Lieovy grupy a levoinvariantní pole

Def G je Lieova grupa $\equiv$

G je grupa a diferenc. varieta

$$\left.\begin{array}{lll} G\times G \to G & g,h \to gh \\ G \to G & g \to g^{-1} \\ G \to \{e\} & g \to e \end{array}\right\} \text{ jsou hladké}$$

Pozn: místo trojice zobrazení lze požadovat hladkost zobr.
$G\times G \to G \qquad g,h \to gh^{-1}$

Pozn: hladkost C^0 implikuje existenci analytické struktury
($C^0 \Rightarrow$ anal - těžké - Hilbertův problém, $C^2 \Rightarrow$ anal - snadné)

Def: L_g a R_g jsou levé a pravé násobení (posun) na G $\equiv$

$L_g: G\to G \qquad L_g h = gh$

$R_g: G\to G \qquad R_g h = hg$

AD_g je adjoint zobr. (konjugace) G na G $\equiv$

$AD_g: G\to G \qquad AD_g h = g h g^{-1}$ tj. $AD_g = L_g R_{g^{-1}}$

Lemma: $L_g R_h = R_h L_g \qquad \forall g,h\in G$

$\forall g\in G \quad L_g H = H L_g \Rightarrow \exists h\in G \quad H = R_h$

$\forall g\in G \quad R_g H = H R_g \Rightarrow \exists h\in G \quad H = L_h$

D: nechť $h = He$

$Hg = HL_g e = L_g H e = L_g h = gh = R_h g \Rightarrow H = R_h$

Def: tenz. pole A na G je levoinvariantní $\equiv$

$\forall g\in G \quad L_{g*} A = A$

obdobně pravoinvariant-ost

Def: $l[A] = l_A \in \mathcal{T}^a_b G$ je levoinv. rozmesení tenzoru $A\in \mathbf{T}^a_{e\,b} G$ do G $\equiv$

$l[A]\big|_g = L_{g*} A$

obdobně $r[A] = r_A$ značí pravoinv. rozmesení

Pozn: zvláště pro $m\in \mathbf{T}_e G$ budeme užívat l_m

Lema: levoinv. rozneseni $l[A]$ je levoinvariantní
obdobně pro pravoinv.

D: $(L_{g*} L[A])|_h = L_{g*}(L[A]|_{g^{-1}h}) = L_{g*} L_{g^{-1}h*} A|_e = L_{h*} A|_e = L[A]|_h$

Lema: levo/pravo-invariant-ost komutuje s
$\otimes$, C, $\wedge$, d

D: plyne z vlastností indukc. zobr. (např. $\phi^* d = d\phi^*$)

Lema: $a, b \in TM$ levoinvariantní vekt. pole $\Rightarrow$
$[a,b]$ je levoinvariantní pole
obdobně pro pravoinv.

D: plyne opět z vl. indukc. zobr $\phi_*[a,b] = [\phi_* a, \phi_* b]$

Theorem $A \in T^{\,g}_{\,e}M$ levo/pravo-invariantní pole
je jednoznačně dáno hodnotou $A|_e$ v e a to
levo/pravo-invariantním roznesením $A = l[A|_e]$ resp. $r[A|_e]$

D: $A|_g = (L_{g*}A)|_g = L_{g*}(A|_e) = l[A]|_g$

Lema: nechť E_α, E^α jsou levoinv. báze, $A = A^{\alpha\dots}_{\beta\dots} E_\alpha \dots E^\beta \dots$
A je levoinv. $\Leftrightarrow$ $A^{\alpha\dots}_{\beta\dots}$ jsou konstanty

Pozn: lze definovat levo/pravoinvariantní míru roznosem objem. el. z e
- určená až na násobek konstantou
- často (např. kompaktní, abelovské, souvislé nilpotentní) je biinvariantní
$\to$ Haarova míra

Theorem: $A \in T^{\,g}_{\,e}G$ $g \in G$

$L_{g*}\, l[A] = l[A]$
$R_{g*}\, l[A] = l[AD_{g^{-1}*}A]$
$AD_{g*}\, l[A] = l[AD_{g*}A]$
$r[A]|_g = l[AD_{g^{-1}*}A]|_g$

$R_{g*}\, r[A] = r[A]$
$L_{g*}\, r[A] = r[AD_{g*}A]$
$AD_{g*}\, r[A] = r[AD_{g*}A]$
$l[A]|_g = r[AD_{g*}A]|_g$

D: $(R_{g*}\, l[A])|_h = R_{g*}\, l[A]|_{hg^{-1}} = R_{g*} L_{hg^{-1}*} A = L_h AD_{g^{-1}*}A = l[AD_{g^{-1}*}A]|_h$
$AD_{g*}\, l[A] = L_{g*} R_{g^{-1}*} l[A] = l[AD_{g^{-1}*}A]$
$r[A]|_g = R_{g*}A = R_{g*}\, l[A]|_e = l[AD_{g^{-1}*}A]|_g$ $\Rightarrow$ $l[A]|_g = r[AD_{g*}A]|_g$

Theorem: $A \in T^{\,g}_{\,e}G$ A levoinvariantní
A je bi-invariantní (levo i pravo-inv.) $\Leftrightarrow$
$A|_e$ je AD-invariantní tj. $A|_e = AD_{g*}A|_e$

D: $R_{g*}A = R_{g*}\, l[A|_e] = l[AD_{g^{-1}*}A|_e] = l[A|_e] = A$

Lieova algebra LG a exponenciální zobrazení

Def: $\mathfrak{g}$ je Lieova algebra LG G $\equiv$

$\mathfrak{g} = T_e G$ prostor prvků LA

$[a,b] = [l_a, l_b]|_e$ Lieova závorka na LA

Poz.: LA $\mathfrak{g}$ je isomorfní s LA levoinv. polí se stand. Lieovou závorkou a to skrze levoinv. roz. Vztah $a \to l_a$ $[l_a, l_b] = l_{[a,b]}$

Def: $c_{\alpha\beta}{}^{\gamma}$ je strukturní tenzor LA $\mathfrak{g}$ $\equiv$

$[a,b]^{\gamma} = a^{\alpha} b^{\beta} c_{\alpha\beta}{}^{\gamma}$

$c_{\alpha\beta}{}^{\gamma}$ rozneseme na celou G levoinv. rozšíření

komponenty $c_{\alpha\beta}{}^{\gamma}$ vůči levoinv. bázi E_α nazýv. strukt. konst.

