Gauss polar coordinates on a 2-surface

On an arbitrary 2-dim. manifold we can introduce such coordinates ρ, φ in which the metric reads

$$ds^2 = d\rho^2 + [f(\rho,\varphi)]^2 d\varphi^2 \qquad (*)$$

Note: we can always do a coordinate transformation $x^{1'} = h(x^1, x^2)$, $x^{2'} = g(x^1, x^2)$ such that in the new coordinates $g'_{11} = 1$ and $g'_{12} = 0$, so $(*)$ is general enough

for $f = \rho$ in $(*)$ we get $ds^2 = d\rho^2 + \rho^2 d\varphi^2$ which is the flat-space metric in cylindrical coordinates

$\rho \in (0, \infty)$
$\varphi \in (0, 2\pi)$

for $f = a \sin\frac{\rho}{a}$, $\rho \equiv a\vartheta$, $a = \text{const}$, $\vartheta \in \langle 0, \pi \rangle$

$(*)$ implies

$$ds^2 = a^2 d\vartheta^2 + a^2 \sin^2\frac{a\vartheta}{a} d\varphi^2 =$$

$$= a^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right)$$

which is the metric on the 2-sphere with radius a

As the basis vector choose unit vectors $\vec{e}_{(1)}, \vec{e}_{(2)}$ (so not natural basis $\delta^\alpha_\cdot$)

so $\vec{e}_{(1)} = \left(\underbrace{\frac{1}{\sqrt{g_{11}}}}_{=1}, 0\right)$, $\vec{e}_{(2)} = \left(0, \frac{1}{\sqrt{g_{22}}}\right) = \left(0, \frac{1}{f}\right)$

$$\vec{e}_{(2)} \cdot \vec{e}_{(2)} = g_{\alpha\beta} e^\alpha_{(2)} e^\beta_{(2)} = g_{22} \left(e^2_{(2)}\right)^2 = f^2 \frac{1}{f^2} = 1$$

So $g_{ab} = \vec{e}_{(a)} \vec{e}_{(b)} = g_{\alpha\beta} \, e_{(a)}^{\alpha} e_{(b)}^{\beta} = \delta_{ab} = \text{const}!$

$\Rightarrow g^{ab} = \delta^{ab}$

'covariant' basis (dual to $\vec{e}_{(a)}$) is

$$\vec{e}^{(a)} = g^{ab} \vec{e}_{(b)} = \vec{e}_{(a)}$$

Denoting $dx^{\alpha} = (d\rho, d\varphi)$ an arbitrary 'shift' we have

$$\theta^1(d\vec{x}) = e^{(1)}_{\alpha} dx^{\alpha} = d\rho$$

$$\theta^2(d\vec{x}) = e^{(2)}_{\alpha} dx^{\alpha} = f \, d\varphi$$

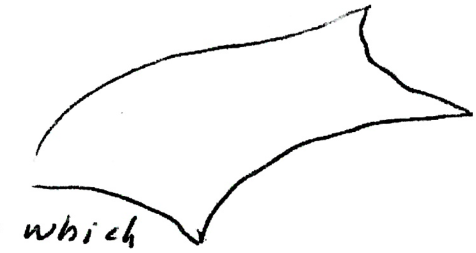
So the metric $(*)$ can be written as

$$\boxed{ds^2 = (\theta^1)^2 + (\theta^2)^2}$$

where it is understood that $\theta^1(d\vec{x}) = d\rho$, $\theta^2(d\vec{x}) = f d\varphi$

General 2-dimensional surface

One can always introduce
- at least at some region



Gauss polar coordinates in which

$$\boxed{ds^2 = d\rho^2 + [f(\rho,\varphi)]^2 d\varphi^2} \qquad (\sim)$$

for $f = \rho \Rightarrow$ plane in cylindrical coordinates

for $f = a\sin\frac{\rho}{a}$, $\rho \equiv a\vartheta$, $a = \text{const}$

$$\Rightarrow \quad ds^2 = a^2 d\vartheta^2 + a\sin^2\vartheta\, d\varphi^2 = a^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

line element on 2-sphere of radius a

We found in detail (see G2) that it is useful to put $\boxed{\theta^1 = d\rho, \quad \theta^2 = f\,d\varphi}$

recall frame comp. $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

the metric $(\sim)$ can thus be written simply as

$$ds^2 = (\theta^1)^2 + (\theta^2)^2$$

since $dg_{ab} = \omega_{ab} + \omega_{ba}$ and $g_{ab} = \text{const}$

the only non-vanishing connexion 1-form

is $\quad \omega_{12} = -\omega_{21} \qquad (\omega_{11} = \omega_{22} = 0)$

We shall „guess" solution from Cartan's first eqs:

$$d\theta^a = -\omega^a{}_b \wedge \theta^b$$

$a=1$

$$d\theta^1 = -\omega^1{}_1 \wedge \theta^1 - \omega^1{}_2 \wedge \theta^2$$

but $\omega^1{}_1 = g^{1a}\omega_{a1} = g^{11}\underbrace{\omega_{11}}_{=0} = 0$

and $d\theta^1 = d(d\rho) = d^2\rho = 0$

So $\quad 0 = -\omega^1{}_2 \wedge f\,d\varphi$

In order this to be true, $\omega^1{}_2 \sim d\varphi$

for $a=2$:

$$d\theta^2 = -\omega^2{}_1 \wedge \theta^1 \qquad (\omega^2{}_2 = g^{2a}\omega_{a2} = 0)$$

$$d\theta^2 = d(f\,d\varphi) = df \wedge d\varphi + f\underbrace{d^2\varphi}_{=0} =$$

$$= f_{,\rho}\,d\rho \wedge d\varphi + f_{,\varphi}\underbrace{d\varphi \wedge d\varphi}_{=0} = \longrightarrow = f_{,\rho}\,d\rho \wedge d\varphi$$

