Summary

Instead of the basis vectors $\vec{e}_{(1)} \dots \vec{e}_{(n)}$, $\vec{e}^{(1)} \dots \vec{e}^{(n)}$
we use independent 1-forms θ^a, $a = 1, \dots, n$
they form basis of all 1-forms, $\theta^a \wedge \theta^b$ of 2forms
etc

connection 1-forms $\omega^a{}_b$
satisfy 1-st Cartan equations

$$\boxed{d\theta^a = -\omega^a{}_b \wedge \theta^b}$$

$ds^2 = g_{ab}\, \theta^a \cdot \theta^b$, if $g_{ab} = \text{const}$,
$\underline{\omega_{ab} + \omega_{ba} = 0}$

frame components
of metric

Curvature 2-forms 2nd Cartan eqs of structure

$$\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b}$$

$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \theta^c \wedge \theta^d$

frame components of Riemann

Identity pro křivost

V tomto Cartanově formalismu vyjdou Bianchiho identity velice jednoduše jako podmínky integrability 2. rovnic struktury

(*) $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$

Z $\Omega^a{}_b$ nyní uděláme vnější diferenciál – přitom potřebujeme, že

$d^2\omega^a{}_b = 0$, a že pro 1-formy α, β platí: $d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$

Je tedy

$$d\Omega^a{}_b = \underbrace{d^2\omega^a{}_b}_{=0} + \underbrace{d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b}_{\text{dosadím z (*) zpětně pomocí } \Omega^a{}_c}$$

$$= (\Omega^a{}_c - \omega^a{}_d \wedge \omega^d{}_c) \wedge \omega^c{}_b - \omega^a{}_c \wedge (\Omega^c{}_b - \omega^c{}_d \wedge \omega^d{}_b)$$

spolu vypadnou

$\Longrightarrow$

$$\boxed{d\Omega^a{}_b = \Omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \Omega^c{}_b}$$

snadno ukážeme, že toto jsou Bianchiho identity: specialisujeme se na souř. basi a Riem. souřadnice, takže

$\omega^\alpha{}_\beta = \underbrace{\Gamma^\alpha{}_{\beta\gamma}}_{=\Gamma^\alpha{}_{\beta\gamma}} \underbrace{dx^\gamma}_{=\theta^\gamma} = 0$ v nějakém pevném vybraném bodě

Pak z předchozího vztahu dostaneme

$$\partial_\varepsilon R^\alpha{}_{\beta\gamma\delta}\, dx^{[\varepsilon} dy^\gamma dz^{\delta]} = 0$$

$\Longleftrightarrow$ $R^\alpha{}_{\beta[\gamma\delta;\varepsilon]} = 0$, tj. Bianchiho identity.

Další identitu dostaneme z 1. rovnic struktury

$d\theta^a = -\omega^a{}_b \wedge \theta^b$ – uděláme-li vnější diferenciály těchto rovnic, zjistíme, že musí platit

$$\underbrace{d^2\theta^a}_{=0} = \underbrace{-d\omega^a{}_b \wedge \theta^b}_{-\Omega^a{}_b \wedge \theta^b + \cancel{X}} + \omega^a{}_b \wedge \underbrace{d\theta^b}_{\cancel{-\omega^b{}_c \wedge \theta^c}}$$

$$\boxed{\Omega^a{}_b \wedge \theta^b = 0}$$

což lze přepsat do tvaru $R_{abcd}\, \theta^b \wedge \theta^c \wedge \theta^d = 0$ – a to není nic jiného než $R_{a[bcd]} = 0$.

Lze dále ukázat, že tato cyklická identita spolu s antisym. Riem. tenzoru v obou párech implikuje: $R_{abcd} = R_{cdab}$

$\Rightarrow$ všechny symetrie Riem. tensoru jsou shrnuty ve dvou identitách, které splňují 2-formy křivosti:

$$\boxed{\Omega_{ab} = -\Omega_{ba}\ , \qquad \Omega^a{}_b \wedge \theta^b = 0.}$$

Důkaz, že z

$$\boxed{\begin{aligned} &\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \\ \text{a} \quad &dg_{ab} = \omega_{ab} + \omega_{ba} \\ \Longrightarrow \quad &\Omega_{ab} = -\Omega_{ba}\,; \end{aligned}}$$

(pracuji jen v indexech $a, b, \ldots$, tj. zvyšuji a snižuji pomocí g_{ab}); užívám jen $d(f\Omega) = df \wedge \Omega + f\, d\Omega$

$$\Omega_{ab} = g_{as}\Omega^s{}_b = g_{as}\, d\omega^s{}_b + \omega_{ac} \wedge \omega^c{}_b$$

Pak

$$\Omega_{ab} + \Omega_{ba} = g_{as}\, d\omega^s{}_b + \omega_{ac} \wedge \omega^c{}_b +$$

$$+ g_{bs}\, d\omega^s{}_a + \omega_{bc} \wedge \omega^c{}_a =$$

$$= d(g_{as}\omega^s{}_b) - dg_{as} \wedge \omega^s{}_b + d(g_{bs}\omega^s{}_a) - dg_{bs} \wedge \omega^s{}_a$$

$$+ \omega_{ac} \wedge \omega^c{}_b + \omega_{bc} \wedge \omega^c{}_a =$$

$$= d\omega_{ab} - (\omega_{as} + \omega_{sa}) \wedge \omega^s{}_b + d\omega_{ba} - (\omega_{bs} + \omega_{sb}) \wedge \omega^s{}_a$$

$$+ \omega_{as} \wedge \omega^s{}_b + \omega_{bs} \wedge \omega^s{}_a =$$

$$= \underbrace{d\omega_{ab} + d\omega_{ba}}_{=0} - \omega_{sa} \wedge \omega^s{}_b - \omega_{sb} \wedge \omega^s{}_a =$$

neb $d^2 g_{ab} = d\omega_{ab} + d\omega_{ba} = 0$

$$= -\omega_{sa} \wedge \omega^s{}_b + \underbrace{\omega^s{}_a \wedge \omega_{sb}}_{= \omega_{sa} \wedge \omega^s{}_b} = 0 \quad \text{q.e.d.}$$

