Differential forms, Riemann geometry, GR

Reminder (also because of English)

Diff. form: multilinear scalar function associated with a completely antisymmetric („skew") tensor

Example: $F_{\alpha\beta} = -F_{\beta\alpha}$

2-form $\Phi(d\vec{x}, d\vec{y}) = 2!\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]} = 2 F_{\alpha\beta}\, dx^{\alpha} dy^{\beta}$

1-form $\theta(d\vec{x}) = A_\alpha dx^\alpha$

0-form ... scalar

Wedge product (or exterior product) „$\wedge$"

of p-form and q form is $(p+q)$-form

Example: $\psi = \phi \wedge \theta$

3-form $\psi(d\vec{x}, d\vec{y}, d\vec{z}) = 3!\, F_{\alpha\beta} A_\gamma\, dx^{[\alpha} dy^{\beta} dz^{\gamma]}$

$$= 3!\, \underbrace{F_{[\alpha\beta} A_{\gamma]}}_{\text{complete antisymmetrization}}\, dx^\alpha dy^\beta dz^\gamma$$

Thm. α ... degree a, β ... degree b then

$$\alpha \wedge \beta = (-1)^{ab}\, \beta \wedge \alpha$$

$\Rightarrow \Theta \wedge \Theta = 0$ if Θ has odd degree

Note: Vector product in E_3 in forms

$$\theta = A_\alpha dx^\alpha \quad , \quad \theta_2 = B_\beta dy^\beta$$

$$\theta_1 \wedge \theta_2 = 2! \, A_\alpha B_\beta \, dx^{[\alpha} dy^{\beta]} = \underbrace{(A_\alpha B_\beta - A_\beta B_\alpha)}_{\equiv \, C_{\alpha\beta}} dx^\alpha dy^\beta$$

let $C_{\gamma} = \frac{1}{2!} \varepsilon_{\gamma\alpha\beta} C_{\alpha\beta}$

Then $\vec{C} = \vec{A} \times \vec{B}$

Exterior differential „d"

... of p-form is $p+1$ form

Ex: $\phi = 2! \, F_{\alpha\beta} \, dx^{[\alpha} dy^{\beta]}$ .... 2 form

3-form $d\phi(d\vec{x}, d\vec{y}, d\vec{z}) = 3! \, \partial_\alpha F_{\beta\gamma} \, dx^{[\alpha} dy^{\beta} dz^{\gamma]}$

$$= 3! \, F_{\beta\gamma;\alpha} \, dx^{[\alpha} dy^{\beta} dz^{\gamma]}$$

Thms:

f - scalar f. $\quad df(d\vec{x}) = \partial_\alpha f \, dx^\alpha$

for arbitrary scal. f and form Ω:

$$\boxed{d(f\Omega) = f\, d\Omega + df \wedge \Omega}$$

More generally

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^a \alpha \wedge d\beta \qquad a = \deg \alpha$$

for any Ω:

$$\boxed{d^2 \Omega = 0}$$

Differentials of coordinates

Choosing a coordinate system $x^\alpha = (x^1, \dots x^n)$

each of x^μ considered as scalar field (0-form)

e.g. $\rightarrow$ $\underbrace{dx^1}_{\text{1-form}}(\underbrace{d\vec{y}}_{\text{vector argument}}) = (\partial_\alpha x^1)\, dy^\alpha = \frac{\partial x^1}{\partial x^\alpha} dy^\alpha = \underbrace{\delta^1_\alpha}_{\vec{e}^{(1)}_\alpha} dy^\alpha = dy^1$

$\vec{e}^{(1)}_\alpha$ components of covariant basis in natural coordinates

can write also

$\underbrace{dx^1}_{\text{1-form}}(\underbrace{d\vec{x}}_{\text{vector}}) = \underbrace{dx^1}_{\text{1st component of vector}}$

(" notation as in $\underbrace{y(x)}_{\text{function}} = \underbrace{y}_{\text{value of } y}$ ")

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present standard geom. notation:

$f(x^\alpha)$ smooth, vector $\underline{t} = t^\alpha \frac{\partial}{\partial x^\alpha}$,

$\frac{\partial}{\partial x^\alpha}$ ... basis in $T(M)$ (tangent space to manif. M)

dual basis dx^α

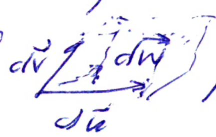
$$dx^\alpha\left(\frac{\partial}{\partial x^\beta}\right) = \frac{\partial \overset{f}{x^\alpha}}{\partial x^\beta} = \delta^\alpha_\beta$$

every 1-form can be written as $\omega = \left(\sum \omega_\alpha dx^\alpha\right) = \omega_\alpha dx^\alpha$

