Conformal scalar field

`NTMF059` – Credit assignment 2021

The *Weyl transformation* can be understood as a local rescaling of the metric by a positive function, i.e., replacement of the original metric g_{ab} on the manifold M by new metric $\tilde{g}_{ab}$ on the same manifold defined as

$$g_{ab} \mapsto \tilde{g}_{ab} = \Omega^2 \, g_{ab} \,, \tag{1}$$

where Ω is a real positive function on M called *conformal factor*. The Weyl transformation changes lengths of vectors, but it preserves their mutual angels. It can be useful to express conformal factor in the form

$$\Omega = e^{\omega} \,, \tag{2}$$

where ω is arbitrary function.

The metric g_{ab} is canonically associated with the Levi-Civita covariant derivative ∇_a (torsion-free metric covariant derivative). Analogously, with the conformally rescaled metric $\tilde{g}_{ab}$ we can associate the Levi-Civita covariant derivative $\tilde{\nabla}_a$. The difference tensor Q^a_{bc} inducing $\tilde{\nabla}_b - \nabla_b$ is given by

$$Q^a_{bc} = \delta^a_b \Upsilon_c + \delta^a_c \Upsilon_b - g_{bc} \, g^{ad} \Upsilon_d \,, \tag{3}$$

where

$$\Upsilon_a = d_a \ln \Omega = d_a \omega \,, \tag{4}$$

and g^{ab} is the inverse of the metric g_{ab}.

1. Prove that the Ricci tensor is transformed as

$$\widetilde{\mathrm{Ric}}_{ab} = \mathrm{Ric}_{ab} - (n-2)\nabla_a \nabla_b \, \omega - g_{ab} \, \Box \omega + (n-2) \Upsilon_a \Upsilon_b - (n-2) \, g_{ab} \, \Upsilon_c \Upsilon_d \, g^{cd} \,, \tag{5}$$

   where $\Box \equiv g^{ab} \, \nabla_a \nabla_b$ represents the d'Alembert operator.

2. What is the transformation relation for the scalar curvature $\mathcal{R}$?

The field equations for arbitrary field X are called *conformally invariant*, if there exists a rescaling $\tilde{X} = \Omega^s X$ such that the rescaled field $\tilde{X}$ satisfies the same form of the field equations, however, derived using the rescaled metric $\tilde{g}$. Then, a quantity s (if it exists) is called *conformal weight*.

3. Assume the vacuum Maxwell equations in a spacetime of arbitrary dimension n,

$$\nabla_{[a}F_{bc]} = 0\,, \quad g^{cb}\nabla_c F_{ab} = 0\,, \tag{6}$$

a conformally rescaled field $\tilde{F}_{ab} = \Omega^s F_{ab}$, and a conformal invariance of the Maxwell equations, i.e.,

$$\tilde{\nabla}_{[a}\tilde{F}_{bc]} = 0\,, \quad \tilde{g}^{cb}\tilde{\nabla}_c \tilde{F}_{ab} = 0\,. \tag{7}$$

To satisfy the conformal invariance assumption the spacetime dimension n and conformal weight s have to take specific values. What are these values?

4. Find the transformation of the d'Alembert operator acting on a scalar field ϕ with conformal weight s, i.e., find $\tilde{\Box}\tilde{\phi}$. Prove that the d'Alembert equation $\Box\phi = 0$ is not, in general, for $n > 2$ conformally invariant.

5. Prove that the combination

$$\left[\Box - \frac{n-2}{4(n-1)}\,\mathcal{R}\right]\phi = 0 \tag{8}$$

is conformally invariant for a suitable choice of the conformal weight s of the scalar field ϕ. What is the suitable weight s value?

A solution to the equation (8) is called *conformal scalar field*. Its energy-momentum tensor in the case $n = 4$ can be written as

$$T_{ab} = \frac{2}{3}\,(\nabla_a\phi)(\nabla_b\phi) - \frac{1}{6}g_{ab}\,g^{cd}\,(\nabla_c\phi)(\nabla_d\phi) + \frac{1}{6}\big(\mathrm{Ric}_{ab} - \frac{1}{2}\mathcal{R}g_{ab}\big)\phi^2 - \frac{1}{3}\phi\,\nabla_a\nabla_b\phi + \frac{1}{3}g_{ab}\,\phi\Box\phi\,. \tag{9}$$

6. Find the trace of T_{ab} under the assumption that the field equation (8) is satisfied.

7. Under the same assumption prove that the energy-momentum tensor divergence is vanishing, i.e., $g^{ab}\nabla_a T_{bc} = 0$.