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Élie Cartan

Undoubtedly one of the greatest mathematicians of this century, Élie Cartan's career was nevertheless characterized by a rare harmony of genius and modesty. He was born on April 9, 1869, in Dolomieu (Isère), a village in the south of France. His father was a blacksmith. Cartan's elementary education was made possible by one of the state stipends for gifted children. In 1888 he entered the École Normale Supérieure, where he learned higher mathematics from such masters as Tannery, Picard, Darboux, and Hermite. His research work started with his famous thesis on continuous groups, a subject suggested to him by his fellow student Tresse, recently returned from studying with Sophus Lie in Leipzig. Cartan's first teaching position was at Montpellier, where he was *maître de conférences;* he then went successively to Lyons, to Nancy, and finally in 1909 to Paris. He was made a professor at the Sorbonne in 1912. The report on his work which was the basis for this promotion was written by Poincaré; this was one of the circumstances in his career of which he seemed to have been genuinely proud. He remained at the Sorbonne until his retirement in 1940.

Cartan was an excellent teacher; his lectures were gratifying intellectual experiences, which left the student with a generally mistaken idea that he had grasped all there was on the subject. It is therefore the more surprising that for a long time his ideas did not exert the influence they so richly deserved to have on young mathematicians. This was partly due to Cartan's extreme modesty. But in 1939, the celebration of Cartan's scientific jubilee, J. Dieudonné could rightly say to him: "*. . . vous êtes un 'jeune,' et vous comprenez les jeunes*"—it was then beginning to be true that the young understood Cartan.

In foreign countries, particularly in Germany, his recognition as a great mathematician came earlier. It was perhaps H. Weyl's fundamental papers on group representations published around 1925 that established Cartan's reputation among mathematicians not in his own field. Meanwhile, the development of abstract algebra naturally helped to attract attention to his work on Lie algebra.

Cartan was elected to the French Academy in 1931. In his later years he received several other honors. Thus he was a foreign member of the National Academy of Sciences, U.S.A., and a foreign Fellow of the (British) Royal Society. In 1936 he was awarded an honorary degree by Harvard University.

Closely interwoven with Cartan's life as a scientist and teacher had been his family life, which was filled with an atmosphere of happiness and serenity. He had four children, three sons, Henri, Jean, and Louis, and a daughter, Hélène. Jean Cartan oriented himself towards music, and already appeared to be one of the most gifted composers of his generation, when he was cruelly taken by death. Louis Cartan was a physicist; arrested by the Germans at the beginning of the Resistance, he was murdered by them after a long period of detention. Henri Cartan followed in the footsteps of his father to become a leading mathematician.

Cartan's mathematical work can be roughly classified under three main headings: group theory, systems of differential equations, and geometry. These themes are, however, constantly interwoven with each other in his work. Almost everything Cartan did is more of less connected with the theory of Lie groups.

Sophus Lie introduced the groups of transformations which were named after him. The idea of considering the *abstract group* which underlies a given group of transformations came only later; it appears quite explicity in the first paper by Cartan. Whereas, for Lie, the problem of classification consisted in finding all possible transformation groups on a given number of variables (a far too difficult problem in the present stage of mathematics as soon as the number of variables is not very small), for Cartan the problem was to find all possible abstract structures of continuous groups. He solved the problem completely for "simple" groups (those having no proper normal subgroups). Once the structures of all simple groups were known, it became possible to look for all possible realizations of any one of these structures by transformations of a specific nature, and,

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in particular, for their realizations as groups of linear transformations. This is the problem of the determination of the representations of a given group; it was solved completely by Cartan for simple groups. The solution led in particular to the discovery, as early as 1913, of the *spinors*, which were to be rediscovered later in a special case by the physicists.

Cartan also investigated the infinite Lie groups, i.e., the groups of transformations whose operations depend *not* on a finite number of continuous parameters, but on arbitrary functions. In that case, one does not have the notion of the abstract underlying group. Cartan and Vessiot found, at about the same time and independently of each other, a substitute notion which consists in defining when two infinite Lie groups are to be considered as isomorphic. Cartan then proceeded to classify all possible types of non-isomorphic infinite Lie groups.

Cartan also paid much attention to the study of topological properties of groups considered in the large. He showed how many of these topological problems may be reduced to purely algebraic questions; by so doing, he discovered the very remarkable fact that many properties of the group in the large may be read from the infinitesimal structure of the group, i.e., are already determined when some arbitrarily small piece of the group is given. His work along these lines resembles that of the paleontologist reconstructing the shape of a prehistoric animal from the peculiarities of some small bone.

The idea of studying the abstract structure of mathematical objects which hides itself beneath the analytical clothing was also the mainspring of Cartan's theory of differential systems. He insisted on having a theory of differential equations which is invariant under arbitrary changes in variables. Only in this way can the theory uncover the specific properties of the objects one studies by means of the differential equations they satisfy, in contradistinction to what depends only on the particular representation of these objects by numbers or sets of numbers. In order to achieve such an invariant theory, Cartan made a systematic use of the notion of the *exterior differential* of a differential form, a notion which he helped to create and which has the required property of being invariant with respect to any change of variables.