Def: $k_{\alpha\beta}$ je Killingova bi-lin forma (metrika) $\equiv$

$k_{\alpha\beta} = -\frac{1}{2} c_{\alpha\mu}{}^{\nu} c_{\beta\nu}{}^{\mu}$

Poz.: koeficient "$-\frac{1}{2}$" je konvenční. Pro kompaktní prostor bude $k_{\alpha\beta}$ pozitivně definitní.
$k_{\alpha\beta}$ je metrikou (nedegenerovaná) pro poloprosté gr.

Lema: $c_{\alpha\beta}{}^{\gamma}$ a $k_{\alpha\beta}$ jsou levoinvariantní

$[l_a, l_b]^{\gamma} = l_a^{\alpha} l_b^{\beta} c_{\alpha\beta}{}^{\gamma}$

Theorem: $c_{\alpha\beta}{}^{\gamma}$ a $k_{\alpha\beta}$ jsou bi-invariantní

D: $Ad_g a = AD_{g*} a$ viz níže , $AD_g e = e$

$Ad_g [a,b] = Ad_g [l_a, l_b]|_e = [Ad_g l_a, Ad_g l_b]|_e = [l_{Ad_g a}, l_{Ad_g b}]|_e = [Ad_g a, Ad_g b]$

$\Rightarrow AD_{g*} c_{\alpha\beta}{}^{\gamma} = c_{\alpha\beta}{}^{\gamma}$ $\Rightarrow$ $c_{\alpha\beta}{}^{\gamma}$ je bi-invariantní

$\Rightarrow k_{\alpha\beta}$ bi-invariantní

Def: exp je exponenc. zobr $\equiv$

$\exp : \mathfrak{g} \to G$

$\exp((\alpha+\beta)m) = \exp(\alpha m)\exp(\beta m)$

$\exp(0) = e \qquad \frac{D}{d\varepsilon}\exp(\varepsilon m)\big|_{\varepsilon=0} = m$

Pozn.: $\exp(\alpha m)$ tak tvoří 1-param. abel. podgrupu G vycházející z e ve směru m

Theorem

$\exp(\alpha m)$ je integr. kř. polí l_m i r_m procházející e

D: nechť $\tilde{l}_m(\alpha)$, $\tilde{r}_m(\alpha)$ jsou integr. kř. polí l_m, r_m skrze e
tak že $\tilde{l}_m(0) = \tilde{r}_m(0) = e$
ukážeme, že $\tilde{l}_m(\alpha)$ i $\tilde{r}_m(\alpha)$ splňují vlastnosti $\exp(\alpha m)$
Lemma: $\gamma(\alpha)$ je integr. kř. l_m mající společný bod s $\tilde{l}_m(\alpha)$ $\Rightarrow$ $\gamma(\alpha) = \tilde{l}_m(\alpha+\alpha_0)$
plyne z jedn. řešení dif. rovnic prvního řádu

$\tilde{r}_m(0) = e$ - předpoklad $\qquad \frac{D\tilde{r}_m}{d\alpha}\big|_{\alpha=0} = r_m|_e = m$

nechť $k(\alpha) = \tilde{r}_m(\alpha)\tilde{r}_m(\beta) \qquad k(0) = \tilde{r}_m(\beta)$

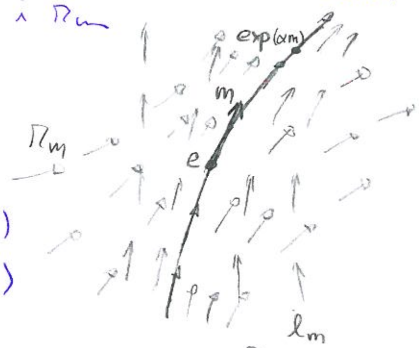
$\frac{Dk}{d\alpha}\big|_{\alpha_0} = R_{\tilde{r}_m(\beta)*}\frac{D\tilde{r}_m}{d\alpha}\big|_{\alpha_0} = R_{\tilde{r}_m(\beta)*}\, r_m|_{\tilde{r}_m(\alpha_0)} = r_m|_{\tilde{r}_m(\alpha_0)\tilde{r}_m(\beta)} = r_m|_{k(\alpha_0)}$

$\Rightarrow k(\alpha)$ integr. kř. r_m $\Rightarrow$ $k(\alpha) = \tilde{r}_m(\alpha+\beta)$ $\Rightarrow$ $\tilde{r}_m(\alpha+\beta) = \tilde{r}_m(\alpha)\tilde{r}_m(\beta)$

$r_m|_{\tilde{l}_m(\alpha_0)} = R_{\tilde{l}_m(\alpha_0)*}\, m = R_{\tilde{l}_m(\alpha_0)*}\frac{D\tilde{l}_m}{d\varepsilon}\big|_{\varepsilon=0} = \frac{D}{d\varepsilon}\left(\tilde{l}_m(\varepsilon)\, l_m(\alpha_0)\right)\big|_{\varepsilon=0} = \frac{D}{d\varepsilon}\tilde{l}_m(\alpha_0+\varepsilon)\big|_{\varepsilon=0} = l_m|_{\tilde{l}_m(\alpha_0)}$

