Geometrical Methods of Theoretical Physics I – NTMF059 – 2023 – tutorials

Exercise: Assume a sphere S^2 covered by the coordinates ϑ and φ. Calculate the Lie brackets $[Y, Z]$ and $[X, Y]$, where the vector fields X, Y, and Z are defined as

$$X = -\sin\varphi\frac{\partial}{\partial\vartheta} - \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\,, \qquad Y = \cos\varphi\frac{\partial}{\partial\vartheta} - \sin\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\,, \qquad Z = \frac{\partial}{\partial\varphi}\,. \tag{1}$$

Exercise: Assume the spherical coordinates

$$x = r\sin\vartheta\cos\varphi\,, \qquad y = r\sin\vartheta\sin\varphi\,, \qquad z = r\cos\vartheta\,. \tag{2}$$

Find the inverse relations and express the related coordinate basis

$$\frac{\partial}{\partial r}\,, \frac{\partial}{\partial\vartheta}\,, \frac{\partial}{\partial\varphi}\,, \qquad \text{in term of} \qquad \frac{\partial}{\partial x}\,, \frac{\partial}{\partial y}\,, \frac{\partial}{\partial z}\,. \tag{3}$$

The vector ∂_φ generates rotation around z axis. Permutation $z\,x\,y \;\rightarrow\; x\,y\,z$ then leads to the rotation generator around x axis. Express such generator using $\{\partial_r\,, \partial_\vartheta\,, \partial_\varphi\}$.

Exercise: Torsion of a dyad derivative in polar coordinates. Assume the 2-dimensional Euclidean metric

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\,, \tag{4}$$

and the orthonormal dyad

$$e^\rho = \mathrm{d}\rho\,,\; e^\varphi = \rho\mathrm{d}\varphi\,, \qquad \text{i.e.}\,, \qquad e_\rho = \frac{\partial}{\partial\rho}\,,\; e_\varphi = \frac{1}{\rho}\frac{\partial}{\partial\varphi}\,, \tag{5}$$

and the associated dyad derivative $\eth e^\rho = 0$ and $\eth e^\varphi = 0$.

Express the second derivative of a function $\eth\eth f$ and their commutator $\eth_a\eth_b f - \eth_b\eth_a f$.

Determine the torsion $t = \mathrm{Tor}[\eth]$ using the above commutator.

Calculate the torsion using $t^j = \mathrm{d}\,e^j$.

Find the torsion using the Lie bracket $t_{kl} = -[e_k,\, e_l]$.

Exercise: Find an explicit form of the vector ξ that is parallelly transported along a circle of constant latitude on S^2, i.e., along a line parametrized by $\vartheta = \vartheta_0 = \text{const}$ and angle φ. The metric on S^2 is given by

$$g = r_0^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\,, \tag{6}$$

and the non-trivial components of the tensor connecting coordinate and Levi-Civita covariant derivative (Christoffel's symbols) are

$$\Gamma^\varphi_{\vartheta\varphi} = \Gamma^\varphi_{\varphi\vartheta} = \cot\vartheta\,, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta\,. \tag{7}$$

The condition for ξ to be parallelly transported along a constant latitude circle is

$$\frac{\nabla}{\mathrm{d}\varphi}\xi = 0\,. \tag{8}$$

Find components of ξ using the coordinate basis ∂_ϑ and ∂_φ, and the normalized dyad $e_{\hat\vartheta}$ and $e_{\hat\varphi}$, respectively,

$$\xi = \xi^\vartheta\frac{\partial}{\partial\vartheta} + \xi^\varphi\frac{\partial}{\partial\varphi} = \xi^{\hat\vartheta}e_{\hat\vartheta} + \xi^{\hat\varphi}e_{\hat\varphi}\,. \tag{9}$$

Exercise: Geodesics on S^2. Explicitly calculate the equations constraining $z(\alpha)$ to be a geodesic,

$$\frac{\nabla}{\mathrm{d}\alpha}\frac{\mathrm{D}\,z}{\mathrm{d}\alpha} = 0\,, \qquad \text{i.e.} \qquad \frac{\mathrm{d}^2\,z^a}{\mathrm{d}\alpha^2} + \Gamma^a_{bc}\frac{\mathrm{d}z^b}{\mathrm{d}\alpha}\frac{\mathrm{d}z^c}{\mathrm{d}\alpha} = 0\,. \tag{10}$$

Find the solution in the form $\vartheta(\alpha)$ and $\varphi(\alpha)$, or $\vartheta(\varphi)$.