Raised in the French geometrical tradition, Cartan had a constant interest in differential geometry. He had the unusual combination of a vast knowledge of Lie groups, a theory of differential systems whose invariant character was particularly suited for geometrical investigations, and, most important of all, a remarkable geometrical intuition. As a result, he was able to see the geometrical content of very complicated calculations, and even to substitute geometrical arguments for some of the computations.

In the 1920's the general theory of relativity gave a new impulse to differential geometry. This gave rise to a feverish search for spaces with a suitable local structure. The most notable example of such a local structure is a Riemann metric. It can be generalized in various ways, by modifying the form of the integral which defines the arc length in Riemannian geometry (Finsler geometry), by studying only those properties pertaining to the geodesics or paths (geometry of paths of Eisenhart, Veblen, and T. Y. Thomas), by studying the properties of a family of Riemann metrics whose fundamental forms differ from each other by a common factor (conformal geometry), etc. While in all these directions the definition of a parallel displacement is considered to be the major concern, the approach of Cartan to these problems is most original and satisfactory. Again the notion of group plays the central role. Roughly speaking, a generalized space in the sense of Cartan is a space of tangent spaces such that two infinitely near tangent spaces are related by an infinitesimal transformation of a given Lie group. Such a structure is known as a connection.

Besides several books, Cartan published about 200 mathematical papers. His major specialties, in addition to geometry, were group theory and differential equations. Cartan's papers on group theory fall into two categories, distinguished from each other both by the nature of the question treated and by the time at which they were written.

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The papers of the first cycle are purely algebraic in character; they are more concerned with what are now called Lie algebras than with group theory proper. The work of Cartan's second group-theoretic period is concerned with the groups themselves, and not with their Lie algebras, and in general with the global aspect of the group.

For an account of his algebraic discoveries we return to J. H. C. Whitehead once more.*

synovec A. N. Whiteheada

In the years 1897, 1898 Cartan turned his attention from Lie algebras to linear associative algebras. In 1898 he proved the Wedderburn structure theorem [Chapter 28] for algebras over the real and complex fields. The methods which Wedderburn (1908) used in proving his theorem are more suitable than Cartan's to the problems of linear associative algebra. Indeed this paper of Wedderburn's is one of the outstanding contributions to the subject and it is reasonable to associate the theorem with his name. But the fundamental importance of Cartan's paper, which Wedderburn duly acknowledged, should not be forgotten.

During the years 1904 to 1909, there are his papers on infinite transformation groups, as defined by Lie. Such a group is "infinite" in the sense that its general transformation cannot be expressed in terms of a finite set of parameters. In general it is only defined as a local group, or pseudo-group, whose transformations operate on different open subsets of Cartesian space. Finally a (local) group, G, of this kind is defined as the totality of transformations which leave invariant a given set of differential equations. The transformations in G are themselves given by a set D, of differential equations. Since G is infinite the general solution of D is not expressed in terms of a finite set of parameters.

There have been very few, if any, new contributions to the general theory of infinite groups since these papers of Cartan. This is doubtless due to the difficulty of the subject and also to the appearance of temporary finality in Cartan's work. That is to say, there does not seem to have been much hope of greatly extending his theory with the methods which have been available during the last forty years. At the present time the obvious questions are those concerning global infinite groups, acting on n-dimensional manifolds. It may be that a theory of such groups will be constructed on the basis of Cartan's local theory. In this case it would not be surprising if the latter were eventually considered to be his greatest work.

From 1916 onwards Cartan's papers, with one or two exceptions, were on differential geometry, including the theory of generalized spaces and differential geometry in the large. This work on differential geometry would, by itself, have been sufficient to establish Cartan among the leading mathematicians of this half-century. It is remarkable that he embarked on it when he was nearly fifty years old and maintained a steady output of first-class work throughout the subsequent thirty years.

As for Cartan's work on differential equations, probably the best authority is his own 1945 book, *Les systèmes différentiels extérieurs et leurs applications géométriques*. But we now return to the Chern-Chevalley biography for further details of Cartan's geometric work, which is the major theme of the present chapter.†

Einstein's theory of general relativity gave a new impetus to differential geometry. In their efforts to find an appropriate model of the universe, geometers have broadened

* Royal Society of London, *Obituary Notices of Fellows*, *loc. cit.*

† Shing-Shen Chern and C. Chevalley, *loc. cit.*

their horizon from the study of submanifolds in classical spaces (Euclidean, non-Euclidean, projective, conformal, etc.) to that of more general spaces intrinsically defined. The result is an extension of the work of Gauss and Riemann on Riemannian geometry to spaces with a "connection," which may be an affine connection, a Weyl connection, a projective connection, or a conformal connection. In these generalizations, sometimes called non-Riemannian geometry, an important tool is the absolute differential calculus of Ricci and Levi-Civita. The results achieved are of considerable geometric interest. For instance, in the theory of projective connections, developed independently by Cartan, Veblen, Eisenhart, and Thomas, it is shown that when the space has a system of paths defined by a system of differential equations of the second order, a generalized projective geometry can be defined in the space which reduces to ordinary projective geometry when the differential system is that of the straight lines. Numerous other examples can be cited. The problem at this stage is twofold: (1) to give a definition of "geometry" which will include most of the existing spaces of interest; (2) to develop analytic methods for the treatment of the new geometries, it being increasingly clear that the absolute differential calculus is inadequate.