$\Rightarrow$ $\tilde{l}_m(\alpha)$ i $\tilde{r}_m(\alpha)$ jsou int. kř. obou polí l_m i r_m

$\Rightarrow \exp(\alpha m) = \tilde{l}_m(\alpha) = \tilde{r}_m(\alpha)$



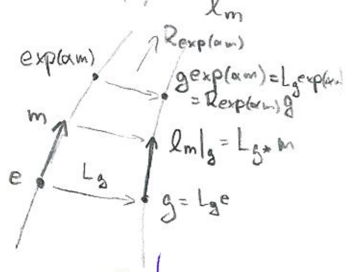
Theorem

l_m je generátor 1-par. gr. diff $R_{\exp(\alpha m)}$

r_m je generátor 1-par. gr. diff $L_{\exp(\alpha m)}$

D: l_m je gener. $R_{\exp(\alpha m)}$ $\Leftrightarrow$ $l_m|_g = \frac{D}{d\varepsilon}R_{\exp(\varepsilon m)}\, g\big|_{\varepsilon=0}$

$\frac{D}{d\varepsilon}R_{\exp(\varepsilon m)}\, g\big|_{\varepsilon=0} = \frac{D}{d\varepsilon}\, g\exp(\varepsilon m)\big|_{\varepsilon=0} = L_{g*}\frac{D}{d\varepsilon}\exp(\varepsilon m)\big|_{\varepsilon=0} = L_{g*}m = l_m|_g$



Lemma: $A \in T^{a}_{b\,e}\,G$

$\mathcal{L}_{l_m} A = -\frac{d}{d\alpha}R_{\exp(\alpha m)*}A\big|_{\alpha=0} \qquad \mathcal{L}_{r_m} A = -\frac{d}{d\alpha}L_{\exp(\alpha m)*}A\big|_{\alpha=0}$

Theorem: $A \in T^{a}_{b\,e}\,G$

A je levoinvar $\Leftrightarrow$ $\forall m \;\; \mathcal{L}_{r_m}A = 0$

A je pravoinvar $\Leftrightarrow$ $\forall m \;\; \mathcal{L}_{l_m}A = 0$

Theorem

$$[l_a, l_b]^\gamma = l_a^\alpha \, l_b^\beta \, c_{\alpha\beta}{}^\gamma = l_{[a,b]}^\gamma$$

$$[l_a, r_b]^\gamma = 0$$

$$[r_a, r_b]^\gamma = - r_a^\alpha \, r_b^\beta \, c_{\alpha\beta}{}^\gamma = - r_{[a,b]}^\gamma$$

důkaz:

$[l_a, l_b] = l_{[a,b]}$ viz výše

$$[l_a, r_b] = \mathcal{L}_{l_a} r_b = - \frac{d}{d\varepsilon} R_{\exp(\varepsilon a)*} \, r_b \Big|_{\varepsilon=0} = - \frac{d}{d\varepsilon} r_b \Big|_{\varepsilon=0} = 0$$

$$[r_a, r_b] = \mathcal{L}_{r_a} r_b = - \frac{d}{d\varepsilon} L_{\exp(\varepsilon a)*} \, r_b \Big|_{\varepsilon=0} = - \frac{d}{d\varepsilon} r\left[\mathrm{Ad}_{\exp(\varepsilon a)} b \right] \Big|_{\varepsilon=0}$$

$$= - r[\mathrm{ad}_a b] \overset{!}{=} - r_{[a,b]}$$ důkaz až na LGA-7

$$r_{[a,b]}^\gamma \big|_g = R_{g*} [a,b]^\gamma = R_{g*} (a^\alpha b^\beta c_{\alpha\beta}{}^\gamma) \underset{\uparrow}{=} r_a^\alpha \, r_b^\beta \, c_{\alpha\beta}{}^\gamma$$ pouze vyjádření pomocí c

bi-invariance $c_{\alpha\beta}{}^\gamma$

Theorem

$$[a,b] = \frac{D}{d\tau} \left[\exp(\sqrt{\tau} a) \exp(\sqrt{\tau} b) \exp(-\sqrt{\tau} a) \exp(-\sqrt{\tau} b) \right] \Big|_{\tau=0}$$

Adjoint reprezentace

Def AD_g je adjoint zobr. (konjugace) G na G ≡

$AD_g : G \to G \qquad AD_g h = g h g^{-1}$

Def Ad_g je adjoint zobr (přidružené) G na $\mathfrak{g}$ ≡

$Ad_g : \mathfrak{g} \to \mathfrak{g} \qquad Ad_g = AD_{g*}$

Lema:

$AD_g = L_g R_{g^{-1}} \qquad Ad_g = L_{g*} R_{g^{-1}*}$

Lema:

$Ad_g m = \frac{D}{d\varepsilon} AD_g h(\varepsilon)\Big|_{\varepsilon=0}$ kde $\frac{Dh}{d\varepsilon}\Big|_{\varepsilon=0} = m$

např. pro $h(\alpha) = \exp(\alpha m)$

Theorem

AD_g je homomorf. gr. G $\qquad$ AD je repres. G na G

Ad_g je homomorf alg. $\mathfrak{g}$ $\qquad$ Ad je repres. G na $\mathfrak{g}$

D: $AD_g(h_1 h_2) = g h_1 h_2 g^{-1} = g h_1 g^{-1} g h_2 g^{-1} = AD_g h_1 \, AD_g h_2$

$Ad_g [a,b] = [Ad_g a, Ad_g b] \Leftarrow C$ je AD-invariantní

$AD_{g_1 g_2} = AD_{g_1} AD_{g_2} \qquad AD_{g^{-1}} = (AD_g)^{-1} \qquad AD_e = id \qquad$ triviální