Example: x^1, x^2, x^3 Cartesian coord. in E_3

$dx^1 \wedge dx^2 \wedge dx^3$ ... is 3-form

$dx^1 \wedge dx^2 \wedge dx^3 (d\vec{u}, d\vec{v}, d\vec{w}) = 3!\, \delta^1_\alpha \delta^2_\beta \delta^3_\gamma\, du^{[\alpha} dv^\beta dw^{\gamma]}$

$= \varepsilon_{\alpha\beta\gamma}\, du^\alpha dv^\beta dw^\gamma$ volume of 

parallelepiped

Maxwell equations in forms

Let $F_{\alpha\beta}$ is the Maxwell elmag. tensor

Maxwell eqs. (1) $\boxed{F^{\mu\nu}{}_{,\nu} = \frac{4\pi}{c} j^{\mu}, \quad F_{\mu\nu}{}^{,\nu} = \frac{4\pi}{c} j_{\mu}}$

4 current

(2) $$\boxed{\begin{aligned} &F_{[\alpha\beta,\gamma]\,cykl} = \\ &= F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \end{aligned}}$$ (2)

Dual

$$F^{*}_{\alpha\beta} = \frac{1}{2!} \varepsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}$$

$\varepsilon_{\alpha\beta\gamma\delta}$ - Levi-Civita symbol

Equivalent form of Maxw. eqs. in terms of $F^{*}_{\alpha\beta}$:

(1') $$F^{*}_{[\mu\nu,\alpha]} = \frac{4\pi}{c} \frac{1}{3!} {}^{*}j_{[\mu\nu\alpha]}$$

where $${}^{*}j_{\varrho\sigma\tau} = \varepsilon_{\xi\varrho\sigma\tau} j^{\xi}$$

(2') $$F^{*}_{\mu\nu,}{}^{\nu} = 0$$

Define $$F = \frac{1}{2} F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} = \frac{1}{2} F_{\mu\nu}\, 2!\, du^{[\mu} dv^{\nu]}$$

$$dF = \frac{1}{2}\, 3!\, F_{\mu\nu,\alpha}\, du^{[\alpha} dv^{\mu} dw^{\nu]}$$

$$= \frac{1}{2} F_{\mu\nu,\alpha}\, dx^{\alpha} \wedge dx^{\mu} \wedge dx^{\nu}$$

Eq. (2) follows from $\boxed{dF = 0}$

satisfied if $F = dA$ since $d^2 A = 0$

Define $\quad {}^*F = \frac{1}{2} F^*_{\mu\nu}\, dx^\mu \wedge dx^\nu = \frac{1}{2} F^*_{\mu\nu}\, 2!\, du^{[\mu} dv^{\nu]}$

Equation (1') $\Longleftrightarrow$ $\boxed{\underbrace{d\,{}^*F}_{3\text{form}} = \frac{4\pi}{c}\underbrace{{}^*J}_{3\text{-form}}}$ $\quad(\tilde{1'})$

where 3-form

$${}^*J = \frac{1}{2}\, {}^*J_{\mu\nu\alpha}\, du^{[\alpha} dv^{\mu} dw^{\nu]}$$

$$= \frac{1}{2}\, \varepsilon_{\rho\mu\nu\alpha}\, J^\rho\, du^{[\alpha} dv^{\mu} dw^{\nu]}$$

$$= \frac{1}{2} J^\rho \varepsilon_{\rho\mu\nu\alpha} \frac{1}{3!} \Big(du^\alpha dv^\mu dw^\nu + du^\mu dv^\nu dw^\alpha$$

$$+ du^\nu dv^\alpha dw^\mu - du^\mu dv^\alpha dw^\nu$$

$$- du^\nu dv^\mu dw^\alpha - du^\alpha dv^\nu dw^\mu \Big)$$

Eq. $(\tilde{1'})$ in forms follows from (1') $\times$

$$-\frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\, {}^*F_{\mu\nu,\beta} = \frac{4\pi}{c}\left(-\frac{1}{3!}\right)\varepsilon^{\alpha\rho\sigma\tau}\, {}^*J_{\rho\sigma\tau}$$

multiply by $\varepsilon_{\alpha\gamma\delta\varepsilon}$ and use
the relations for $\varepsilon^{\cdots}\varepsilon_{\cdots}$
– see the following page

