^2\ \mathbf{d}\psi\,\mathbf{d}\psi + \rho^2\ \mathbf{d}\varphi\,\mathbf{d}\varphi$$
$$= \frac{c^2-\ell^2v^2}{c^2-\ell^2}\ \mathbf{d}c\,\mathbf{d}c + \frac{c^2-\ell^2v^2}{1-v^2}\ \mathbf{d}v\,\mathbf{d}v + (c^2-\ell^2)\,(1-v^2)\ \mathbf{d}\varphi\,\mathbf{d}\varphi$$
$$dV = \rho\sigma^2\,d\eta\,d\psi\,d\varphi = (c^2-\ell^2v^2)\,dc\,dv\,d\varphi$$

Laméovy koeficienty:

$$h_\eta = \sigma \qquad h_\psi = \sigma \qquad h_\varphi = \rho$$
$$h_c^2 = \frac{c^2-\ell^2v^2}{c^2-\ell^2} \qquad h_v^2 = \frac{c^2-\ell^2v^2}{1-v^2} \qquad h_\varphi^2 = (c^2-\ell^2)\,(1-v^2)$$

Vztah eliptických a kartézských bází vektorů:

$$\boldsymbol{e}_x = \cos\varphi\,\big(\cos\alpha\,\boldsymbol{e}_\psi + \sin\alpha\,\boldsymbol{e}_\eta\big) - \sin\varphi\,\boldsymbol{e}_\varphi \qquad \boldsymbol{e}_\eta = \boldsymbol{e}_c = \cos\alpha\,\boldsymbol{e}_z + \sin\alpha\,\big(\cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y\big)$$
$$\boldsymbol{e}_y = \sin\varphi\,\big(\cos\alpha\,\boldsymbol{e}_\psi + \sin\alpha\,\boldsymbol{e}_\eta\big) + \cos\varphi\,\boldsymbol{e}_\varphi \qquad -\boldsymbol{e}_\psi = \boldsymbol{e}_v = \sin\alpha\,\boldsymbol{e}_z - \cos\alpha\,\big(\cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y\big)$$
$$\boldsymbol{e}_z = -\sin\alpha\,\boldsymbol{e}_\psi + \cos\alpha\,\boldsymbol{e}_\eta \qquad \boldsymbol{e}_\varphi = -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y$$

kde $\sin\alpha = \dfrac{\ell}{\sigma}\,\cosh\eta\,\sin\psi = \dfrac{c\sqrt{1-v^2}}{\sqrt{c^2-\ell^2v^2}} = \dfrac{\rho}{\sigma}\sqrt{\dfrac{\sigma^2+r^2+\ell^2}{\sigma^2+r^2-\ell^2}}$

$$\cos\alpha = \frac{\ell}{\sigma}\,\sinh\eta\,\cos\psi = \frac{v\sqrt{c^2-\ell^2}}{\sqrt{c^2-\ell^2v^2}} = \frac{z}{\sigma}\sqrt{\frac{\sigma^2+r^2-\ell^2}{\sigma^2+r^2+\ell^2}}$$

Derivace bázových vektorů a forem:

$$\begin{aligned}
\nabla \boldsymbol{e}_\eta &= \frac{1}{\sigma}\coth\eta\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\varphi - \frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\,\boldsymbol{e}^\eta\,\boldsymbol{e}_\psi + \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\,\boldsymbol{e}^\psi\,\boldsymbol{e}_\psi\\
\nabla \boldsymbol{e}_\psi &= \frac{1}{\sigma}\cot\psi\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\varphi - \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\,\boldsymbol{e}^\psi\,\boldsymbol{e}_\eta + \frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\,\boldsymbol{e}^\eta\,\boldsymbol{e}_\eta\\
\nabla \boldsymbol{e}_\varphi &= -\frac{1}{\sigma}\coth\eta\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\eta - \frac{1}{\sigma}\cot\psi\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\psi
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot\boldsymbol{e}_\eta &= \frac{\ell^2}{\sigma^3}\coth\eta\big(2\sinh^2\eta+\sin^2\psi\big) &&= \nabla\cdot\boldsymbol{e}_c = \frac{c\,(2c^2-\ell^2v^2-\ell^2)}{\sigma^3\sqrt{c^2-\ell^2}}\\
\nabla\cdot\boldsymbol{e}_\psi &= \frac{\ell^2}{\sigma^3}\cot\psi\big(\sinh^2\eta+2\sin^2\psi\big) &&= -\nabla\cdot\boldsymbol{e}_v = \frac{v\,(c^2-2\ell^2v^2+\ell^2)}{\sigma^3\sqrt{1-v^2}}\\
\nabla\cdot\boldsymbol{e}_\varphi &= 0 &&= \nabla\cdot\boldsymbol{e}_\varphi = 0
\end{aligned}$$

