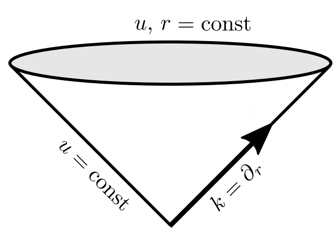
Curvature calculation for the Vaidya spacetime

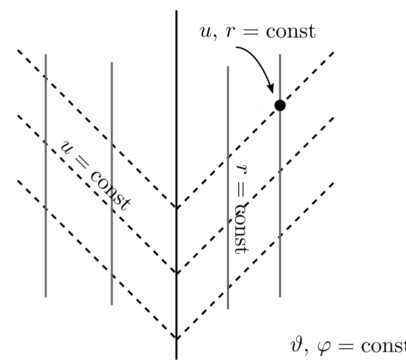
NTMF059 – Credit assignment problem 2022

Vaidya[1] *spacetime* is represented by the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(u)}{r}\right)\mathrm{d}u^2 - \mathrm{d}u\mathrm{d}r - \mathrm{d}r\mathrm{d}u + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{1}$$

where we assume $(-+++)$ signature. Coordinate u in (1) is the retarded time introducing null foliation of the spacetime, r is the affine parameter along null geodesic congruence tangent (and simultaneously normal) to a given null hypersurface $u = \mathrm{const}$ (its generator is $k = \partial_r$), finally, ϑ a φ cover 2-space with $u = \mathrm{const}$ and $r = \mathrm{const}$.





The Vaidya geometry is exact spherically symmetric *non-vacuum* solution[2] to Einstein's gravitational field equations, where the energy-momentum tensor corresponds to the pure radiation and can be written in the form

$$T = \mu(u,r)\,k\,k\,. \tag{2}$$

The Vaidya spacetime is thus non-static generalization of the Schwarzschild (black-hole) solution, see $m(u)$ dependence. Motivation for studies of such a generalization is the ambition to describe elementary model of a radiating star. The density of radiation is then given by the function $\mu(u,r)$. Its relation to the parameters of metric (1) will be clarified in the last part of this homework.

[1] Prahalad Chunnilal Vaidya (1918–2010).

[2] Due to the Birkhoffovu theorem, the only spherically symmetric vacuum solution to general relativity with vanishing cosmological constant is the Schwarzschild spacetime.

1. For spacetime (1), appropriate *frame of metric 1-forms* e^m has to be choosen. We thus require constant components g_{mn} of metric (1) in such a frame, where $g = g_{mn}e^m e^n$. By explicit calculation of g_{mn} show that a reasonable choice is

$$e^0 = \mathrm{d}u\,, \quad e^1 = \mathrm{d}r + \frac{1}{2}\left(1 - \frac{2m(u)}{r}\right)\mathrm{d}u\,, \quad e^2 = r\,\mathrm{d}\vartheta\,, \quad e^3 = r\sin\vartheta\,\mathrm{d}\varphi\,. \tag{3}$$

   Find also the dual basis of vectors e_k.

2. Calsulate the exterior derivative of the frame 1-forms $\mathrm{d}e^m$. The result should be expressed in thers of the wedge products of the original frame 1-forms. In this way you obtain explicit form of the *first Cartan structure equations*, namely

$$\mathrm{d}e^m = -\omega^m{}_n \wedge e^n\,. \tag{4}$$

3. By solving these equation find[3] the connection 1-forms $\omega^m{}_n$.

4. Using the *second Cartan structure equations* find the curvature 2-forms $\Omega^m{}_n$,

$$\Omega^m{}_n = \mathrm{d}\omega^m{}_n + \omega^m{}_k \wedge \omega^k{}_n\,. \tag{5}$$

   Express $\Omega^m{}_n$ in terms of the wedge products of the frame 1-forms.

5. Identify the frame components of the Riemann tensor $R_{kl}{}^m{}_n$,

$$\Omega^m{}_n = \frac{1}{2}R_{kl}{}^m{}_n\, e^k \wedge e^l\,, \tag{6}$$

   where the coefficient $\frac{1}{2}$ compensates normalization of the basis $e^k \wedge e^l$ in the k, l summation.

6. By their contraction find the frame components of the Ricci tensor and then the scalar curvature R.

7. Construct also the coordinate components of the Ricci tensor $R_{\bar{a}\bar{b}}$. Here, the bar denotes original coordinates, i.e., $\bar{a} = u, r, \vartheta, \varphi$.

8. Substituting the result into the non-vacuum Einstein equations $R_{\bar{a}\bar{b}} - \frac{1}{2}Rg_{\bar{a}\bar{b}} = 8\pi T_{\bar{a}\bar{b}}$ explicitly express the function $\mu(u, r)$ entering (2) and encoding the radiation density.

[3]Notice that $\omega_{mn} = \omega_{[mn]}$, where $\omega_{mn} = g_{mk}\omega^k{}_n$.