Exercise: Geometry on S^3. Assume the metric

$$g = \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\,, \tag{11}$$

and calculate the Christoffel symbols Γ^a_{bc} (i.e., components of the tensor connecting coordinate and Levi-Civita covariant derivatives). Then calculate the divergence $\nabla_a\alpha^a$ with α given as a 1-form

$$\alpha = f(\chi)\sin^{-2}\chi\mathrm{d}\chi\,. \tag{12}$$

Exercise: Geometry of 2-dimensional surfaces. Assume an axially symmetric surface N given in terms of 3-dimensional Euclidean space

$$g_{\mathbb{E}^3} = \mathrm{d}Z^2 + \mathrm{d}\mathrm{P}^2 + \mathrm{P}^2\mathrm{d}\Phi^2 \tag{13}$$

as

$$Z = z(\rho)\,, \qquad \mathrm{P} = \rho\,, \qquad \Phi = \varphi\,. \tag{14}$$

Calculate the metric g restricted on such a surface, i.e., $g = g_{\mathbb{E}^3}|_N$. The result should be

$$g = (1 + z'^2)\mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\,. \tag{15}$$

Consider the Schwarzschild metric

$$g_{Sch} = -\left(1 - \frac{r_0}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{r_0}{r}}\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\,. \tag{16}$$

Find the static equatorial plane geometry, i.e., $t = \text{const}$ and $\vartheta = \frac{\pi}{2}$, and its embedding into $\mathbb{E}^3$.

Assume a 2-dimensional surface N and the metric in the form

$$g = \alpha^2\mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\,, \qquad \text{with} \qquad \alpha = \alpha(\rho)\,. \tag{17}$$

Calculate associated Levi-Civita covariant derivative ∇ on N, i.e., express the Christoffel symbols Γ^a_{bc} determining its relation to the coordinate derivative.

Then, calculate the curvature of ∇ on N, i.e., the Riemann tensor components R_{abcd}, Ricci tensor Ric_{ab}, and the scalar curvature $\mathcal{R} = 2K$.

Find the function α such that the curvature $\mathcal{R}$ will be constant.

Discuss the possible embeddings of such constant curvature geometries into the Euclidean space $\mathbb{E}^3$, i.e., solve $\alpha^2 = 1 + z'^2$ with respect to $z(\rho)$. Various subcases have to be distinguished with respect to the values of integration constants.

Exercise: Prove the identity

$$\nabla_m C_{ab}{}^m{}_c = (d-3)Cot_{cab}\,. \tag{18}$$

Exercise: Prove the identity

$$(p+1)^{[p+1]}\boldsymbol{\delta}^{\boldsymbol{b_0\ldots b_p}}_{\boldsymbol{a_0\ldots a_p}} = \boldsymbol{\delta}^{\boldsymbol{b_0}}_{\boldsymbol{a_0}}{}^{[p]}\boldsymbol{\delta}^{\boldsymbol{b_1\ldots b_p}}_{\boldsymbol{a_1\ldots a_p}} - p\boldsymbol{\delta}^{[\boldsymbol{b_1}}_{\boldsymbol{a_0}}\boldsymbol{\delta}^{|\boldsymbol{b_0}|}_{[\boldsymbol{a_1}}\boldsymbol{\delta}^{\boldsymbol{b_2}}_{\boldsymbol{a_2}}\ldots\boldsymbol{\delta}^{\boldsymbol{b_p}]}_{\boldsymbol{a_p}]} \tag{19}$$

and rewrite it in diagramatic notation. By expressing the first term on the right-hand side, adding this term p-times to both sides of the new equation, and dividing by $p+1$ we arrive at the decomposition

$$^{(1)[p]}\boldsymbol{\delta} = \boldsymbol{A} + \boldsymbol{S}\ . \tag{20}$$

Using diagrammatic notation prove that all terms in this equation are projectors. Note that $\boldsymbol{S}^2 = \boldsymbol{S}$ actually follows also from $\left({}^{(1)[[p]}\boldsymbol{\delta}\right)^2 = {}^{(1)[[p]}\boldsymbol{\delta}$, $\boldsymbol{A}^2 = \boldsymbol{A}$, $\boldsymbol{A}\bullet\boldsymbol{S} = 0$, but the exercises is to prove $\boldsymbol{S}^2 = \boldsymbol{S}$ directly (at least for $p=2$). [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-tenzory%20a%20tenzorova%20pole`, page 12-13.]