$$\begin{aligned}
\nabla\times\boldsymbol{e}_\eta &= -\frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\,\boldsymbol{e}_\varphi &&= \nabla\times\boldsymbol{e}_c = -\frac{\ell^2\,v\sqrt{1-v^2}}{\sigma^3}\,\boldsymbol{e}_\varphi\\
\nabla\times\boldsymbol{e}_\psi &= \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\,\boldsymbol{e}_\varphi &&= -\nabla\times\boldsymbol{e}_v = \frac{c\sqrt{c^2-\ell^2}}{\sigma^3}\,\boldsymbol{e}_\varphi\\
\nabla\times\boldsymbol{e}_\varphi &= \frac{1}{\sigma}\cot\psi\,\boldsymbol{e}_\eta - \frac{1}{\sigma}\coth\eta\,\boldsymbol{e}_\psi &&= \nabla\times\boldsymbol{e}_\varphi = \frac{1}{\sigma}\Big(\frac{v}{\sqrt{1-v^2}}\,\boldsymbol{e}_c + \frac{c}{\sqrt{c^2-\ell^2}}\,\boldsymbol{e}_v\Big)
\end{aligned}$$

Operátory v eliptických souřadnicích:

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_\eta = a_c = \frac{1}{\sigma}\,f_{,\eta} = \frac{\sqrt{c^2-\ell^2}}{\sigma}\,f_{,c} \qquad a_\psi = -a_v = \frac{1}{\sigma}\,f_{,\psi} = -\frac{\sqrt{1-v^2}}{\sigma}\,f_{,v} \qquad a_\varphi = \frac{1}{\rho}\,f_{,\varphi} = \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,f_{,\varphi}$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$\begin{aligned}
f &= \frac{1}{\sigma}\,a^\eta_{,\eta} + \frac{1}{\sigma}\,a^\psi_{,\psi} + \frac{1}{\rho}\,a^\varphi_{,\varphi} + \frac{\ell^2}{\sigma^3}\coth\eta\big(2\sinh^2\eta+\sin^2\psi\big)\,a^\eta + \frac{\ell^2}{\sigma^3}\cot\psi\big(\sinh^2\eta+2\sin^2\psi\big)\,a^\psi\\
&= \frac{\sqrt{c^2-\ell^2}}{\sigma}\,a^c_{,c} + \frac{\sqrt{1-v^2}}{\sigma}\,a^v_{,v} + \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,a^\varphi_{,\varphi} + \frac{c\,(2c^2-\ell^2v^2-\ell^2)}{\sigma^3\sqrt{c^2-\ell^2}}\,a^c + \frac{v\,(c^2-2\ell^2v^2+\ell^2)}{\sigma^3\sqrt{1-v^2}}\,a^v
\end{aligned}$$

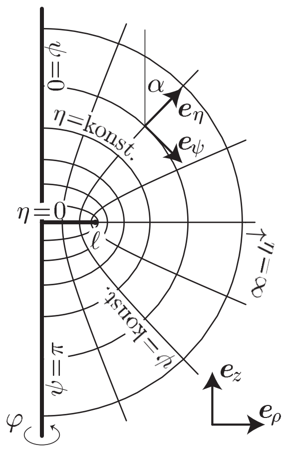
rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

