24.4.20

Milí studenti,

Úplně na závěr je začátek výpočtu Riemannova tenzoru obecného sféricky symetrického prostoročasu. Zkuste ho dopočítat se všemi složkami – pro Schwarzschildův prostoročas můžete pak srovnat s výsledky z přednášky z Relativity či třeba s MTW.

Pozdravy a na zdraví

Jiří Bičák

Riemannův tenzor sféricky symetrického prostoročasu

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

fce $\Phi(t,r)$, $\Lambda(t,r)$ zatím obecné

$$\theta^0 = e^{\Phi}dt$$

$$\theta^1 = e^{\Lambda}dr$$

$$\theta^2 = r\,d\theta \qquad g_{ab} = (-1, 1, 1, 1)$$

$$\theta^3 = r\sin\theta\, d\varphi \qquad \Rightarrow \omega_{ab} = -\omega_{ba}$$

$$\boxed{d\theta^a = -\omega^a{}_b \wedge \theta^b}$$

$$d\theta^0 = e^{\Phi}[\dot{\Phi}dt + \Phi' dr]\wedge dt = e^{\Phi}\Phi' dr\wedge dt = -e^{\Phi}\Phi' dt\wedge dr$$

$$d\theta^1 = e^{\Lambda}[\dot{\Lambda}dt + \Lambda' dr]\wedge dr = e^{\Lambda}\dot{\Lambda}\, dt\wedge dr \quad (\circ)$$

$$d\theta^2 = dr\wedge d\theta$$

$$d\theta^3 = dr\wedge\sin\theta\, d\varphi + r\cos\theta\, d\theta\wedge d\varphi$$

$$\Rightarrow \quad d\theta^0 = -\omega^0{}_a\wedge\theta^a \quad \Rightarrow \omega^0{}_1 = +e^{\Phi-\Lambda}\Phi' dt + a\,dr$$

$$d\theta^1 = -\omega^1{}_a\wedge\theta^a = -\omega^1{}_0\wedge\theta^0 \qquad \omega^0{}_2 = 0$$

ale $\underline{\omega^1{}_0} = \omega_{10} = -\omega_{01} = +\underline{\omega^0{}_1} \Rightarrow$

$$d\theta^1 = -[e^{\Phi-\Lambda}\Phi' dt + a\,dr]\wedge e^{\Phi}dt = a e^{\Phi}dt\wedge dr$$

$$(\circ) \Rightarrow a = e^{\Lambda-\Phi}\dot{\Lambda}$$

$\Rightarrow$

$$\omega^0{}_1 = e^{\Phi-\Lambda}\Phi' dt + e^{\Lambda-\Phi}\dot{\Lambda}\, dr$$

$$\omega^0{}_1 = e^{\Phi-\Lambda}\left[\Phi' dt\right] + e^{\Lambda-\Phi}\left[\dot{\Lambda}\, dr\right] = \omega^1{}_0$$

$$d\theta^2 = dr \wedge d\vartheta = e^{-\Lambda}\theta^1 \wedge \frac{1}{r}\theta^2 = \frac{e^{-\Lambda}}{r}\theta^1 \wedge \theta^2 =$$

$$= -\frac{e^{-\Lambda}}{r}\theta^2 \wedge \theta^1$$

$$d\theta^2 = -\omega^2{}_a \wedge \theta^a \Rightarrow \omega^2{}_1 = \frac{e^{-\Lambda}}{r}\theta^2 \Rightarrow$$

$$\omega^2{}_1 = e^{-\Lambda} d\vartheta$$

$$\omega^3{}_1 : \quad d\theta^3 = e^{-\Lambda}\theta^1 \wedge r^{-1}\theta^3 + r\cos\vartheta\, r^{-1}\theta^2 \wedge \frac{1}{r\sin\vartheta}\theta^3$$

$$d\theta^3 = -\frac{e^{-\Lambda}}{r}\theta^3 \wedge \theta^1 - \frac{1}{r}\operatorname{cotg}\vartheta\, \theta^3 \wedge \theta^2$$

$$d\theta^3 = -\omega^3{}_a \wedge \theta^a$$

$$\omega^3{}_1 = \frac{e^{-\Lambda}}{r}\theta^3 = e^{-\Lambda}\sin\vartheta\, d\varphi$$

$$\omega^3{}_2 = \frac{1}{r}\operatorname{cotg}\vartheta\, \theta^3 = \frac{1}{r}\operatorname{cotg}\vartheta\, r\sin\vartheta\, d\varphi = \cos\vartheta\, d\varphi$$