Exercise: Prove that the 1-form

$$\boldsymbol{\omega} = -\frac{y}{\rho^2}\mathbf{d}x - \frac{x}{\rho^2}\mathbf{d}y \tag{21}$$

is closed in $\mathbb{E}^2 \setminus \{0\}$ covered by Cartesian coordinates x, y with $\rho^2 = x^2 + y^2$. Show that $\boldsymbol{\omega}$ cannot be exact in $\mathbb{E}^2 \setminus \{0\}$ by calculating $\mathbf{d}\varphi$, where $\tan\varphi = y/x$. [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-vnejsi%20kalkulus`, page 8.]

Exercise: Consider a homogeneous metric on $\mathbb{S}^2$,

$$\boldsymbol{g} = r_0^2\left(\mathbf{d}\vartheta^2 + \sin^2\vartheta\mathbf{d}\varphi^2\right)\ , \tag{22}$$

and calculate its curvature using the Cartan's structure equations

$$\begin{aligned} \mathbf{d}\boldsymbol{e}^k + \boldsymbol{\omega}^k{}_l \wedge \boldsymbol{e}^l = 0\ , \quad \boldsymbol{\omega}^k{}_l = -\boldsymbol{\omega}^l{}_k\ , \\ \boldsymbol{\Omega}^k{}_l = \mathbf{d}\boldsymbol{\omega}^k{}_l + \boldsymbol{\omega}^k{}_j \wedge \boldsymbol{\omega}^j{}_l\ , \quad \boldsymbol{\Omega}^k{}_l = -\boldsymbol{\Omega}^l{}_k\ , \end{aligned} \tag{23}$$

where $\boldsymbol{e}_k$ is the orhtonormal vector frame and $\boldsymbol{e}^k$ its dual coframe. The Riemann tensor is given by

$$\boldsymbol{R_{ab}}^{\boldsymbol{c}}{}_{\boldsymbol{d}} = \boldsymbol{\Omega_{ab}}^k{}_l e_k^{\boldsymbol{c}} e^l_{\boldsymbol{d}}\ . \tag{24}$$

Calculate Riemann tensor with all indices down ${}^\flat\boldsymbol{R}$, the Ricci tensor $\boldsymbol{Ric}$, and the Ricci scalar $\mathcal{R}$. (It is also instructive to find components $R_{\vartheta\varphi}{}^\vartheta{}_\phi$ and $R_{\vartheta\varphi}{}^\varphi{}_\vartheta$.) [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-vnejsi%20kalkulus`, page 9-10.]

Exercise: Consider the embedding $\iota : N \to M$ of a 2-dimensional manifold N covered by coordinates x, y into a 3-dimensional manifold M with flat metric covered by Cartesian coordinates,

$$\boldsymbol{g} = \mathbf{d}X^2 + \mathbf{d}Y^2 + \mathbf{d}Z^2\ . \tag{25}$$

The embedding ι is given by

$$X = x\ , \quad Y = y, \quad Z = \mathcal{Z}(x,y)\ . \tag{26}$$

Calculate its differential $\mathrm{D}\iota$ and determine the induced metric $\boldsymbol{g}|_N = \iota^*\boldsymbol{g}$ for a generic function $\mathcal{Z}(x,y)$. Now, consider the half-sphere $\mathcal{Z} = \sqrt{R_0^2 - x^2 - y^2}$ and show that the calculation can be simplified when done in spherical coordinates. [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-indukovana%20zobrazeni,%20toky%20a%20Lieova%20derivace`, page 11.]