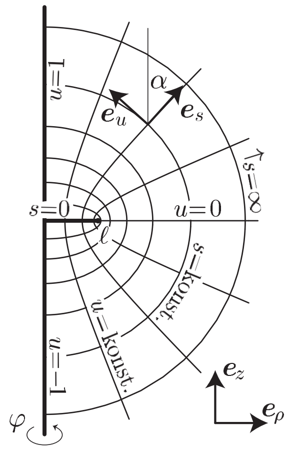
$$\begin{aligned}
b_\eta &= \frac{1}{\sigma}\,a_{\varphi,\psi} - \frac{1}{\rho}\,a_{\psi,\varphi} + \frac{\cot\psi}{\sigma}\,a_\varphi = \quad b_c = \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,a_{v,\varphi} - \frac{\sqrt{1-v^2}}{\sigma}\,a_{\varphi,v} + \frac{v}{\sigma\sqrt{1-v^2}}\,a_\varphi\\
b_\psi &= \frac{1}{\rho}\,a_{\eta,\varphi} - \frac{1}{\sigma}\,a_{\varphi,\eta} - \frac{\coth\eta}{\sigma}\,a_\varphi = -b_v = \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,a_{c,\varphi} - \frac{\sqrt{c^2-\ell^2}}{\sigma}\,a_{\varphi,c} - \frac{c}{\sigma\sqrt{c^2-\ell^2}}\,a_\varphi\\
b_\varphi &= \frac{1}{\sigma}\big(a_{\psi,\eta} - a_{\eta,\psi}\big) + \frac{\ell^2}{\sigma^3}\big(\cosh\eta\sinh\eta\,a_\psi - \cos\psi\sin\psi\,a_\eta\big)\\
&\qquad = \quad b_\varphi = \frac{\sqrt{1-v^2}}{\sigma}\,a_{c,v} - \frac{\sqrt{c^2-\ell^2}}{\sigma}\,a_{v,c} - \frac{c\sqrt{c^2-\ell^2}}{\sigma^3}\,a_v - \frac{\ell^2 v\sqrt{1-v^2}}{\sigma^3}\,a_c
\end{aligned}$$

laplace:

$$\begin{aligned}
\nabla^2 f &= \frac{1}{\sigma^2}\,f_{,\eta\eta} + \frac{1}{\sigma^2}f_{,\psi\psi} + \frac{1}{\rho^2}f_{,\varphi\varphi} + \frac{1}{\sigma^2}\big(\coth\eta\,f_{,\eta} + \cot\psi f_{,\psi}\big)\\
&= \frac{c^2-\ell^2}{\sigma^2}\,f_{,cc} + \frac{1-v^2}{\sigma^2}\,f_{,vv} + \frac{1}{(c^2-\ell^2)(1-v^2)}f_{,\varphi\varphi} + \frac{2}{\sigma^2}\big(c\,f_{,c} - v\,f_{,v}\big)
\end{aligned}$$

Oblé elipsoidální souřadnice





Hodnoty souřadnic:

$$\eta \in \mathbb{R}^+ \quad \psi \in \big(0, \pi\big) \qquad\qquad s \in \mathbb{R}^+ \quad u \in (-1, 1)$$

Vztah ke kartézským souřadnicím:

$$\begin{aligned}
x &= \ell \cosh\eta \, \sin\psi \, \cos\varphi &&= \sqrt{s^2+\ell^2}\,\sqrt{1-u^2}\,\cos\varphi \\
y &= \ell \cosh\eta \, \sin\psi \, \sin\varphi &&= \sqrt{s^2+\ell^2}\,\sqrt{1-u^2}\,\sin\varphi \\
z &= \ell \sinh\eta \, \cos\psi &&= s\,u
\end{aligned}$$

$$\cosh\eta = \frac{\sqrt{s^2+\ell^2}}{\ell} = \frac{\sqrt{r^2+\sigma^2+\ell^2}}{\sqrt{2}\,\ell} = \frac{\sqrt{z^2+(\rho+\ell)^2}+\sqrt{z^2+(\rho-\ell)^2}}{2\,\ell} \qquad \sinh\eta = \frac{s}{\ell} = \frac{\sqrt{r^2+\sigma^2-\ell^2}}{\sqrt{2}\,\ell}$$