Souhr:

$$\omega^1{}_0 = \omega^0{}_1 = e^{\Phi-\Lambda}\phi' dt + e^{\Lambda-\phi}\dot{\Lambda}\, dr$$

$$= e^{-\Lambda}\phi'\theta^0 + e^{-\phi}\dot{\Lambda}\,\theta^1$$

$$\omega^2{}_1 = \frac{e^{-\Lambda}}{r}\theta^2, \qquad \omega^3{}_1 = \frac{e^{-\Lambda}}{r}\theta^3, \qquad \omega^3{}_2 = \frac{1}{r}\operatorname{cotg}\vartheta\, \theta^3$$

$$\omega^1{}_2 = \omega_{12} = -\omega_{21} = -\omega^2{}_1$$

$$\omega^1{}_3 = -\omega^3{}_1, \quad \omega^2{}_3 = -\omega^3{}_2$$

další jsou 0

$$R_{0101} = - R^{0}{}_{101}$$

$$R_{\alpha\beta\gamma\delta} = e^{(a)}_{\alpha} e^{(b)}_{\beta} e^{(c)}_{\gamma} e^{(d)}_{\delta} R_{abcd}$$

$$R_{0101} = e^{(0)}_{0} e^{(1)}_{1} e^{(0)}_{0} e^{(1)}_{1} R_{0101} =$$

$$= e^{2\Phi} e^{2\Lambda} R_{(0101)} =$$

$$= \ldots$$

Dále se specializuj na Schw. metriku

nebo aspoň $\phi(r)$, $\Lambda(r)$

$$\Omega^0{}_2 = \underbrace{d\omega^0{}_2}_{=0} + \omega^0{}_1 \wedge \omega^1{}_2 = \underbrace{\left(e^{-\Lambda}\phi'\theta^0 + e^{-\phi}\dot{\Lambda}\theta^1\right)}_{\omega^0{}_1} \wedge \left(-\frac{e^{-\Lambda}}{r}\theta^2\right)$$

$$= e^{\phi-\Lambda}[\phi' \underbrace{dt}_{e^{-\phi}\theta_0}] + (e^{\Lambda-\phi})\dot{\Lambda}\underbrace{dr}_{e^{-\Lambda}\theta_1} \wedge \left(-\frac{e^{-\Lambda}}{r}\theta^2\right) =$$

$$-\frac{1}{r}e^{\phi-2\Lambda}\left[\phi' e^{-\phi}\theta^0 \wedge \theta^2 - \dot{\Lambda}e^{-\Lambda}\theta^1 \wedge \theta^2\right]$$

$$= -\frac{1}{r}e^{-2\Lambda}\phi'\theta^0 \wedge \theta^2 + e^{-\Lambda-\phi}\dot{\Lambda}\frac{1}{r}\theta^1 \wedge \theta^2$$

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pro statickou metriku:

$$R^0{}_{202} = -\frac{1}{r}e^{-2\Lambda}\phi'$$

$$\Rightarrow R_{0202} = \frac{1}{r}e^{-2\Lambda}\phi'$$

$$\underbrace{R_{0202}}_{\text{souř.}} = e^{2\phi} r^2 \frac{1}{r} e^{-2\Lambda}\phi' = \underline{r\, e^{2\phi-2\Lambda}\phi'}$$

$$e^{2\phi} = 1 - \frac{2M}{r}, \quad e^{2\Lambda} = \frac{1}{1-\frac{2M}{r}}$$

$$\cancel{2}\, e^{2\phi}\phi' = \frac{\cancel{2}M}{r^2}$$

$$\boxed{\begin{aligned} R_{t\theta t\theta} &= r\frac{M}{r^2}\left(1-\frac{2M}{r}\right) \\ &= \frac{M}{r}\left(1-\frac{2M}{r}\right) \end{aligned}}$$

OK