$\boxed{\varepsilon_{\alpha\beta\gamma\delta}}$

Properties of Lev.-Civ. $\varepsilon_{\alpha\beta\gamma\delta} =$
- $+1$ for $(\alpha\beta\gamma\delta) = 0\,1\,2\,3$ and even per
- -1 for ... $0\,1\,2\,?$
- 0 if 2 indices are equal

and $\varepsilon^{\alpha\beta\gamma\delta} = -\,\varepsilon_{\alpha\beta\gamma\delta}$ ... thought as individual)

here $\varepsilon_{0123} = +1 \qquad \varepsilon^{0123} = -1$ all other by permutations

Products are as follows:

1)
$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\kappa\lambda} = -\begin{vmatrix} \delta^\alpha_\mu & \delta^\alpha_\nu & \delta^\alpha_\kappa & \delta^\alpha_\lambda \\ \delta^\beta_\mu & \delta^\beta_\nu & \delta^\beta_\kappa & \delta^\beta_\lambda \\ \delta^\gamma_\mu & \delta^\gamma_\nu & \delta^\gamma_\kappa & \delta^\gamma_\lambda \\ \delta^\delta_\mu & \delta^\delta_\nu & \delta^\delta_\kappa & \delta^\delta_\lambda \end{vmatrix}$$

2)
$$\varepsilon^{\alpha\beta\gamma\sigma}\varepsilon_{\mu\nu\kappa\sigma} = -\begin{vmatrix} \delta^\alpha_\mu & \delta^\alpha_\nu & \delta^\alpha_\kappa \\ \delta^\beta_\mu & \delta^\beta_\nu & \delta^\beta_\kappa \\ \delta^\gamma_\mu & \delta^\gamma_\nu & \delta^\gamma_\kappa \end{vmatrix} \equiv -\,\delta^{\alpha\beta\gamma}_{\mu\nu\kappa}$$

in MTW, etc.

3)
$$\varepsilon^{\alpha\beta\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = -2\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right)$$

4)
$$\varepsilon^{\alpha\lambda\rho\sigma}\varepsilon_{\mu\lambda\rho\sigma} = -6\,\delta^\alpha_\mu = -3!\,\delta^\alpha_\mu$$

5)
$$\varepsilon^{\alpha\lambda\rho\sigma}\varepsilon_{\alpha\lambda\rho\sigma} = -24 = -4!$$

Riemannian geometry in forms, or

Cartan calculus

Traditional Riem. Geom.: metric $g_{\alpha\beta} \rightsquigarrow \Gamma^{\alpha}_{\beta\gamma}$ (Christofell's)

$\rightsquigarrow$ curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$, in n dimensions

Cartan:

n basic 1-forms θ^a $\quad a = 1, \dots n$

n^2 1-forms of connection $\omega^a{}_b$

n^2 2-forms $\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \theta^c \wedge \theta^d$ of curvature

Cartan's structure equations:

„First equations of structure":

$$d\theta^a = -\omega^a{}_b \wedge \theta^b \qquad (I)$$

„Second equations of structure":

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad (II)$$

Important are also Ricci rotation coefficients $\gamma^a{}_{bc}$

$\omega^a{}_b = \gamma^a{}_{bc}\, \theta^c$

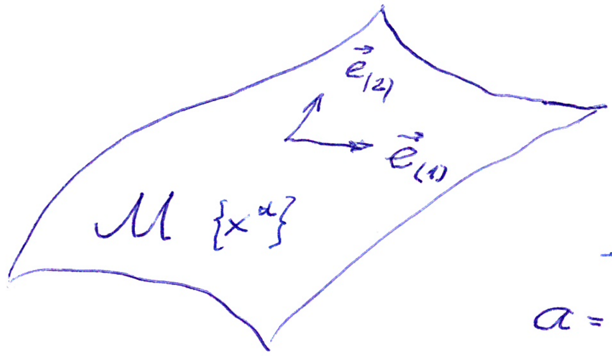
The essence of whole Riem. geometry is summarized in the equations of structure (I), (II)

connecting $d\theta^a$ with $\omega^a{}_b$ and

$d\omega^a{}_b$ with $\Omega^a{}_b$

Advantage: less quantities („indices"), calculations

Basic 1-forms θ^a



manifold $\mathcal{M}$; metric $g_{\mu\nu}$

coordinates x^α $\alpha = 1, \dots n$

$\vec{e}_{(a)}(x^\mu)$ ... n linearly independ. vector fields „frames"

$a = 1, \dots, n$

in coordinates x^α the components are

$$e^{\alpha}_{(a)}$$

component α

enumerate frame vector

Frame components of tensor

$$T_{ab\dots} = T_{\alpha\beta\dots}\, e^{\alpha}_{(a)} e^{\beta}_{(b)} \dots$$

scalars!