$$\sin\psi = \sqrt{1-u^2} = \frac{\sqrt{r^2-\sigma^2+\ell^2}}{\sqrt{2}\,\ell} = \frac{\sqrt{z^2+(\rho+\ell)^2}-\sqrt{z^2+(\rho-\ell)^2}}{2\,\ell} \qquad \cos\psi = u = \frac{\sqrt{2}\,z}{\sqrt{\sigma^2+r^2-\ell^2}}$$

$$\tan\varphi = \frac{y}{x} \qquad\qquad x = \rho\cos\varphi \qquad y = \rho\sin\varphi$$

pomocné funkce:

$$\begin{aligned}
\rho^2 &= x^2+y^2 = \ell^2\cosh^2\eta\,\sin^2\psi = (s^2+\ell^2)\,(1-u^2) \\
r^2 &= \rho^2+z^2 = \ell^2\big(\cosh^2\eta-\cos^2\psi\big) = \ell^2\big(\sinh^2\eta+\sin^2\psi\big) = s^2-\ell^2u^2+\ell^2 \\
\sigma^2 &= \sqrt{z^2+(\rho+\ell)^2}\,\sqrt{z^2+(\rho-\ell)^2} = \sqrt{(r^2-\ell^2)^2+4\ell^2z^2} = \sqrt{(r^2+\ell^2)^2-4\ell^2\rho^2} \\
&= \ell^2\big(\cosh^2\eta-\sin^2\psi\big) = \ell^2\big(\sinh^2\eta+\cos^2\psi\big) = s^2+\ell^2u^2
\end{aligned}$$

Metrika a objemový element:

$$\begin{aligned}
\boldsymbol{q} &= \sigma^2\,\mathbf{d}\eta\,\mathbf{d}\eta + \sigma^2\,\mathbf{d}\psi\,\mathbf{d}\psi + \rho^2\,\mathbf{d}\varphi\,\mathbf{d}\varphi \\
&= \frac{s^2+\ell^2u^2}{s^{+}\ell^2}\,\mathbf{d}s\,\mathbf{d}s + \frac{s^2+\ell^2u^2}{1-u^2}\,\mathbf{d}u\,\mathbf{d}u + (s^2+\ell^2)\,(1-u^2)\,\mathbf{d}\varphi\,\mathbf{d}\varphi
\end{aligned}$$

$$dV = \rho\sigma^2\,d\eta\,d\psi\,d\varphi = (s^2+\ell^2u^2)\,ds\,du\,d\varphi$$

Laméovy koeficienty:

$$h_\eta = \sigma \qquad h_\psi = \sigma \qquad h_\varphi = \rho$$

$$h_s^2 = \frac{s^2+\ell^2u^2}{s^2+\ell^2} \qquad h_u^2 = \frac{s^2+\ell^2u^2}{1-u^2} \qquad h_\varphi^2 = (s^2+\ell^2)\,(1-u^2)$$

Vztah eliptických a kartézských bází vektorů:

$$\begin{aligned}
\boldsymbol{e}_x &= \cos\varphi\,\big(\cos\alpha\,\boldsymbol{e}_\psi+\sin\alpha\,\boldsymbol{e}_\eta\big)-\sin\varphi\,\boldsymbol{e}_\varphi & \boldsymbol{e}_\eta = \boldsymbol{e}_s &= \cos\alpha\,\boldsymbol{e}_z+\sin\alpha\,\big(\cos\varphi\,\boldsymbol{e}_x+\sin\varphi\,\boldsymbol{e}_y\big) \\
\boldsymbol{e}_y &= \sin\varphi\,\big(\cos\alpha\,\boldsymbol{e}_\psi+\sin\alpha\,\boldsymbol{e}_\eta\big)+\cos\varphi\,\boldsymbol{e}_\varphi & -\boldsymbol{e}_\psi = \boldsymbol{e}_u &= \sin\alpha\,\boldsymbol{e}_z-\cos\alpha\,\big(\cos\varphi\,\boldsymbol{e}_x+\sin\varphi\,\boldsymbol{e}_y\big) \\
\boldsymbol{e}_z &= -\sin\alpha\,\boldsymbol{e}_\psi+\cos\alpha\,\boldsymbol{e}_\eta & \boldsymbol{e}_\varphi &= -\sin\varphi\,\boldsymbol{e}_x+\cos\varphi\,\boldsymbol{e}_y
\end{aligned}$$