$Ad_{g_1 g_2} = Ad_{g_1} Ad_{g_2} \qquad Ad_{g^{-1}} = (Ad_g)^{-1} \qquad Ad_e = \mathbb{1} \qquad \Leftarrow Ad_g = AD_{g*}$

Theorem

$$\exp(Ad_g m) = AD_g \exp m$$

D: $AD_g \exp(\alpha m)$ splňuje vlastn. exp:

$AD_g \exp(\alpha m)\big|_{\alpha=0} = e$

$AD_g \exp((\alpha+\beta) m) = AD_g(\exp(\alpha m)\exp(\beta m)) = (AD_g \exp(\alpha m))(AD_g \exp(\beta m))$

$\frac{D}{d\alpha} AD_g \exp(\alpha m)\big|_{\alpha=0} = Ad_g m$

$\Rightarrow AD_g \exp m = \exp(Ad_g m)$

Def: ad_m je adjoint (přidružené) zobr $\mathfrak{g} \to \mathfrak{g}$ $\equiv$

$\equiv$ ad_m je generátor Ad_g tj.

$$ad_m a = \frac{d}{d\varepsilon} Ad_{g_\varepsilon} a \Big|_{\varepsilon=0}$$ kde $\frac{Dg_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = m$ $g_0 = e$

např. $g_\varepsilon = \exp(\varepsilon m)$

Theorem:

$$ad_a b = [a,b] \qquad \text{tj.} \quad {ad_a}^{\kappa}{}_{\underline{\gamma}} = a^{\alpha} c_{\underline{\alpha}\gamma}{}^{\kappa}$$

důkaz:

$$[a,b] = [l_a, l_b]\big|_e = \mathcal{L}_{l_a} l_b\big|_e = \frac{d}{d\varepsilon} R_{\exp(-\varepsilon a)*} l_b\Big|_{\varepsilon=0}\Big|_e = l\left[\frac{d}{d\varepsilon} Ad_{\exp(\varepsilon a)} b\Big|_{\varepsilon=0}\right]\Big|_e = ad_a b$$

$R_{g*} l_b = l[Ad_{g^{-1}} b]$

Theorem: ad je reprezentace $\mathfrak{g}$ na $\mathfrak{g}$ tj.

$$ad_{[a,b]} = [ad_a, ad_b] \equiv ad_a \cdot ad_b - ad_b \cdot ad_a$$

důkaz:

$$ad_{[a,b]} c - [ad_a, ad_b] \cdot c = [[a,b],c] - [a,[b,c]] + [b,[a,c]] =$$
$$= [[a,b],c] + [[b,c],a] + [[c,a],b] \overset{J.I.}{=} 0$$

Theorem:

$$Ad_{\exp a} = \exp(ad_a)$$

kde $\exp M$ pro $M \in \mathfrak{g}^1_1$ splňuje

$\exp(0) = \delta$ $\quad \frac{d}{d\alpha} \exp(\alpha M)\Big|_{\alpha=\alpha_0} = M \cdot \exp(\alpha_0 M)$ $\quad \exp((\alpha+\beta)M)$

a lze napsat jako řadu

$$\exp M = \delta + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$

důkaz: $Ad_{\exp(\alpha a)}$ se musí dokázat vlastnosti $\exp(\alpha ad_a)$

$Ad_{\exp(0a)} = \delta$ $\quad \frac{D}{d\alpha} Ad_{\exp(\alpha a)}\Big|_{\alpha=0} = ad_a$

$Ad_{\exp((\alpha+\beta)m)} = Ad_{\exp(\alpha m)\exp(\beta m)} = Ad_{\exp(\alpha m)} \cdot Ad_{\exp(\beta m)}$

Def:

G je prostá L. gr. $\equiv$

G je nekomutující

G nemá vlastní invariantní podgrupu, tj.

$$\mathrm{Ad}_G S = S \Rightarrow (S = G \vee S = \{e\})$$

G je poloprostá L. gr. $\equiv$

$$G = \bigotimes_{k}{}_{\text{L. gr.}} G_k \qquad G_k \text{ prosté}$$

Def:

$\mathfrak{g}$ je prostá L. alg $\equiv$

$\mathfrak{g}$ je neábelovská

$\mathfrak{g}$ nemá vlastní ideál, tj.

$$[I, \mathfrak{g}] = I \Rightarrow (I = \mathfrak{g} \vee I = \{0\})$$

$\mathfrak{g}$ je poloprostá L. alg. $\equiv$

$$\mathfrak{g} = \bigoplus_{k}{}_{\text{L. alg.}} \mathfrak{g}_k \qquad \mathfrak{g}_k \text{ prosté}$$

Theorem:

G L. gr., $\mathfrak{g}$ její L. alg

G je prostá $\Leftrightarrow$ $\mathfrak{g}$ je prostá

G je poloprostá $\Leftrightarrow$ $\mathfrak{g}$ je poloprostá

C

Theorem

$\mathfrak{g}$ prostá L. alg $\Rightarrow$ adj repr. $\mathcal{C}$ je ireducibilní

$\mathfrak{g}$ poloprostá L. alg $\Rightarrow$ adj. repr. $\mathcal{C}$ je věrná

$\mathfrak{g}$ poloprostá L. alg $\Rightarrow$ Kill. metrika k je nedegenerovaná

Theorem

G poloprostá L. gr., $\mathfrak{g}$ její L. alg., k Killing. metrika

G kompaktní $\Rightarrow$ k positivně definitní

Def:

$\mathfrak{g}$ poloprostá L. alg., k Killing. metrika

$\mathfrak{g}$ je kompaktní L. alg $\equiv$ k je positivně definitní

Def: $\mathrm{Der}\,\mathfrak{g}$ je prostor všech derivací na Lieově alg. $\mathfrak{g}$

lin. oper. δ je (algebraická) derivace na $\mathfrak{g}$ $\equiv$ $\delta[a,b] = [\delta a, b] + [a, \delta b]$