Introduce matrix

$$g_{ab} = \vec{e}_{(a)} \cdot \vec{e}_{(b)} \quad \dots \text{scalar product}$$

i.e.

$$\boxed{g_{ab} = g_{\alpha\beta}\, e^{\alpha}_{(a)} e^{\beta}_{(b)}}$$

frame components of metric $g_{\alpha\beta}$

$\vec{e}_{(a)}$ independent $\Rightarrow$ exists g^{ab} satisfying $g^{ab} g_{ac} = \delta^b_c$

Define dual basis

$$\boxed{\vec{e}^{\,(a)} = g^{ab}\, \vec{e}_{(b)}}$$

summing in "b"

$$\vec{e}_{(c)} \cdot \vec{e}^{\,(a)} = g^{ab}\, \vec{e}_{(b)} \vec{e}_{(c)} = g^{ab} g_{bc} = \delta^a_c$$

$\rightsquigarrow$ It is consistent to raise/lower indices $a, b, \dots$ by g_{ab}/g^{ab}

So $\quad e_{(c)}^{\alpha} e^{(a)}_{\alpha} = \delta^a_c$

also $\quad e^{\alpha}_{(a)} e^{(a)}_{\gamma} = \delta^{\alpha}_{\gamma}$ $\quad$ „orthogonality" relations

since $\quad g_{ab} = g_{\alpha\beta}\, e^{\alpha}_{(a)} e^{\beta}_{(b)} \;/ e^{(a)}_{\gamma}$

$$e_{(b)\gamma} = e_{(b)\alpha} \underbrace{e^{\alpha}_{(a)} e^{(a)}_{(\gamma)}}_{\text{must be } = \delta^{\alpha}_{\gamma}}$$

Introduce 1-forms θ^a, $a = 1, \dots n$, associated with dual basis:

$$\boxed{\begin{aligned} \theta^a &= e^{(a)}_{\alpha}\, dx^{\alpha} \\ dx^{\alpha} &= e^{\alpha}_{(a)}\, \theta^a \end{aligned}} \qquad \theta^a(d\vec{x}) = e^{(a)}_{\alpha} dx^{\alpha} \quad (*)$$

2 possible interpretations: 1) numerical relations where $d\vec{x}$ $(= dx^{\alpha})$ is a vector argument substituted into θ^a, i.e. $(*)$, or 2) linear relation between 1-forms θ^a and dx^{α} $\quad$ ($e^{(a)}_{\alpha} = \delta^a_{\alpha}$ in "natural" basis)

θ^a form basis of all 1-forms:

$$\alpha = A_{\alpha}\, dx^{\alpha} = A_{\alpha}\, e^{\alpha}_{(a)}\, \theta^a = A_a\, \theta^a;$$

$\theta^a \wedge \theta^b$ form basis of all 2-forms, etc.

$$\phi = 2!\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]} = F_{ab}\, \theta^a \wedge \theta^b$$

in natural basis dx^{α}; $dx^{\alpha} \wedge dx^{\beta}, \dots$ form the basis

Expressing the interval $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$:

$$\boxed{ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = \underbrace{g_{ab}\, \theta^a \theta^b}_{\text{numerical relation}}}$$

Indeed, $g_{ab}\, \theta^a \theta^b = g_{\alpha\beta}\, e^\alpha_{(a)} e^\beta_{(b)} e^{(a)}_\gamma dx^\gamma e^{(b)}_\delta dx^\delta$

$= \delta^\alpha_\gamma \qquad = \delta^\beta_\delta$

$$= g_{\gamma\delta}\, dx^\gamma dx^\delta$$

Connection 1-forms

introduce „covariant differential":

$$[D\vec{e}_{(a)}]^\alpha \stackrel{\text{def.}}{=} e^\alpha_{(a);\gamma}\, dx^\gamma \qquad (\text{assume } g, \Gamma \text{ given})$$

it is a vector – can be written as a linear comb. of $\vec{e}_{(a)}$

$$\boxed{D\vec{e}_{(b)} = \omega^c{}_b\, \vec{e}_{(c)}}$$

$\omega^c{}_b\,(dx)$ ... connection 1-forms ('enumarated' by $^c{}_b$)

$\times\, \vec{e}^{(a)}$: $\vec{e}^{(a)} D\vec{e}_{(b)} = \omega^c{}_b\, \underbrace{e_{(c)} \vec{e}^{(a)}}_{\delta^a_c}$

but $\underbrace{D(\vec{e}^{(a)} \vec{e}_{(b)})}_{\text{number } \delta^a_b} = \vec{e}^{(a)} D\vec{e}_{(b)} + \vec{e}_{(b)} D\vec{e}^{(a)} = 0$

$$\Rightarrow \boxed{\omega^a{}_b} = -\vec{e}_{(b)} D\vec{e}^{(a)} = -e^\beta_{(b)} [D\vec{e}^{(a)}]_\beta =$$

$$= -e^\beta_{(b)}\, e^{(a)}_{\beta;\gamma}\, dx^\gamma = -e^{(a)}_{\beta;\gamma}\, e^\beta_{(b)}\, dx^\gamma$$