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$$\begin{aligned}
\sin\alpha &= \frac{\ell}{\sigma}\,\sinh\eta\,\sin\psi = \frac{s\sqrt{1-u^2}}{\sqrt{s^2+\ell^2u^2}} = \frac{\rho}{\sigma}\sqrt{\frac{\sigma^2+r^2-\ell^2}{\sigma^2+r^2+\ell^2}} \\
\cos\alpha &= \frac{\ell}{\sigma}\,\cosh\eta\,\cos\psi = \frac{u\sqrt{s^2+\ell^2}}{\sqrt{s^2+\ell^2u^2}} = \frac{z}{\sigma}\sqrt{\frac{\sigma^2+r^2+\ell^2}{\sigma^2+r^2-\ell^2}}
\end{aligned}$$

Derivace bázových vektorů a forem:

$$\nabla \boldsymbol{e}_\eta = \frac{1}{\sigma}\tanh\eta\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\varphi + \frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\, \boldsymbol{e}^\eta\, \boldsymbol{e}_\psi + \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\, \boldsymbol{e}^\psi\, \boldsymbol{e}_\psi$$

$$\nabla \boldsymbol{e}_\psi = \frac{1}{\sigma}\cot\psi\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\varphi - \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\, \boldsymbol{e}^\psi\, \boldsymbol{e}_\eta - \frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\, \boldsymbol{e}^\eta\, \boldsymbol{e}_\eta$$

$$\nabla \boldsymbol{e}_\varphi = -\frac{1}{\sigma}\tanh\eta\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\eta - \frac{1}{\sigma}\cot\psi\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\psi$$

$$\nabla\cdot \boldsymbol{e}_\eta = \frac{\ell^2}{\sigma^3}\tanh\eta\big(2\cosh^2\eta - \sin^2\psi\big) \quad = \nabla\cdot\boldsymbol{e}_s = \frac{s\,(2s^2+\ell^2u^2+\ell^2)}{\sigma^3\sqrt{s^2+\ell^2}}$$

$$\nabla\cdot \boldsymbol{e}_\psi = \frac{\ell^2}{\sigma^3}\cot\psi\big(\cosh^2\eta - 2\sin^2\psi\big) \quad = -\nabla\cdot\boldsymbol{e}_u = \frac{u\,(s^2+2\ell^2u^2-\ell^2)}{\sigma^3\sqrt{1-u^2}}$$

$$\nabla\cdot \boldsymbol{e}_\varphi = 0 \quad = \nabla\cdot\boldsymbol{e}_\varphi = 0$$

$$\nabla\times \boldsymbol{e}_\eta = \frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\,\boldsymbol{e}_\varphi \quad = \nabla\times\boldsymbol{e}_s = \frac{\ell^2\,u\sqrt{1-u^2}}{\sigma^3}\,\boldsymbol{e}_\varphi$$

$$\nabla\times \boldsymbol{e}_\psi = \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\,\boldsymbol{e}_\varphi \quad = -\nabla\times\boldsymbol{e}_u = \frac{s\sqrt{s^2+\ell^2}}{\sigma^3}\,\boldsymbol{e}_\varphi$$

$$\nabla\times \boldsymbol{e}_\varphi = \frac{1}{\sigma}\cot\psi\,\boldsymbol{e}_\eta - \frac{1}{\sigma}\tanh\eta\,\boldsymbol{e}_\psi \quad = \nabla\times\boldsymbol{e}_\varphi = \frac{1}{\sigma}\Big(\frac{u}{\sqrt{1-u^2}}\,\boldsymbol{e}_s + \frac{s}{\sqrt{s^2+\ell^2}}\,\boldsymbol{e}_u\Big)$$