Exercise: Consider the embedding $\iota : N \to M$ of a 2-dimensional manifold N covered by coordinates z, φ into a 3-dimensional manifold M with flat metric covered by cylindrical coordinates,

$$\boldsymbol{g} = \mathbf{d}P^2 + \mathbf{d}Z^2 + P^2\mathbf{d}\Phi^2. \tag{27}$$

The embedding ι is axially symmetric and given by

$$Z = z\ , \quad \Phi = \varphi\ ,, \quad P = \mathcal{P}(z)\ . \tag{28}$$

Calculate the induced metric $\boldsymbol{q} = \boldsymbol{g}|_N = \iota^*\boldsymbol{g}$. Evaluate $\pounds_{\boldsymbol{\partial}_z}\boldsymbol{q}$ and find under which conditions $\boldsymbol{\partial}_z$ is a Killing vector and under which it is a conformal Killing vector. Find the flow u_ξ generated by the vector field $\boldsymbol{\partial}_z$, calculate $U_{\xi*}\boldsymbol{q}$ for the above metric $\boldsymbol{q}$ with the conformal symmetry, and use it to compute $\pounds_{\boldsymbol{\partial}_z}\boldsymbol{q}$ from definition of the Lie derivative. [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-indukovana%20zobrazeni,%20toky%20a%20Lieova%20derivace`, page 12-13.]

Exercise: Consider a homogeneous metric on $\mathbb{S}^2$,

$$\boldsymbol{g} = \mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2\ , \tag{29}$$

and show by calculating $\pounds_{\boldsymbol{X}}\boldsymbol{g}$, $\pounds_{\boldsymbol{Y}}\boldsymbol{g}$, $\pounds_{\boldsymbol{Z}}\boldsymbol{g}$ that the vector fields

$$\begin{aligned} \boldsymbol{X} &= -\sin\varphi\,\boldsymbol{\partial}_\vartheta - \cos\varphi\cot\vartheta\,\boldsymbol{\partial}_\varphi\ ,\\ \boldsymbol{Y} &= \cos\varphi\,\boldsymbol{\partial}_\vartheta - \sin\varphi\cot\vartheta\,\boldsymbol{\partial}_\varphi\ ,\\ \boldsymbol{Z} &= \boldsymbol{\partial}_\varphi\ , \end{aligned} \tag{30}$$

are the Killing vectors of $\boldsymbol{g}$. [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-indukovana%20zobrazeni,%20toky%20a%20Lieova%20derivace`, page 14.]

Exercise: Consider a homogeneous metric on $\mathbb{S}^2$,

$$\boldsymbol{g} = r_0^2\left(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2\right)\ , \tag{31}$$

and determine the Levi-Civita tensor $\boldsymbol{\epsilon}$ from the orthonormal frame. Compute $\pounds_{\boldsymbol{\partial}_\vartheta}\boldsymbol{\epsilon}$ and $\pounds_{\boldsymbol{\partial}_\varphi}\boldsymbol{\epsilon}$. Find a function $f(\vartheta)$ so that $\pounds_{f\boldsymbol{\partial}_\vartheta}\boldsymbol{\epsilon} = 0$. (Notice that $\pounds_{\boldsymbol{\xi}}\boldsymbol{g} = 0 \implies \pounds_{\boldsymbol{\xi}}\boldsymbol{\epsilon} = 0$ but not the other way around.) [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-metricka%20struktura`, page 12.]

Exercise: Consider a homogeneous metric on $\mathbb{S}^3$,

$$\boldsymbol{g} = \mathbf{d}\chi^2 + \sin^2\chi\left(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2\right)\ , \tag{32}$$

and calculate $\boldsymbol{\epsilon}$ and $^\sharp\boldsymbol{\epsilon}$. Let us now consider the 1-form

$$\boldsymbol{\alpha} = f(\chi)\sin^{-2}\chi\,\mathbf{d}\chi \tag{33}$$

and find $*\mathbf{d}*\boldsymbol{\alpha}$. [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-metricka%20struktura`, page 12-13.]

Exercise: Calculate the length of the pseudo-circle $z(\tau)$, $\tau \in (0, \tilde{\tau})$ in 2-dimensional Minkowski spacetime,

$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2\ , \tag{34}$$

that is parametrized by

$$\begin{aligned} t(\tau) &= b\sinh(\tau)\ ,\\ x(\tau) &= b\cosh(\tau)\ . \end{aligned} \tag{35}$$

[The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-metricka%20struktura`, page 13.]