Also: $D\vec{e}^{(a)} = -\omega^a{}_b\, \vec{e}^{(b)}$

Ricci rotation coefficients:

$$\underline{\gamma^a{}_{bc} = -e^{(a)}{}_{\beta;\gamma}\, e^{\beta}_{(b)}\, e^{\gamma}_{(c)}} \quad \cdots \text{ scalars !}$$

projection of covariant deriv.
of dual basis's

The following relation is simple and very useful:

$$\boxed{\omega^a{}_b = \gamma^a{}_{bc}\, \theta^c} \qquad \left(= -e^{(a)}{}_{\beta;\gamma}\, e^{\beta}_{(b)}\, e^{\gamma}_{(c)} \times e^{(c)}{}_{\mu}\, dx^{\mu}\right)$$

$$\uparrow \text{connection 1 forms} \qquad \uparrow \text{scalars} \qquad \uparrow \text{basis 1 forms}$$

$= \delta^{\gamma}_{\mu}$ dx^{γ}

cp. p. F10

<u>In the coordinate (natural) basis:</u>

$$e^{(a)}{}_{\alpha} = \delta^a_{\alpha}, \quad e^{\alpha}_{(a)} = \delta^{\alpha}_a \qquad dx^{\alpha} = \theta^{\alpha}$$

i.e. frame components of any tensor are numerically equal to „ordinary" coordinate components – and

$$\gamma^a{}_{bc} = -e^{(a)}{}_{\beta;\gamma}\, e^{\beta}_{(b)}\, e^{\gamma}_{(c)} =$$

$$= -\Big[\underbrace{e^{(a)}{}_{\beta,\gamma}}_{=0 \text{ since } e^{(a)}{}_{\beta} = \delta^a_{\beta}} - \Gamma^{\delta}{}_{\beta\gamma}\, \underbrace{e^{(a)}{}_{\delta}}_{\delta^a_{\delta}}\Big]\, \underbrace{e^{\beta}_{(b)}}_{\delta^{\beta}_{b}}\, \underbrace{e^{\gamma}_{(c)}}_{\delta^{\gamma}_{(c)}}$$

$$= +\Gamma^a{}_{bc}$$

Start from $g_{ab} = \vec{e}_{(a)} \cdot \vec{e}_{(b)}$

each g_{ab} is scalar, so

$$\underbrace{d}_{\text{„exterior"}} g_{ab} = D g_{ab} = \vec{e}_{(a)} D\vec{e}_{(b)} + \vec{e}_{(b)} D\vec{e}_{(a)}$$

$$= \underbrace{\vec{e}_{(a)} \vec{e}_{(c)}}_{g_{ac}} \omega^{c}{}_{b} + \underbrace{\vec{e}_{(b)} \vec{e}_{(c)}}_{g_{bc}} \omega^{c}{}_{a}$$

$\Longrightarrow$

$$\boxed{d g_{ab} = \omega_{ab} + \omega_{ba}} \qquad (x)$$

Symmetrized 1-forms $2\omega_{(ab)}$ are equal to exterior differentials of 0-forms g_{ab}

In natural basis (coordinates) $e^{\alpha}_{(a)} = \delta^{\alpha}_{a}$

$g_{\alpha\beta} = g_{ab}$ numerically and (x) $\Rightarrow$

$$d g_{\alpha\beta} = (\underbrace{\gamma_{\alpha\beta\gamma}}_{g_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma}} + \underbrace{\gamma_{\beta\alpha\gamma}}_{\dots\dots}) \underbrace{\theta^{\gamma}}_{dx^{\gamma}}$$

$$\Rightarrow \quad \partial_{\gamma} g_{\alpha\beta} = g_{\alpha\mu} \Gamma^{\mu}_{\beta\gamma} + g_{\mu\beta} \Gamma^{\mu}_{\alpha\gamma}$$

$$\Rightarrow \quad g_{\alpha\beta;\gamma} = 0$$

First Cartan structure equations

calculate exterior derivatives of 1-forms

$$\theta^a = e^{(a)}_{\ \beta}\, dx^\beta$$

$$d\theta^a = 2!\, \underbrace{e^{(a)}_{\ \beta;\gamma}}_{= -\gamma^a_{\ bc}\, e^{(c)}_{\ \gamma}\, e^{(b)}_{\ \beta}}\, dx^{[\gamma} dy^{\beta]}$$