For this purpose Cartan developed what seems to be the most comprehensive and satisfactory program and demonstrated its advantages in a decisive way. This contribution clearly illustrates his geometric insight and we consider it to be the most important among his works on differential geometry. It can be best explained by means of the modern notion of a "fiber bundle."

A *fiber bundle* is merely the generalization of a simpler concept considered earlier in this book (Chapter 7), namely, the idea of a *Cartesian product*. Let us recall that if there are two sets of numbers, $A = \{1, 2, 3\}$ and $B = \{7, 8\}$, then the Cartesian product symbolized by $A \times B$ and read as "*A cross B*" consists of all possible ordered number pairs formed from selecting the first number from A and the second from B. Thus, "A cross B," or $A \times B$, is the set of *six* number pairs, $\{(1, 7), (1, 8), (2, 7), (2, 8), (3, 7), (3, 8)\}$. Again, if A is the set of all real numbers between 0 and 2—that is, the interval $[0, 2]$ on the X-axis in Figure 20.5—and B is the set of all real numbers in the interval $[0, 1]$ on the Y-axis, then $A \times B$ is the shaded rectangle in Figure 20.5.

If A consists of all points of a circle and B of all points of a line segment, then $A \times B$ is the surface of a cylinder. To clarify this, suppose that the equation of the circle A is $x^2 + y^2 = 25$ (Figure 20.6), and that B is the interval $[0, 1]$ on

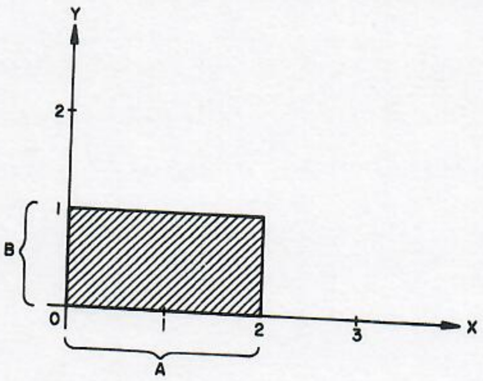

Figure 20.5 $A \times B$ for $A = [0, 2]$ and $B = [0, 1]$

THE NATURE AND GROWTH OF MODERN MATHEMATICS

by

EDNA E. KRAMER

HAWTHORN BOOKS, Inc.
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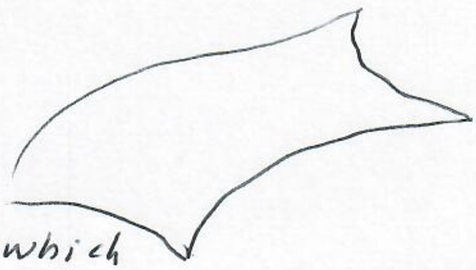
New York

General 2-dimensional surface

One can always introduce
- at least at some region



Gauss polar coordinates in which

$$\boxed{ds^2 = d\rho^2 + [f(\rho,\varphi)]^2 d\varphi^2} \qquad (\sim)$$

for $f = \rho \Rightarrow$ plane cylindrical coordinates

for $f = a \sin\frac{\rho}{a}$, $\rho \equiv a\vartheta$, $a = \text{const}$

$$\Rightarrow \quad ds^2 = a^2 d\vartheta^2 + a \sin^2\vartheta \, d\varphi^2 = a^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2)$$

line element on 2-sphere of radius a

We found last time in detail (see 13) that it is useful to put $\boxed{\theta^1 = d\rho, \quad \theta^2 = f\, d\varphi}$

recall frame comp. $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

the metric $(\sim)$ can thus be written simply as

$$\underline{ds^2 = (\theta^1)^2 + (\theta^2)^2}$$

since $dg_{ab} = \omega_{ab} + \omega_{ba}$ and $g_{ab} = \text{const}$

the only non-vanishing connexion 1-form

is $\omega_{12} = -\omega_{21}$ $\quad (\omega_{11} = \omega_{22} = 0)$

We shall „guess" solution from Cartan's first eqs:

$$d\theta^a = -\omega^a{}_b \wedge \theta^b$$

$a=1$

$$d\theta^1 = -\omega^1{}_1 \wedge \theta^1 - \omega^1{}_2 \wedge \theta^2$$

but $\omega^1{}_1 = g^{1a}\omega_{a1} = g^{11}\underbrace{\omega_{11}}_{=0} = 0$

and $d\theta^1 = d(d\rho) = d^2\rho = 0$

So $\quad 0 = -\omega^1{}_2 \wedge f\, d\varphi$

In order this to be true, $\omega^1{}_2 \sim d\varphi$

for $a=2$,

$$d\theta^2 = -\omega^2{}_1 \wedge \theta^1 \qquad (\omega^2{}_2 = g^{2a}\omega_{a2} = 0)$$

$$d\theta^2 = d(f\, d\varphi) = df \wedge d\varphi + f\underbrace{d^2\varphi}_{=0} =$$

$$= f_{,\rho}\, d\rho \wedge d\varphi + f_{,\varphi}\underbrace{d\varphi \wedge d\varphi}_{=0} = \longrightarrow = f_{,\rho}\, d\rho \wedge d\varphi$$