Def: $\mathrm{ad}_{\mathfrak{g}}$ je prostor všech lin. oper. tvaru ad_k $k \in \mathfrak{g}$

Theorem

$\mathfrak{g}$ poloprostá Lieova alg. $\Rightarrow$ $\mathrm{Der}\,\mathfrak{g} = \mathrm{ad}_{\mathfrak{g}}$

Geometrie Lieových grup

Theorem $\overset{L}{E}_\alpha$ a $\overset{R}{E}_\alpha$ levoinv. a pravoinv. báze, $\overset{L}{E}_\alpha|_e = \overset{R}{E}_\alpha|_e = E_\alpha$ báze v $\mathfrak{g} = T_e G$

$k_{\alpha\beta}$ a $c_{\alpha\beta}{}^\kappa$ jsou biinvariantní tenzory na G

$\overset{L}{k}_{\alpha\beta} = \overset{R}{k}_{\alpha\beta} = k_{\alpha\beta}$ jsou konst., kde $\overset{L}{k}_{\alpha\beta}, \overset{R}{k}_{\alpha\beta}$ jsou komponenty vůči $\overset{L}{E}_\alpha$, $\overset{R}{E}_\alpha$

obdobně $\overset{L}{c}_{\alpha\beta}{}^\kappa = \overset{R}{c}_{\alpha\beta}{}^\kappa = c_{\alpha\beta}{}^\kappa$ a $k_{\alpha\beta}$ jsou komponenty k vůči E_α v $\mathfrak{g}$

Theorem:

G poloprosté $\Leftrightarrow$ $k_{\alpha\beta}$ nedegenerovaná

G poloprosté kompaktní $\Rightarrow$ $k_{\alpha\beta}$ poz. definitní

Theorem $c_{\alpha\beta\gamma} = c_{[\alpha\beta\gamma]}$ kde $c_{\alpha\beta\gamma} = c_{\alpha\beta}{}^\mu k_{\mu\gamma}$ D: viz LA

Theorem l_m a r_m jsou Killingovy vektory metriky $k_{\alpha\beta}$

D biinvariantnost $k_{\alpha\beta}$ $\Leftrightarrow$ $\mathcal{L}_{l_m} k = 0$ $\mathcal{L}_{r_m} k = 0$ $\Leftrightarrow$ Killingy

Theorem G poloprostá LG, ε Levi-Civitův tenzor k

ε je biinvariantní, $\varepsilon = |\varepsilon| = \operatorname{Det}^{\frac{1}{2}} k$ je biinv. obj. element

Def

$\overset{L}{\nabla}$ kov. derivace zachovávající levoinv. tenzory

$\overset{R}{\nabla}$ kov. derivace zachovávající pravoinv. tenzory

Theorem

$\overset{L}{\nabla}, \overset{R}{\nabla}$ existují a jsou dány jednoznačně

torze $\overset{L}{T}_{\mu\nu}{}^\kappa = -c_{\mu\nu}{}^\kappa$ $\overset{R}{T}_{\mu\nu}{}^\kappa = c_{\mu\nu}{}^\kappa$

křivost $\overset{L}{R}_{\alpha\beta\mu}{}^\nu = \overset{R}{R}_{\alpha\beta\mu}{}^\nu = 0$

jejich rozdíl. tenzor je $c_{\mu\nu}{}^\kappa$ tj. $\overset{R}{\nabla} - \overset{L}{\nabla} = \operatorname{Meas}[c] = c$

D: $\overset{L}{\nabla}$ je dáno podm. $\overset{L}{\nabla} l_m = 0$ $\Rightarrow$ m-ádová der. levoinv. b. $\overset{L}{E}_\alpha$

$\Rightarrow \overset{L}{T}_{\alpha\beta}{}^\mu = \overset{L}{\nabla}_{E_\alpha} \overset{L}{E}{}^\mu_\beta - \overset{L}{\nabla}_{E_\beta} \overset{L}{E}{}^\mu_\alpha - [\overset{L}{E}_\alpha, \overset{L}{E}_\beta]^\mu = -c_{\alpha\beta}{}^\kappa \overset{L}{E}{}^\mu_\kappa \Rightarrow \overset{L}{T}_{\alpha\beta}{}^\mu = -c_{\alpha\beta}{}^\mu$

$\Rightarrow \overset{L}{R}_{\alpha\beta}{}^\mu{}_\nu = \overset{L}{\nabla}_\alpha \overset{L}{\nabla}_\beta \overset{L}{E}{}^\mu_\nu - \overset{L}{\nabla}_\beta \overset{L}{\nabla}_\alpha \overset{L}{E}{}^\mu_\nu - \overset{L}{T}_{\alpha\beta}{}^\kappa \overset{L}{\nabla}_\kappa \overset{L}{E}{}^\mu_\nu = 0 \Rightarrow \overset{L}{R}_{\alpha\beta}{}^\mu{}_\nu = 0$

pro pravoinv. bázi $\overset{R}{E}_\alpha$ platí $[\overset{R}{E}_\alpha, \overset{R}{E}_\beta] = -c_{\alpha\beta}{}^\kappa \overset{R}{E}_\kappa \Rightarrow \overset{R}{T}_{\alpha\beta}{}^\mu = c_{\alpha\beta}{}^\mu$