(from $\gamma^a_{\ bc} = -e^{(a)}_{\ \beta;\gamma}\, e^{\beta}_{\ (b)}\, e^{\gamma}_{\ (c)} \quad | \times e^{(b)}_{\ \delta}\, e^{(c)}_{\ \varepsilon}$

$$\Rightarrow \gamma^a_{\ bc}\, e^{(b)}_{\ \delta}\, e^{(c)}_{\ \varepsilon} = -e^{(a)}_{\ \beta;\gamma}\, \delta^\beta_\delta\, \delta^\gamma_\varepsilon =$$

$$= -e^{(a)}_{\ \delta;\varepsilon} \quad \checkmark \;)$$

$$\delta \to \beta \qquad \varepsilon \to \gamma$$

$$\Rightarrow d\theta^a = -\gamma^a_{\ bc}\, \theta^c \wedge \theta^b$$

but $\gamma^a_{\ bc}\, \theta^c = \omega^a_{\ b}$ (viz F11)

$$\Rightarrow \boxed{d\theta^a = -\omega^a_{\ b} \wedge \theta^b}$$

E. Cartan calls this the "first structure eqs."

$\rightarrow$ can calculate $\gamma^a_{\ [bc]}$ ... as curls of dual basis $\vec{e}^{(a)}$

Theorem

Assume 1-forms θ^a and the matrix g_{ab} are given as functions of x^μ. Then the connection 1-forms $\omega^a{}_b$ are fully (uniquely) determined by equations

$$dg_{ab} = \omega_{ab} + \omega_{ba} \qquad (1)$$

$$d\theta^a = -\omega^a{}_b \wedge \theta^b \qquad (2)$$

Proof: (i) $\gamma_{(ab)c}$ are uniquely determined by (1) as numerical coefficients:

$$dg_{ab} = \gamma_{abc}\,\theta^c + \gamma_{bac}\,\theta^c = 2\gamma_{(ab)c}\,\theta^c$$

knowing $dg_{ab} \Rightarrow \gamma_{(ab)c}$ as coefficients of linear comb.

(ii) multiply (2) by g_{ab} (changing first $a \to b$, $b \to c$ in (2))

$$g_{ab}\,d\theta^b = -g_{ab}\,\omega^b{}_c \wedge \theta^c =$$

$$= -g_{ab}\,\gamma^b{}_{cd}\,\theta^d \wedge \theta^c = -\gamma_{acd}\,\theta^d \wedge \theta^c$$

$$= -\gamma_{a[\underset{\downarrow b}{c}\underset{\downarrow c}{d}]}\,\theta^d \wedge \theta^c = -\gamma_{a[bc]}\,\theta^c \wedge \theta^b$$

$$\Rightarrow \gamma_{a[bc]}$$

$$\Rightarrow \omega_{ab} = \gamma_{abc}\,\theta^c =$$

$$= \left[\gamma_{(ab)c} + \gamma_{(ac)b} - \gamma_{(bc)a} + \gamma_{a[bc]} + \gamma_{b[ca]} - \gamma_{c[ab]}\right]\theta^c$$

In practice (see our examples later) one can choose g_{ab} constant (indep. of x^μ)

$$\Rightarrow \quad dg_{ab} = \omega_{ab} + \omega_{ba} = 0$$

$$\Rightarrow \omega_{ab} = -\omega_{ba}$$

Note: g_{ab} is constant for any orthonormal basis as well as for a null tetrad (chosen by Sachs) in grav. radiation theory

It "remains" to solve $d\theta^a = -\omega^a{}_b \wedge \theta^b$ the solution can "usually" be guessed (see our examples later) and from Theorem on page F.14 we know the solution is unique.

F 16

Curvature 2-forms $\Omega^a{}_b$

connect curvature tensor with ext. diff. of $\omega^a{}_b$

$$\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b}$$

Second Cartan structure equations

$\Omega^a{}_b$ defined by $\boxed{\Omega^a{}_b \overset{\text{def.}}{=} \frac{1}{2} \underbrace{R^a{}_{bcd}}\, \theta^c \wedge \theta^d}$

frame components
of the Riemann tensor
(scalars)