So

$$\underbrace{f_{,\rho}\, d\rho \wedge d\varphi}_{} = -\omega^2{}_1 \wedge d\rho$$

$$= \frac{f_{,\rho}}{f}\theta^1 \wedge \theta^2 = -\omega^2{}_1 \wedge \theta^1 = +\theta^1 \wedge \omega^2{}_1$$

$$\Rightarrow \boxed{\omega^2{}_1 = \frac{f_{,\rho}}{f}\theta^2}, \quad \Rightarrow \omega_{21} = -\omega_{12} = -\omega^1{}_2 =$$

$$\Rightarrow \omega^1{}_2 = -\frac{f_{,\rho}}{f}\theta^2$$

indeed is $\sim d\varphi$

Curvature 2-form

$$\Omega^2{}_1 = d\omega^2{}_1 + \underbrace{\omega^2{}_1 \wedge \omega^1{}_2}_{\sim\theta^2 \wedge \sim\theta^2 \;=0} + \underbrace{\omega^2{}_2}_{} \wedge \omega^2{}_1$$

$= 0$ because $\omega_{22} = 0$

$$\Rightarrow \Omega^2{}_1 = d\omega^2{}_1 = d(f_{,\rho}\, d\varphi) = f_{,\rho\rho}\, d\rho \wedge d\varphi + \underbrace{\sim d^2\varphi}_{=0}$$

there is no $f_{,\rho\varphi}\, d\varphi$ since $\wedge\, d\varphi = 0$

$$\Rightarrow \Omega^2{}_1 = f_{,\rho\rho} \underbrace{d\rho}_{\theta^1} \wedge \underbrace{d\varphi}_{\frac{1}{f}\theta^2} = \frac{f_{,\rho\rho}}{f}\, \theta^1 \wedge \theta^2$$

$$\boxed{\Omega^2{}_1 = \frac{f_{,\rho\rho}}{f}\, \theta^1 \wedge \theta^2}$$

$$\Omega^2{}_1 = \frac{1}{2} R^2{}_{1cd}\, \theta^c \wedge \theta^d = \frac{1}{2} R^2{}_{112}\, \theta^1 \wedge \theta^2 + \frac{1}{2} R^2{}_{121}\, \theta^2 \wedge \theta^1 = R^2{}_{112}\, \theta^1 \wedge \theta^2$$

$$\Rightarrow \boxed{R^2{}_{112} = \frac{f_{,\rho\rho}}{f}}$$ other components from symmetries

for flat plane $f = \rho$ and, of course, $R^{\cdot}{}_{\cdot\cdot\cdot} = 0$

for a sphere of radius a,

$$R^2{}_{112} \sim \frac{(a \sin\frac{\rho}{a})_{,\rho\rho}}{a \sin\frac{\rho}{a}} = \frac{(\cancel{a}\frac{1}{\cancel{a}} \cos\frac{\rho}{a})_{,\rho}}{a \sin\frac{\rho}{a}} = -\frac{\frac{1}{a}\sin\frac{\rho}{a}}{a \sin\frac{\rho}{a}}$$

so $\boxed{R^2{}_{112} = -\frac{1}{a^2}}$ as it should be, $\boxed{R^2{}_{121} = +\frac{1}{a^2}}$

correct! $a \to \infty$ $R \to 0$

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Vaidya SpTime
(solution)

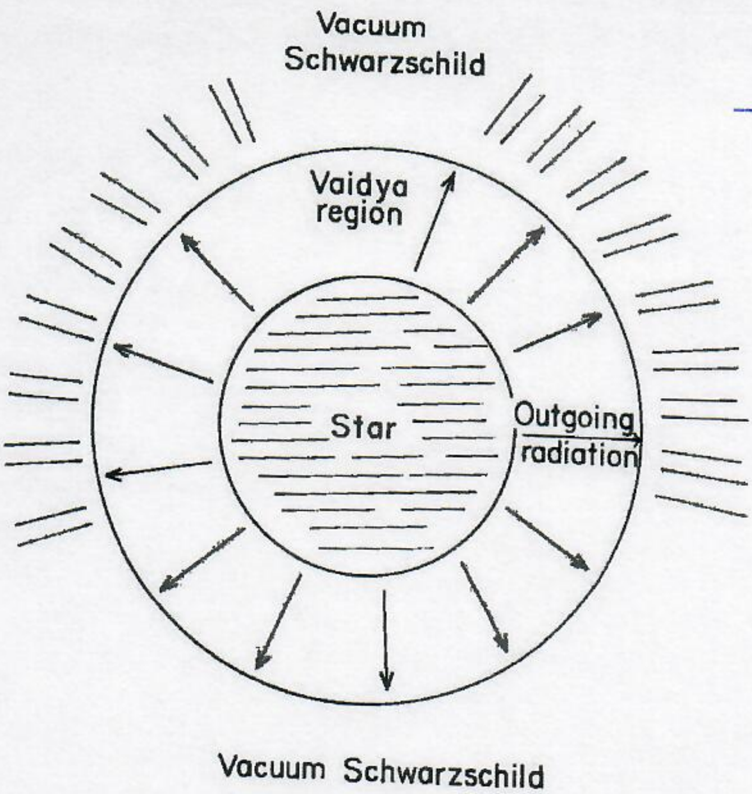

Fig. 19 A schematic diagram for the radiating star configuration. The interior of the star is matched with the Vaidya metric outside which describes the outwards flowing radiation. This is joined smoothly to the vacuum Schwarzschild geometry at the boundary of the radiation zone.

to the body having a charge. In the case of a normal star, the effect of radiation on the overall exterior metric could be considered negligible when effects such as rotation, magnetic fields and so on, are considered which cause perturbations from spherical symmetry. However, the radiation effects would be relevant during the later stages of gravitational collapse when the star would be throwing away considerable mass in the form of radiation or when abundant supply of neutrinos is radiated from a collapsing supernova core (see for example, Kahana, Baron and Cooperstein, 1984).