$\overset{L}{\nabla}_{l_m} r_n = \overset{L}{\nabla}_{l_m} r_n - \overset{L}{\nabla}_{r_n} l_m - [l_m, r_n] = l_m \cdot \overset{L}{T} \cdot r_n = l_m \cdot c \cdot r_n \Rightarrow \overset{L}{\nabla}_\mu r_n^\nu = -c_{\mu\kappa}{}^\nu r_n^\kappa$

nechť $\overset{R}{\nabla} = \overset{L}{\nabla} + A$ $\Rightarrow 0 = \overset{R}{\nabla}_\mu r_m^\nu = \overset{L}{\nabla}_\mu r_m^\nu + A_{\mu\kappa}{}^\nu r_m^\kappa = (A_{\mu\kappa}{}^\nu - c_{\mu\kappa}{}^\nu) r_m^\kappa \Rightarrow A_{\mu\nu}{}^\kappa = c_{\mu\nu}{}^\kappa$

Theorem $\overset{L}{E}{}^\alpha$ a $\overset{R}{E}{}^\alpha$ jsou levo a pravoinv. báze v T^*G, $\overset{R}{E}{}^\alpha|_e = \overset{L}{E}{}^\alpha|_e$

$d\overset{L}{E}{}^\alpha + \frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{L}{E}{}^\kappa \wedge \overset{L}{E}{}^\lambda = 0$

$d\overset{R}{E}{}^\alpha - \frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{R}{E}{}^\kappa \wedge \overset{R}{E}{}^\lambda = 0$

D. $d_\mu \overset{L}{E}{}^\alpha_\nu = \overset{L}{\nabla}_\mu \wedge \overset{L}{E}{}^\alpha_\nu + \overset{L}{T}_{\mu\nu}{}^\beta \overset{L}{E}{}^\alpha_\beta = -c_{\mu\nu}{}^\beta \overset{L}{E}{}^\alpha_\beta = -c_{\kappa\lambda}{}^\alpha \overset{L}{E}{}^\kappa_\mu \overset{L}{E}{}^\lambda_\nu = -\frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{L}{E}{}^\kappa_\mu \wedge \overset{L}{E}{}^\lambda_\nu$

$d_\mu \overset{R}{E}{}^\alpha_\nu = \overset{R}{\nabla}_\mu \wedge \overset{R}{E}{}^\alpha_\nu + \overset{R}{T}_{\mu\nu}{}^\beta \overset{R}{E}{}^\alpha_\beta = c_{\mu\nu}{}^\beta \overset{R}{E}{}^\alpha_\beta = c_{\kappa\lambda}{}^\alpha \overset{R}{E}{}^\kappa_\mu \overset{R}{E}{}^\lambda_\nu = \frac{1}{2} c_{\kappa\lambda}{}^\alpha \overset{R}{E}{}^\kappa_\mu \wedge \overset{R}{E}{}^\lambda_\nu$

Def:

$$\overset{\lambda}{\nabla} = \overset{L}{\nabla} + \frac{\lambda+1}{2}\mathbf{c} = \overset{R}{\nabla} + \frac{\lambda-1}{2}\mathbf{c} \qquad \nabla \equiv \overset{\lambda=0}{\nabla}$$

tj. $\overset{\lambda}{\nabla}_F a^\kappa = \overset{L}{\nabla}_F a^\kappa + \frac{\lambda+1}{2} c_{\mu\nu}{}^\kappa a^\nu = \overset{R}{\nabla}_F a^\kappa + \frac{\lambda-1}{2} c_{\mu\nu}{}^\kappa a^\nu$

Poz.: $\overset{\lambda}{\nabla}$ je "interpolace" mezi $\overset{L}{\nabla} = \overset{\lambda=-1}{\nabla}$ a $\overset{R}{\nabla} = \overset{\lambda=1}{\nabla}$

tj. na singulu affinní str. prostoru konexí $\overset{\lambda}{\nabla} = \frac{1-\lambda}{2}\overset{L}{\nabla} + \frac{1+\lambda}{2}\overset{R}{\nabla}$

Lemma

$$\overset{\lambda}{\nabla}_F \ell_m^\kappa = \frac{\lambda+1}{2} c_{\mu\nu}{}^\kappa \ell_m^\nu \qquad \overset{\lambda}{\nabla}_F \pi_m^\kappa = \frac{\lambda-1}{2} c_{\mu\nu}{}^\kappa \pi_m^\nu$$

speciálně:

$$\overset{\lambda}{\nabla}_{\ell_m}\ell_n = \frac{\lambda+1}{2}\ell_m\cdot c\cdot\ell_n = \frac{1+\lambda}{2}[\ell_m,\ell_n] = \frac{1+\lambda}{2}\ell_{[m,n]}$$

$$\overset{\lambda}{\nabla}_{\pi_m}\pi_n = \frac{\lambda-1}{2}\pi_m\cdot c\cdot\pi_n = \frac{1-\lambda}{2}[\pi_m,\pi_n] = \frac{1-\lambda}{2}\pi_{[m,n]}$$

D: využití $\overset{L}{\nabla}\ell_m = 0$ a $\overset{R}{\nabla}\pi_m = 0$

platí $\ell_m\cdot c\cdot\ell_n = [\ell_m,\ell_n] = \ell_{[m,n]}$ a $\pi_m\cdot c\cdot\pi_n = -[\pi_m,\pi_n] = -\pi_{[m,n]}$

Theorem

$$\overset{\lambda}{\nabla} c = 0 \qquad \overset{\lambda}{\nabla} k = 0$$

D: biinvariance c $\Rightarrow$ $\overset{L}{\nabla} c = \overset{R}{\nabla} c = 0$

$$\mathbf{c}_\alpha c_{\beta\gamma}{}^\kappa = c_{\alpha\lambda}{}^\kappa c_{\beta\gamma}{}^\lambda - c_{\alpha\beta}{}^\lambda c_{\lambda\gamma}{}^\kappa - c_{\alpha\gamma}{}^\lambda c_{\beta\lambda}{}^\kappa = c_{\alpha\lambda}{}^\kappa c_{\beta\gamma}{}^\lambda + c_{\gamma\lambda}{}^\kappa c_{\alpha\beta}{}^\lambda + c_{\beta\lambda}{}^\kappa c_{\gamma\alpha}{}^\lambda \overset{J.I.}{=} 0$$