Operátory v eliptických souřadnicích:

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_\eta = a_s = \frac{1}{\sigma}f_{,\eta} = \frac{\sqrt{s^2+\ell^2}}{\sigma}f_{,s} \qquad a_\psi = -a_u = \frac{1}{\sigma}f_{,\psi} = -\frac{\sqrt{1-u^2}}{\sigma}f_{,u} \qquad a_\varphi = \frac{1}{\rho}f_{,\varphi} = \frac{1}{\sqrt{s^2+\ell^2}\sqrt{1-u^2}}f_{,\varphi}$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$f = \frac{1}{\sigma}a^\eta_{,\eta} + \frac{1}{\sigma}a^\psi_{,\psi} + \frac{1}{\rho}a^\varphi_{,\varphi} + \frac{\ell^2}{\sigma^3}\tanh\eta\big(2\cosh^2\eta - \sin^2\psi\big)\,a^\eta + \frac{\ell^2}{\sigma^3}\cot\psi\big(\cosh^2\eta - 2\sin^2\psi\big)\,a^\psi$$

$$= \frac{\sqrt{s^2+\ell^2}}{\sigma}a^s_{,s} + \frac{\sqrt{1-u^2}}{\sigma}a^u_{,u} + \frac{1}{\sqrt{s^2+\ell^2}\sqrt{1-u^2}}a^\varphi_{,\varphi} + \frac{s\,(2s^2+\ell^2u^2+\ell^2)}{\sigma^3\sqrt{s^2+\ell^2}}a^s + \frac{u\,(s^2+2\ell^2u^2-\ell^2)}{\sigma^3\sqrt{1-u^2}}a^u$$

rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

$$b_\eta = \frac{1}{\sigma}a_{\varphi,\psi} - \frac{1}{\rho}a_{\psi,\varphi} + \frac{\cot\psi}{\sigma}a_\varphi \;=\; b_s = \frac{1}{\sqrt{s^2+\ell^2}\sqrt{1-u^2}}a_{u,\varphi} - \frac{\sqrt{1-u^2}}{\sigma}a_{\varphi,u} + \frac{u}{\sigma\sqrt{1-u^2}}a_\varphi$$

$$b_\psi = \frac{1}{\rho}a_{\eta,\varphi} - \frac{1}{\sigma}a_{\varphi,\eta} - \frac{\tanh\eta}{\sigma}a_\varphi = -b_u = \frac{1}{\sqrt{s^2+\ell^2}\sqrt{1-u^2}}a_{s,\varphi} - \frac{\sqrt{s^2+\ell^2}}{\sigma}a_{\varphi,s} - \frac{s}{\sigma\sqrt{s^2+\ell^2}}a_\varphi$$

$$b_\varphi = \frac{1}{\sigma}\big(a_{\psi,\eta} - a_{\eta,\psi}\big) + \frac{\ell^2}{\sigma^3}\big(\cosh\eta\sinh\eta\,a_\psi + \cos\psi\sin\psi\,a_\eta\big)$$

$$= \; b_\varphi = \frac{\sqrt{1-u^2}}{\sigma}a_{s,u} - \frac{\sqrt{s^2+\ell^2}}{\sigma}a_{u,s} + \frac{\ell^2u\sqrt{1-u^2}}{\sigma^3}a_s - \frac{s\sqrt{s^2+\ell^2}}{\sigma^3}a_u$$

laplace:

$$\nabla^2 f = \frac{1}{\sigma^2}f_{,\eta\eta} + \frac{1}{\sigma^2}f_{,\psi\psi} + \frac{1}{\rho^2}f_{,\varphi\varphi} + \frac{1}{\sigma^2}\big(\tanh\eta\,f_{,\eta} + \cot\psi f_{,\psi}\big)$$

$$= \frac{s^2+\ell^2}{\sigma^2}f_{,ss} + \frac{1-u^2}{\sigma^2}f_{,uu} + \frac{1}{(s^2+\ell^2)(1-u^2)}f_{,\varphi\varphi} + \frac{2}{\sigma^2}\big(s\,f_{,s} - u\,f_{,u}\big)$$