Such a non-static distribution as the radiating star would then be surrounded by an ever-expanding zone of radiation. One could treat this radiating system, together with its radiation, as forming an isolated system in an otherwise empty, asymptotically flat universe. Then, beyond the zone of pure radiation, the space-time is described by the empty Schwarzschild solution (Fig. 19).

One is thus looking here for a spherically symmetric solution to Einstein equations $G_{ij} = 8\pi T_{ij}$ with the geometrical optics type stress-energy tensor for the radiation with form

$$T_{ij} = \sigma k_i k_j, \tag{3.60}$$

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where k_i is a null vector radially directed outwards. The metric is best given in the null coordinates (u, r, θ, ϕ):

below we shall use signature $+---$!

$$\boxed{ds^2 = -\left(1 - \frac{2m(u)}{r}\right) du^2 - 2dudr + r^2 d\Omega^2,} \tag{3.61}$$

with $m(u)$ being an arbitrary non-increasing function of the retarded time u. (In Section 6.4, where we shall be concerned with the application of Vaidya space-times to examine the cosmic censorship hypothesis, we will consider imploding radiation shells, rather than the outgoing case considered here. Then, the function m will be taken to be non-decreasing and the advanced null coordinate $t + r$ will be used.) The above gives the Vaidya metric in the radiation zone, which is to be matched by the interior metric of the radiating body at the boundary of the star and is matched by the Schwarzschild metric in the exterior.

In the form (3.60) for the energy-momentum tensor, σ is defined to be the energy density of radiation as measured locally by an observer with a four-velocity vector v^i. Thus, σ is the energy flux as well as energy density measured in this frame,

$$\boxed{\sigma \equiv T_{ij} v^i v^j,} \tag{3.62}$$

with $v^i v_i = -1$. Working out the connection coefficients from eqn (3.61), the Ricci tensor in null coordinates is given by

$$R_{ij} = -\frac{2}{r^2} \frac{dm(u)}{du} \delta^0{}_i \delta^0{}_j. \tag{3.63}$$

This implies that the Ricci scalar $R^i_i = R = 0$ and hence the Einstein equations give

$$T_{ij} = -\frac{1}{4\pi r^2} \frac{dm(u)}{du} \delta^0{}_i \delta^0{}_j, \tag{3.64}$$

which is the energy-momentum tensor of a radiating field in the geometric optics form. From eqns (3.62) and (3.64) we get

$$\sigma = -\frac{1}{4\pi r^2} \frac{dm(u)}{du}, \tag{3.65}$$

which is the expression for the energy density of radiation.

In the case when $m(u) =$ const. the relationship of the null coordinates in eqn (3.61) with the Schwarzschild coordinates (t, r, θ, ϕ) is not difficult to see. In such a case, one can use the transformation given by Finkelstein (1958) to diagonalize eqn (3.61)

$$\boxed{u = T - r - 2m \log(r - 2m),} \tag{3.66}$$

this gives the Schwarzschild metric in (T, r, θ, φ) coordinates

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Professor P C Vaidya (* 23.5.1918)

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Veteran Gandhian Mathematician P.C.Vaidya passes away
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*Veteran Gandhian Mathematician P.C.Vaidya passes away
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Veteran Mathematician and Ex-Vice Chancellor of Gujarat University professor Shri Prahalad Chunnilal Vaidya(P.C.Vaidya) -- famous for his Vaidya Metric internationally -- passed away today morning. He was in his early 90s. Internationally renown for his several researches and theories in the field of mathematics, and truly Gandhian PC Vaidya was one of the last educationists of Gandhian generation in Gujarat.

When he was a Vice Chancellor of Gujarat University, he used to ride on a bicycle. Till he was healthy, he used to visit Gujarat University's department that he represented in early days of his career on big black bicycle wearing Gandhi topi. For last several years, Vaidya Saheb was almost limited to his house in Ahmedabad's Shardanagar area due to poor health.

Einstein's theory of gravity is described by a set of rather complicated equations which use the mathematics of Riemannian geometry. It is very difficult to solve these equations, and particularly so to find solutions which describe physically interesting solution. But in 1942 Prahalad Chunnilal Vaidya(PC Vaidya) did pioneering work which led to just such a solution. The Vaidya Metric provided a solution to Einstein's equations which describes the gravitational field of a star which emits a great deal of radiation(search Vaidya Metric in Google and you would be amazed to know what amount of research has been made on the bases of Vaidya Sir's theory).

Being a freedom fighter, Shri Vaidya joined Ahimsak Vyayam Sangh in 1930s. When Vaidya heard Professor V.V.Narlikar(
father of Jayant
Narlikar)'s lecture in 1937 in Mumbai, he wrote a postcard requesting that he wanted to work with him. Shri Narlikar replied positively and invited Vaidya to join him. Thus Shri Vaidya went to Banaras with his wife and 6-month-old child. Vaidya then shifted to Mumbai to work with Shri Homi Bhabha at Tata Institute of Fundamental Research. He then became professor of Mathematics in Vallabh Vidyanagar, then in Visnagar and finally in Ahmedabad. Here he became Vice Chancellor of Gujarat University. Vaidya was instrumental in starting the Gujarat Mathematical Society. He had friendly relation with Shri Vikram Sarabhai in 1960s. A meeting with Sarabhai led to formation of the Vikram Sarabhai Community Science Centre in Ahmedabad where first mathematical laboratory was set up. Shri Vaidya always believed that if could be difficult to teach mathematics but it is certainly not difficult to learn mathematics. He believed that mathematics is something that is in our culture.