$\Rightarrow$ $\overset{\lambda}{\nabla} c = \overset{L}{\nabla} c + \frac{\lambda+1}{2}\mathbf{c}c = 0$

$\Rightarrow$ $\overset{\lambda}{\nabla} k = 0$

Theorem

$$\overset{\lambda}{T}_{\mu\nu}{}^\kappa = \lambda c_{\mu\nu}{}^\kappa \qquad \overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda = -\frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa \qquad \overset{\lambda}{Ric}_{\alpha\beta} = \frac{1-\lambda^2}{2} k_{\alpha\beta}$$

D: $\overset{\lambda}{T}_{\mu\nu}{}^\kappa = \overset{L}{T}_{\mu\nu}{}^\kappa + 2\frac{\lambda+1}{2} c_{[\mu\nu]}{}^\kappa = \lambda c_{\mu\nu}{}^\kappa$

$\overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda = \overset{L}{R}_{\mu\nu}{}^\kappa{}_\lambda + \overset{L}{\nabla}_\mu \overset{\lambda}{A}_{\nu\lambda}^\kappa - \overset{L}{\nabla}_\nu \overset{\lambda}{A}_{\mu\lambda}^\kappa + \overset{L}{T}_{\mu\nu}{}^\alpha \overset{\lambda}{A}_{\alpha\lambda}^\kappa + \overset{\lambda}{A}_{\mu\alpha}^\kappa \overset{\lambda}{A}_{\nu\lambda}^\alpha - \overset{\lambda}{A}_{\nu\alpha}^\kappa \overset{\lambda}{A}_{\mu\lambda}^\alpha = -\frac{\lambda+1}{2} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa + \left(\frac{\lambda+1}{2}\right)^2 \left(c_{\mu\alpha}{}^\kappa c_{\nu\lambda}{}^\alpha - c_{\nu\alpha}{}^\kappa c_{\mu\lambda}{}^\alpha\right)$

$= \left(\frac{\lambda+1}{2}\right)^2 \left(c_{\mu\nu}{}^\alpha c_{\lambda\alpha}{}^\kappa + c_{\nu\lambda}{}^\alpha c_{\mu\alpha}{}^\kappa + c_{\lambda\mu}{}^\alpha c_{\nu\alpha}{}^\kappa\right) - \frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa \overset{J.I.}{=} -\frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa$

alternativně

$\ell_m\cdot\overset{\lambda}{T}\cdot\pi_n = \ell_m\cdot\overset{\lambda}{\nabla}\pi_n - \pi_n\cdot\overset{\lambda}{\nabla}\ell_m - [\ell_m,\pi_n] = \frac{\lambda-1}{2}\ell_m\cdot c\cdot\pi_n - \frac{\lambda+1}{2}\pi_n\cdot c\cdot\ell_m = \lambda\,\ell_m\cdot c\cdot\pi_n$

$\overset{\lambda}{R}(\ell_a,\ell_b)\cdot\ell_c = \overset{\lambda}{\nabla}_{\ell_a}\overset{\lambda}{\nabla}_{\ell_b}\ell_c - \overset{\lambda}{\nabla}_{\ell_b}\overset{\lambda}{\nabla}_{\ell_a}\ell_c - \overset{\lambda}{\nabla}_{[\ell_a,\ell_b]}\ell_c = \frac{\lambda+1}{2}\left(\overset{\lambda}{\nabla}_{\ell_a}\ell_{[b,c]} - \overset{\lambda}{\nabla}_{\ell_b}\ell_{[a,c]} - \overset{\lambda}{\nabla}_{\ell_{[a,b]}}\ell_c\right) - \frac{1-\lambda}{2}\overset{\lambda}{\nabla}_{\ell_{[a,b]}}\ell_c$

$= \left(\frac{1+\lambda}{2}\right)^2 \ell[[a,[b,c]] - [b,[a,c]] - [[a,b],c]] - \frac{1-\lambda^2}{4}\ell_{[[a,b],c]} \overset{J.I.}{=} -\frac{1-\lambda^2}{4}[[\ell_a,\ell_b],\ell_c]$

$\Rightarrow$ $\overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda \ell_a^\mu \ell_b^\nu \ell_c^\lambda = -\frac{1-\lambda^2}{4}\ell_a^\mu \ell_b^\nu c_{\mu\nu}{}^\alpha \ell_c^\lambda c_{\alpha\lambda}{}^\kappa$ $\Rightarrow$ $\overset{\lambda}{R}_{\mu\nu}{}^\kappa{}_\lambda = -\frac{1-\lambda^2}{4} c_{\mu\nu}{}^\alpha c_{\alpha\lambda}{}^\kappa$

Theorem

$\nabla \equiv \overset{0}{\nabla}$ je metr. kov. der. k

$$T = 0 \qquad \nabla k = 0 \qquad R = -\frac{1}{4} c\cdot c \qquad Ric = \frac{1}{2}k \qquad Ein + \frac{D-2}{4}k = 0 \qquad \mathcal{R} = \frac{D}{2}$$

Theorem

$\exp(\alpha m)$ jsou geodetiky ∇ orbity ℓ_m a π_m jsou geodetiky ∇

D: $\nabla_{\ell_m}\ell_m = \frac{1}{2}\ell_{[m,m]} = 0$ $\nabla_{\pi_m}\pi_m = \frac{1}{2}\pi_{[m,m]} = 0$ $\exp(\alpha m)$ je orbita ℓ_m a π_m

Theorem K levo(pravo) invar. 1-forma $\Rightarrow$ K je Killing-Yano forma vůči $\overset{\lambda}{\nabla}$