Proof:

$$\omega^a{}_b = \gamma^a{}_{bc}\,\theta^c = -e^{(a)}{}_{\beta;\gamma}\, e^\beta_{(b)} \underbrace{e^\gamma_{(c)}\, e^{(c)}_\delta}_{\delta^\gamma_\delta} dx^\delta$$

$$= -e^{(a)}{}_{\beta;\gamma}\, e^\beta_{(b)}\, dx^\gamma$$

$$d\omega^a{}_b = -2!\left[e^{(a)}{}_{\beta;\gamma}\, e^\beta_{(b)}\right]_{;\delta} dx^{[\delta} dy^{\gamma]}$$

$$= -2!\left[e^{(a)}{}_{\beta;\gamma;\delta}\, e^\beta_{(b)} + e^{(a)}{}_{\beta;\gamma}\, e^\beta_{(b);\delta}\right] dx^{[\delta} dy^{\gamma]}$$

$$= -2!\left\{e^{(a)}{}_{\beta;[\gamma;\delta]}\, e^\beta_{(b)}\, dx^\delta dy^\gamma + e^{(a)}{}_{\beta;[\gamma}\, e^\beta_{(b);\delta]}\, dx^\delta dy^\gamma\right\}$$

$$= -2!\,\frac{1}{2} R^\alpha{}_{\beta\gamma\delta}\, e^{(a)}_\alpha\, e^\beta_{(b)}\, dx^\delta dy^\gamma \underbrace{- 2!\, e^{(a)}{}_{\beta;[\gamma}\, e^\beta_{(b);\delta]}\, dx^\delta dy^\gamma}_{=(*)}$$

$$= R^\alpha{}_{\beta\delta\gamma}\, e^{(a)}_\alpha e^\beta_{(b)}\, dx^\delta dy^\gamma - (*) = R^\alpha{}_{\beta\delta\gamma}\, e^{(a)}_\alpha\, e^\beta_{(b)}\, dx^{[\delta} dy^{\gamma]} - (*)$$

the term (*):

$$-2!\, e^{(a)}{}_{\beta;[\gamma}\, e^{\beta}{}_{(b);\delta]}\, dx^{\delta} dy^{\gamma} = -2!\, e^{(a)}{}_{\beta;\gamma}\, e^{\beta}{}_{(b);\delta}\, dx^{[\delta} dy^{\gamma]}$$

on the other hand:

$$\omega^{a}{}_{c} \wedge \omega^{c}{}_{b} = 2!\, e^{(a)}{}_{\beta;\gamma}\, e^{\beta}{}_{(c)}\, e^{(c)}{}_{\rho;\sigma}\, e^{\rho}{}_{(b)}\, dx^{[\gamma} dy^{\sigma]}$$

[here use $e^{(c)}{}_{\rho;\sigma}\, e^{\rho}{}_{(b)} = \underbrace{\left(e^{(c)}{}_{\rho}\, e^{\rho}{}_{(b)}\right)_{;\sigma}}_{=\delta^{a}_{b}\,=0} - e^{(c)}{}_{\rho}\, e^{\rho}{}_{(b);\sigma} =$

$= -\, e^{(c)}{}_{\rho}\, e^{\rho}{}_{(b);\sigma}$]

$$\Rightarrow \omega^{a}{}_{c} \wedge \omega^{c}{}_{b} = 2!\, e^{(a)}{}_{\beta;\gamma}\, e^{\beta}{}_{(c)} \left(-e^{(c)}{}_{\rho}\, e^{\rho}{}_{(b);\sigma}\right) dx^{[\gamma} dy^{\sigma]}$$

$$= -2\, e^{(a)}{}_{\beta;\gamma}\, e^{\beta}{}_{(b);\underset{\downarrow \delta}{\sigma}}\, dx^{[\gamma} dy^{\underset{\downarrow \delta}{\sigma}]}$$

it exactly cancels the (*) term at the top

This concludes the proof since

$$\Omega^{a}{}_{b} = \frac{1}{2} R^{a}{}_{bcd}\, \theta^{c} \wedge \theta^{d} =$$

$$= \frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta}\, e^{(a)}{}_{\alpha}\, e^{\beta}{}_{(b)}\, e^{\gamma}{}_{(c)}\, e^{\delta}{}_{(d)}\, 2!\, e^{(c)}{}_{\rho}\, e^{(d)}{}_{\sigma}\, dx^{[\rho} dy^{\sigma]}$$

$$= R^{\alpha}{}_{\beta\gamma\delta}\, dx^{[\gamma} dy^{\delta]}\, e^{(a)}{}_{\alpha}\, e^{\beta}{}_{(b)}$$

which, after renaming the indices $\delta \Leftrightarrow \gamma$ and using asymmetry of $R^{\alpha}{}_{\beta\gamma\delta}$ in $\gamma\delta$ is equal to the first term at the bottom of previous page $\boxed{\text{F16}}$

Identities for curvature

In Cartan's formalism Bianchi identities arise as the integrability conditions for the 2nd structure equations

Start from $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ (•)