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Vaidya ve formách

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$$ds^2 = 2\,du\,dr + \left[1-\frac{2m(u)}{r}\right]du^2 - r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2)$$

chci basi $\Theta^a = e^{(a)}_\alpha dx^\alpha$ tak, že

g_{ab} ve $ds^2 = g_{ab}\Theta^a\Theta^b$ jsou konstanty

$$ds^2 = \underbrace{2\,du}_{\equiv\Theta^0}\underbrace{\left\{dr + \frac{1}{2}\left[1-\frac{2m(u)}{r}\right]du\right\}}_{\equiv\Theta^1} - \underbrace{r^2 d\vartheta^2}_{\equiv(\Theta^2)^2} - \underbrace{r^2\sin^2\vartheta\, d\varphi^2}_{\equiv(\Theta^3)^2}$$

$\Theta^0 \equiv du$

$\Theta^1 \equiv dr + \frac{1}{2}\left[1-\frac{2m(u)}{r}\right]du$

$\Theta^2 \equiv r\,d\vartheta$

$\Theta^3 \equiv r\sin\vartheta\, d\varphi$

$$ds^2 = g_{ab}\Theta^a\Theta^b = 2\Theta^1\Theta^0 - (\Theta^2)^2 - (\Theta^3)^2$$

$\Rightarrow$

$g_{ab} = 0$ až na $\quad g_{10} = g_{01} = 1 = g^{10} = g^{01}$

$-g^{33} = -g^{22} = -g_{22} = -g_{33} = +1 \quad$ i.e. $g_{ab} = g^{ab}$

$d(f\omega) = f\,d\omega + df\wedge\omega$

1. rovnice struktury:

$$\boxed{d\Theta^a = -\omega^a{}_b\wedge\Theta^b}$$

(0) $\boxed{d\Theta^0 = 0}$

(1) $d\Theta^1 = \underbrace{d^2 r}_{=0} + d\left\{\underbrace{\frac{1}{2}\left(1-\frac{2m}{r}\right)}_{\text{skalár}}\underbrace{du}_{\text{1-forma}}\right\} = -d\left(\frac{m}{r}\right)\wedge du = \frac{m}{r^2}\,dr\wedge du$

$$\boxed{= \frac{m}{r^2}\,\Theta^1\wedge\Theta^0}$$

(2) $d\Theta^2 = dr\wedge d\vartheta = \frac{1}{r}\Theta^1\wedge\Theta^2 - \underbrace{\frac{1}{2}\left[1-\frac{2m}{r}\right]du\wedge d\vartheta}$

$$= \frac{1}{2}\left[1-\frac{2m}{r}\right]\frac{1}{r}\,\Theta^0\wedge\Theta^2$$

(2) $\boxed{d\Theta^2 = \frac{1}{r}\Theta^1\wedge\Theta^2 - \frac{1}{2r}\left[1-\frac{2m}{r}\right]\Theta^0\wedge\Theta^2}$

(3) $$\boxed{\begin{aligned} d\theta^3 &= d(r\sin\vartheta\, d\varphi) = \sin\vartheta\, dr\wedge d\varphi + r\cos\vartheta\, d\vartheta\wedge d\varphi = \\ &= r^{-1}\left[\theta^1 - \tfrac{1}{2}\left(1-\tfrac{2m}{r}\right)\theta^0\right]\wedge\theta^3 + r^{-1}\cot\theta\;\theta^2\wedge\theta^3 \end{aligned}}$$

Hádáme řešení (0) – (3):

1) musí $\omega^1{}_0 = 0$, neboť:

$$\omega^1{}_0 = g^{1s}\omega_{s0} = \underbrace{g^{10}}_{=1}\underbrace{\omega_{00}}_{=0} = 0$$

$=0$ neb $\omega_{ab} = -\omega_{ba}$,

neboť $\underbrace{dg_{ab}}_{=0!} = \omega_{ab} + \omega_{ba}$ viz $g_{ab} = \text{const.}$

2) řešení $\boxed{d\theta^1 = \frac{m}{r^2}\,\theta^1\wedge\theta^0}$ (1)

$$d\theta^1 = -\omega^1{}_a\wedge\theta^a \overset{1)}{=} -\omega^1{}_1\wedge\theta^1 - \omega^1{}_2\wedge\theta^2 - \omega^1{}_3\wedge\theta^3$$

=> (1) musí vzniknout z členu $-\omega^1{}_1\wedge\theta^1$, neboť ten obsahuje θ^1 jako jediný, θ^0 není nikde

=>
$$\omega^1{}_1 = \frac{m}{r^2}\,\theta^0\,;$$

a ty další tak, aby členy vypadly

(*) $\omega^1{}_2 = A\theta^2$, $\omega^1{}_3 = B\theta^3$, $\omega^1{}_0 = 0$ viz 1) — zatím neurčené

3) řešení $d\theta^2 = \frac{1}{r}\,\theta^1\wedge\theta^2 - \frac{1}{2r}\left[1-\frac{2m}{r}\right]\theta^0\wedge\theta^2$

$$d\theta^2 = -\omega^2{}_a\wedge\theta^a = -\omega^2{}_0\wedge\theta^0 - \omega^2{}_1\wedge\theta^1 - \omega^2{}_2\wedge\theta^2 - \omega^2{}_3\wedge\theta^3$$

guess:

$$\omega^2{}_1 = \frac{\theta^2}{r},\quad \omega^2{}_2 = g^{2s}\omega_{s2} = 0,\quad \omega^2{}_3 = C\theta^3,\ 2 \ldots$$