D: $\overset{\lambda}{\nabla}_\mu K_\nu = \overset{L}{\nabla}_\mu K_\nu - \frac{\lambda+1}{2} c_{\mu\nu}{}^\alpha K_\alpha = -\frac{\lambda+1}{2} c_{\mu\nu}{}^\alpha K_\alpha = \overset{\lambda}{\nabla}_{[\mu} K_{\nu]}$ $\Rightarrow$ $\overset{\lambda}{\nabla}_\mu K_\nu = \overset{\lambda}{\nabla}_{[\mu} K_{\nu]}$

Theorem

$$\overset{\lambda}{\nabla}_{l_m} = \mathcal{L}_{l_m} + \frac{\lambda-1}{2}\,\mathbf{tens}[l_m\cdot c] \qquad l_m\cdot c = l[\mathrm{ad}_m]$$

$$\overset{\lambda}{\nabla}_{\pi_m} = \mathcal{L}_{\pi_m} + \frac{\lambda+1}{2}\,\mathbf{tens}[\pi_m\cdot c] \qquad \pi_m\cdot c = \pi[\mathrm{ad}_m]$$

konkrétně

$$\overset{R}{\nabla}_{l_m} = \mathcal{L}_{l_m} \qquad \nabla_{l_m} = \mathcal{L}_{l_m} - \frac{1}{2}\,\mathbf{tens}[l_m\cdot c]$$

$$\overset{L}{\nabla}_{\pi_m} = \mathcal{L}_{\pi_m} \qquad \nabla_{\pi_m} = \mathcal{L}_{\pi_m} + \frac{1}{2}\,\mathbf{tens}[\pi_m\cdot c]$$

D:

$$\mathcal{L}_{l_m} = \overset{\lambda}{\nabla}_{l_m} + \overset{\lambda}{\Lambda}_{l_m} \qquad \overset{\lambda}{\Lambda}_{l_m} = -\overset{\lambda}{\nabla} l_m - l_m\cdot\overset{\lambda}{T} = -\frac{\lambda+1}{2} c\cdot l_m - \lambda\, l_m\cdot c = -\frac{\lambda-1}{2}\, l_m\cdot c$$

$$\mathcal{L}_{\pi_m} = \overset{\lambda}{\nabla}_{\pi_m} + \overset{\lambda}{\Lambda}_{\pi_m} \qquad \overset{\lambda}{\Lambda}_{\pi_m} = -\overset{\lambda}{\nabla}\pi_m - \pi_m\cdot\overset{\lambda}{T} = -\frac{\lambda-1}{2} c\cdot\pi_m - \lambda\,\pi_m\cdot c = -\frac{\lambda+1}{2}\,\pi_m\cdot c$$

$$c = l[c] = \pi[c] \qquad m\cdot c|_e = c\,d_m$$

Def E_α báze v e

$\bar{x}^\alpha$ jsou normální souř. na okolí e $\equiv$

$$g = \exp(m^\alpha E_\alpha) \iff \bar{x}^\alpha(g) = m^\alpha$$

Theorem $c_{\alpha\beta}{}^\kappa$ a $k_{\alpha\beta}$ komponenty c a k vzhledem k E_α v e

$$\bar{k}_{\alpha\beta} = k_{\alpha\beta} - \frac{1}{12} c_{\alpha\kappa}{}^\mu c_{\beta\lambda}{}^\nu k_{\mu\nu}\,\bar{x}^\kappa\bar{x}^\lambda + \mathcal{O}(\bar{x}^4)$$

$$\bar{\Gamma}^\mu{}_{\alpha\beta} = -\frac{1}{12}\left(c_{\kappa\alpha}{}^\nu c_{\nu\beta}{}^\mu + c_{\kappa\beta}{}^\nu c_{\nu\alpha}{}^\mu\right)\bar{x}^\kappa + \mathcal{O}(\bar{x}^3)$$

$$\bar{R}_{\alpha\beta}{}^\kappa{}_\lambda = -\frac{1}{4} c_{\alpha\beta}{}^\nu c_{\nu\lambda}{}^\kappa + \mathcal{O}(\bar{x}^2)$$

$\bar{A}^{\alpha\ldots}_{\beta\ldots}$ komponenty vzhledem k souř. bázi $\frac{\partial}{\partial\bar{x}^\alpha}, d\bar{x}^\alpha$

zřejmě $A^{\alpha\ldots}_{\beta\ldots} = \bar{A}^{\alpha\ldots}_{\beta\ldots}|_e$

D: plyne z rozvojů metr., Γ a R v norm. souř.

$$\bar{k}_{\alpha\beta} = \bar{k}_{\alpha\beta}|_e - \frac{1}{3}\bar{R}_{\alpha\kappa\beta\lambda}|_e\,\bar{x}^\kappa\bar{x}^\lambda + \nabla\bar{R}|_e\,\bar{x}^3 + \mathcal{O}(\bar{x}^4)$$

$$\bar{\Gamma}^\mu_{\alpha\beta} = -\frac{1}{3}\left(\bar{R}_{\kappa\alpha}{}^\mu{}_\beta + \bar{R}_{\kappa\beta}{}^\mu{}_\alpha\right)|_e\,\bar{x}^\kappa + \nabla\bar{R}|_e\,\bar{x}^2 + \mathcal{O}(\bar{x}^3)$$

$$\bar{R}_{\alpha\beta}{}^\kappa{}_\lambda = \bar{R}_{\alpha\beta}{}^\kappa{}_\lambda|_e + \nabla\bar{R}|_e\,\bar{x} + \mathcal{O}(\bar{x}^2)$$

využilo se $\bar{A}^{\alpha\ldots}_{\beta\ldots}|_e = A^{\alpha\ldots}_{\beta\ldots}$ a $c_{\alpha\beta\gamma} = c_{[\alpha\beta\gamma]}$ - viz LA