Exterior differential of $\Omega^a{}_b$

using $d(d\omega^a{}_b) = 0$ and for 1-forms

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$$

Therefore,

$$d\Omega^a{}_b = \underbrace{d^2\omega^a{}_b}_{=0} + \underbrace{d\omega^a{}_c}_{\text{back from (•) above}} \wedge \omega^c{}_b - \omega^a{}_c \wedge \underbrace{d\omega^c{}_b}_{\text{back from (•)}}$$

$$\Rightarrow \quad \underline{\underline{d\Omega^a{}_b = \Omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \Omega^c{}_b}} \quad (BI)$$

These are <u>Bianchi identities</u> (in general form)

To see it, specialize to the coordinate basis and local Riemannian coordinates ($\Gamma^\alpha_{\beta\gamma} = 0$) so at the given point $\omega^\alpha{}_\beta = \Gamma^\alpha_{\beta\gamma} dx^\gamma = 0$ and (BI) imply

$$\partial_\varepsilon R^\alpha{}_{\beta\gamma\delta}\, dx^{[\varepsilon} dy^\gamma dz^{\delta]} = 0$$

for any $dx^\varepsilon, dy^\gamma, dz^\delta \Rightarrow R^\alpha{}_{\beta[\gamma\delta;\varepsilon]} = 0$!

The <u>next identity</u> from the 1st structure eqs $d\theta^a = -\omega^a{}_b \wedge \theta^b$

$$\Rightarrow \underbrace{d^2\theta^a}_{=0} = -d\omega^a{}_b \wedge \theta^b + \omega^a{}_b \wedge d\theta^b$$

$$\Rightarrow \boxed{\Omega^a{}_b \wedge \theta^b = 0} \Rightarrow R_{abcd}\, \theta^b \wedge \theta^c \wedge \theta^d = 0$$

$$\Rightarrow R_{a[bcd]} = 0$$

Next relation: $\boxed{\Omega_{ab} = -\Omega_{ba}}$
(a identity)

Proof: use $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, $\Omega_{ab} = -\Omega_{ba}$

and $d(f\Omega) = df \wedge d\Omega + f \cdot d\Omega$

and $dg_{ab} = \omega_{ab} + \omega_{ba}$

So, $\Omega_{ab} = g_{as}\Omega^s{}_b = g_{as}\, d\omega^s{}_b + \omega_{ac} \wedge \omega^c{}_b$

Then

$$\Omega_{ab} + \Omega_{ba} = g_{as}\, d\omega^s{}_b + \omega_{ac} \wedge \omega^c{}_b +$$

$$+ g_{bs}\, d\omega^s{}_a + \omega_{bc} \wedge \omega^c{}_a =$$

$$= d(g_{as}\omega^s{}_b) - dg_{as} \wedge \omega^s{}_b + d(g_{bs}\omega^s{}_a) - dg_{bs} \wedge \omega^s{}_a$$

$$+ \omega_{ac} \wedge \omega^c{}_b + \omega_{bc} \wedge \omega^c{}_a =$$

$$= d\omega_{ab} - (\omega_{as} + \omega_{sa}) \wedge \omega^s{}_b + d\omega_{ba} - (\omega_{bs} + \omega_{sb}) \wedge \omega^s{}_a$$

$$+ \omega_{as} \wedge \omega^s{}_b + \omega_{bs} \wedge \omega^s{}_a =$$

$$= \underbrace{d\omega_{ab} + d\omega_{ba}}_{= d^2 g_{ab} = 0} - \omega_{sa} \wedge \omega^s{}_b - \omega_{sb} \wedge \omega^s{}_a$$

$$= -\omega_{sa} \wedge \omega^s{}_b + \underbrace{\omega^s{}_a \wedge \omega_{sb}}_{\omega_{sa} \wedge \omega^s{}_b} = 0 \checkmark \quad \text{q.e.d.}$$

Summary

Instead of the basis vectors $\vec{e}_{(1)} \dots \vec{e}_{(n)}$, $\vec{e}^{(1)} \dots \vec{e}^{(n)}$

we use independent 1-forms θ^a, $a = 1, \dots, n$

they form basis of all 1-forms, $\theta^a \wedge \theta^b$ of 2forms etc

Connection 1-forms $\omega^a{}_b$

satisfy 1-st Cartan equations

$$\boxed{d\theta^a = -\omega^a{}_b \wedge \theta^b}$$

$ds^2 = g_{ab}\, \theta^a \cdot \theta^b$, if $g_{ab} = \text{const}$,

$$\underline{\omega_{ab} + \omega_{ba} = 0}$$

frame components of metric

Curvature 2-forms 2nd Cartan eqs of structure

$$\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b}$$

$$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \theta^c \wedge \theta^d$$

frame components of Riemann