4) (3)

$$d\theta^3 = r^{-1}\left[\theta^1 - \tfrac{1}{2}\left(1-\tfrac{2m}{r}\right)\theta^0\right]\wedge\theta^3 + r^{-1}\cot\vartheta\,\theta^2\wedge\theta^3$$

$$d\theta^3 = -\omega^3{}_a\wedge\theta^a = -\omega^3{}_0\wedge\theta^0 - \omega^3{}_1\wedge\theta^1 - \omega^3{}_2\wedge\theta^2 \underbrace{-\omega^3{}_3\wedge\theta^3}$$

$g^{33}\omega_{33} = 0$

$$\omega^3{}_1 = \tfrac{1}{r}\theta^3,\quad \omega^3{}_0 = -\tfrac{1}{2r}\left(1-\tfrac{2m}{r}\right)\theta^3,\quad \omega^3{}_3 \equiv 0$$

$$\omega^3{}_2 = r^{-1}\cot\vartheta\,\theta^3$$

Ostatní ze antisym. ω_{ab}:

$\omega^2{}_0 = g^{25}\omega_{50}$
$= g^{22}\omega_{20} =$
$= -\omega_2$

Např. $\omega^1{}_2 = g^{10}\omega_{02} = -\omega_{20} = -\omega^2{}_0 = \tfrac{1}{2}r^{-1}(1-2m/r)\,\theta^2$ ($g^{22} = -1$)

$$\omega^1{}_3 = -\omega^3{}_0 = \tfrac{1}{2}r^{-1}(1-2m/r)\,\theta^3$$

toto je kompatibilní s odhadem (*)

a umožňuje to fixovat A, B

$$\omega^2{}_3 = -\omega^3{}_2 \text{ fixuje } C$$

Podobně dostane

$$\omega^0{}_1 \equiv 0,\quad \omega^0{}_2 \equiv -\omega^2{}_1 = -r^{-1}\theta^2,$$

$$\omega^0{}_3 = -\omega^3{}_1 = -r^{-1}\theta^3,\quad \omega^0{}_0 = -\omega^1{}_1 = -(m/r^2)\,\theta^0$$

odtud $\Rightarrow$ $\omega^0{}_b\wedge\theta^b = 0$ $\Rightarrow$ $\omega^0{}_b$ splňuje rovnici,

totiž: $d\theta^0 = \omega^0{}_b\wedge\theta^b = 0$

$\Rightarrow$ jednoznačné a správné řešení pro $\omega^a{}_b$

zbytek průmočáry:

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

$$\boxed{\Omega^1{}_1} = d\omega^1{}_1 + \underbrace{\omega^1{}_1 \wedge \omega^1{}_1}_{0} + \omega^1{}_2 \wedge \omega^2{}_1 + \omega^1{}_3 \wedge \omega^3{}_1$$

$$= d\left[\left(\tfrac{m}{r^2}\right)\theta^0\right] + 0 + \tfrac{1}{2} r^{-2}(1-2m/r)\left(\underbrace{\theta^2 \wedge \theta^2}_{=0} + \underbrace{\theta^3 \wedge \theta^3}_{=0}\right)$$

$$= -\left(\frac{2m}{r^3}\right) dr \wedge \theta^0 = \boxed{-\left(\frac{2m}{r^3}\right)\theta^1 \wedge \theta^0}$$

Srovnáním s $\Omega^1{}_1 = \frac{1}{2} R^1{}_{1cd}\, \theta^c \wedge \theta^d$

$\Rightarrow$ tetrádové složky $R\ldots$

$$R^1{}_{141} = -R^1{}_{114} = \frac{2m}{r^3}\,, \quad \text{jiné } R^1{}_{1cd} = 0$$

a podobně najdu.

$$\Omega^1{}_2 = \left(\frac{m}{r^2}\right)\theta^2 \wedge \theta^0 + \left(\frac{m}{r^3}\right)\theta^1 \wedge \theta^2$$

$$\Omega^1{}_3 = \left(\frac{m}{r^2}\right)\theta^3 \wedge \theta^0 + \left(\frac{m}{r^3}\right)\theta^1 \wedge \theta^3$$

$$\Omega^2{}_1 = \left(\frac{m}{r^3}\right)\theta^2 \wedge \theta^0$$

$$\Omega^3{}_1 = \left(\frac{m}{r^3}\right)\theta^3 \wedge \theta^0$$

$$\Omega^2{}_3 = -\left(\frac{2m}{r^3}\right)\theta^2 \wedge \theta^3$$

antisym. $\Omega_{ab} = -\Omega_{ba}$ určí další $R\ldots$

Kontrakcí tetrádové složky Ricciho tenz

jelikož $\Omega'_0 = \Omega_{00} \equiv 0$

$$R_{00} = R^2{}_{002} + R^3{}_{003} = -R'_{202} - R'_{303}$$ (jelikož $\Omega^2{}_0 = -\Omega'_2$)

$$= - \cdots = \frac{2\mathring{m}}{r^2}$$

další $R_{..} = 0$

$$R_{\alpha\beta} = R_{ab}\, e^{(a)}{}_\alpha\, e^{(b)}{}_\beta = R_{00}\, e^{(0)}{}_\alpha\, e^{(0)}{}_\beta =$$

$$= 2\left(\frac{\mathring{m}}{r^2}\right)(\partial_\alpha u)(\partial_\beta u)$$