Exercise: Consider a 2-dimensional Minkowski spacetime,

$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2\ , \tag{36}$$

and find all its Killing vectors by solving $\pounds_{\boldsymbol{\xi}}\boldsymbol{g} = 0$ directly in inertial coordinates t, x. Show also, that the boost Killing vector $x\boldsymbol{\partial}_t + t\boldsymbol{\partial}_x$ can be alternatively obtained by rewriting the metric to the accelerated coordinates τ, ρ,

$$\boldsymbol{g} = -\rho^2\mathbf{d}\tau^2 + \mathbf{d}\rho^2\ . \tag{37}$$

Find $\boldsymbol{g}$ and $\boldsymbol{\epsilon}$ in null coordinates,

$$\begin{aligned} u &= t - x \ , \\ v &= t + x \ . \end{aligned} \tag{38}$$

and express $\boldsymbol{\partial}_u$, $\boldsymbol{\partial}_v$ by means of $\boldsymbol{\partial}_t$, $\boldsymbol{\partial}_x$. [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-metricka%20struktura`, page 14-15.]

Exercise: Consider 3-dimensional Minkowski spacetime,

$$\begin{aligned} \boldsymbol{g} &= -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 \ , \\ &= -\mathbf{d}t^2 + \mathbf{d}r^2 + r^2\mathbf{d}\varphi^2 \ , \end{aligned} \tag{39}$$

and determine the induced metrics at pseudospherical surfaces $-t^2 + r^2 = \text{const} < 0$ and $-t^2 + r^2 = \text{const} > 0$. (They represent the hyperbolic 2-space $\mathbb{H}^2$ and the de Sitter spacetime dS_2.) Calculate the subalgebra of $SO(2,1)$ group of symmetries of $\boldsymbol{g}$, i.e., evaluate all Lie brackets of $\boldsymbol{\partial}_\varphi$, $x\boldsymbol{\partial}_t + t\boldsymbol{\partial}_x$, $y\boldsymbol{\partial}_t + t\boldsymbol{\partial}_y$. [The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-metricka%20struktura`, page 16.]

Exercise: An anti-symmetric 2-form $\boldsymbol{\mathcal{F}}$ on the complexification of $\Lambda^2 M$ is called (anti-)self-dual if it satisfies $*\boldsymbol{\mathcal{F}} = \mp\boldsymbol{\mathcal{F}}$, i.e., it is the eigenfunction of $*$ with the eigenvalues $\mp i$. (This is due to the fact that the Hodge dual forms a complex unit, $** = -\text{id}$.) One can see that the real electromagnetic tensor $\boldsymbol{F}$ and its dual $*\boldsymbol{F}$ are related to the self-dual $\boldsymbol{\mathcal{F}}$ and the anti-self-dual $\bar{\boldsymbol{\mathcal{F}}}$ via

$$\begin{aligned} \boldsymbol{F} &= \tfrac{1}{2}\left(\boldsymbol{\mathcal{F}} + \bar{\boldsymbol{\mathcal{F}}}\right) \ , \\ *\boldsymbol{F} &= -\tfrac{i}{2}\left(\boldsymbol{\mathcal{F}} - \bar{\boldsymbol{\mathcal{F}}}\right) \ , \end{aligned} \tag{40}$$

This allows us to rewrite the vacuum Maxwell's equations, $\mathbf{d}\boldsymbol{F} = 0$, $\mathbf{d} * \boldsymbol{F} = 0$, together as $\mathbf{d}\mathcal{F} = 0$. Show that the energy-momentum tensor of the electromagnetic field,

$$\boldsymbol{T_{ab}} = \boldsymbol{F_{am}F_{bn}g^{mn}} - \tfrac{\mathbf{1}}{\mathbf{2}}\boldsymbol{F^2 g_{ab}} \ , \quad F^2 = \boldsymbol{F} \bullet \boldsymbol{F} \ , \tag{41}$$

can be written as

$$\boldsymbol{T_{ab}} = \tfrac{1}{2}\boldsymbol{\mathcal{F}_{am}\bar{\mathcal{F}}_{bn}g^{mn}} \ . \tag{42}$$

[The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-podvariety%20a%20tecne%20distribuce`, page 12.]

Exercise: Consider a 4-dimensional Minkowski spacetime in spherical coordinates,

$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}r^2 + r^2\left(\mathbf{d}\vartheta^2 + \sin^2\vartheta\mathbf{d}\varphi^2\right) \ , \tag{43}$$

and calculate $*\mathbf{d} * \boldsymbol{F}$ for the electromagnetic field given by $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$ with

$$\boldsymbol{A} = -f(r)\mathbf{d}t \ . \tag{44}$$

[The solution is available at `https://utf.mff.cuni.cz/vyuka/NTMF059/2023/poznamky/GM-podvariety%20a%20tecne%20distribuce`, page 13.]