So

$$\underbrace{f_{,\rho}\,d\rho \wedge d\varphi}_{} = -\omega^2{}_1 \wedge d\rho$$

$$= \frac{f_{,\rho}}{f}\theta^1 \wedge \theta^2 = -\omega^2{}_1 \wedge \theta^1 = +\theta^1 \wedge \omega^2{}_1$$

$$\Rightarrow \boxed{\omega^2{}_1 = \frac{f_{,\rho}}{f}\theta^2}, \quad \Rightarrow \omega_{21} = -\omega_{12} = -\omega^1{}_2 =$$

$$\Rightarrow \omega^1{}_2 = -\frac{f_{,\rho}}{f}\theta^2$$

indeed is $\sim d\varphi$

Curvature 2-form

$$\Omega^2{}_1 = d\omega^2{}_1 + \underbrace{\omega^2{}_1 \wedge \omega^1{}_2}_{\sim\theta^2 \wedge \sim\theta^2 \,=\, 0} + \underbrace{\omega^2{}_2}_{=0} \wedge \omega^2{}_1$$

$=0$ because $\omega_{22}=0$

$$\Rightarrow \Omega^2{}_1 = d\omega^2{}_1 = d(f_{,\rho}\, d\varphi) = f_{,\rho\rho}\, d\rho \wedge d\varphi + \underbrace{\sim d^2\varphi}_{=0}$$

there is no $f_{,\varphi\rho}\, d\varphi$ since $\wedge\, d\varphi = 0$

$$\Rightarrow \Omega^2{}_1 = f_{,\rho\rho} \underbrace{d\rho}_{\theta^1} \wedge \underbrace{d\varphi}_{\frac{1}{f}\theta^2} = \frac{f_{,\rho\rho}}{f}\,\theta^1 \wedge \theta^2$$

$$\boxed{\Omega^2{}_1 = \frac{f_{,\rho\rho}}{f}\,\theta^1 \wedge \theta^2}$$

$$\Omega^2{}_1 = \frac{1}{2} R^2{}_{1cd}\,\theta^c \wedge \theta^d = \frac{1}{2} R^2{}_{112}\,\theta^1 \wedge \theta^2 + \frac{1}{2} R^2{}_{121}\,\theta^2 \wedge \theta^1$$
$$= R^2{}_{112}\,\theta^1 \wedge \theta^2$$

$$\Rightarrow \boxed{R^2{}_{112} = \frac{f_{,\rho\rho}}{f}}$$ other components from symmetries

for flat plane $f = \rho$ and, of course, $R^{\cdot}{}_{\cdot\cdot\cdot} = 0$

for a sphere of radius a,

$$R^2{}_{112} \sim \frac{(a \sin\frac{\rho}{a})_{,\rho\rho}}{a \sin\frac{\rho}{a}} = \frac{(\cancel{a}\frac{1}{\cancel{a}} \cos\frac{\rho}{a})_{,\rho}}{a \sin\frac{\rho}{a}} = -\frac{\frac{1}{a}\sin\frac{\rho}{a}}{a \sin\frac{\rho}{a}}$$

so $\boxed{R^2{}_{112} = -\frac{1}{a^2}}$ as it should be, $\boxed{R^2{}_{121} = +\frac{1}{a^2}}$

correct! $a \to \infty$ $R \to 0$

From: E. Kramer: The Nature and Growth of modern mathematics

Élie Cartan

Undoubtedly one of the greatest mathematicians of this 20th century, Élie Cartan's career was nevertheless characterized by a rare harmony of genius and modesty. He was born on April 9, 1869, in Dolomieu (Isère), a village in the south of France. His father was a blacksmith. Cartan's elementary education was made possible by one of the state stipends for gifted children. In 1888 he entered the École Normale Supérieure, where he learned higher mathematics from such masters as Tannery, Picard, Darboux, and Hermite. His research work started with his famous thesis on continuous groups, a subject suggested to him by his fellow student Tresse, recently returned from studying with Sophus Lie in Leipzig. Cartan's first teaching position was at Montpellier, where he was *maître de conférences;* he then went successively to Lyons, to Nancy, and finally in 1909 to Paris. 40 years old He was made a professor at the Sorbonne in 1912. The report on his work which was the basis for this promotion was written by Poincaré; this was one of the circumstances in his career of which he seemed to have been genuinely proud. He remained at the Sorbonne until his retirement in 1940.

Cartan was an excellent teacher; his lectures were gratifying intellectual experiences, which left the student with a generally mistaken idea that he had grasped all there was on the subject. It is therefore the more surprising that for a long time his ideas did not exert the influence they so richly deserved to have on young mathematicians. This was partly due to Cartan's extreme modesty. But in 1939, the celebration of Cartan's scientific jubilee, J. Dieudonné could rightly say to him: "... *vous êtes un 'jeune,' et vous comprenez les jeunes*"—it was then beginning to be true that the young understood Cartan.

In foreign countries, particularly in Germany, his recognition as a great mathematician came earlier. It was perhaps H. Weyl's fundamental papers on group representations published around 1925 that established Cartan's reputation among mathematicians not in his own field. Meanwhile, the development of abstract algebra naturally helped to attract attention to his work on Lie algebra.

Cartan was elected to the French Academy in 1931. In his later years he received several other honors. Thus he was a foreign member of the National Academy of Sciences, U.S.A., and a foreign Fellow of the (British) Royal Society. In 1936 he was awarded an honorary degree by Harvard University.

Closely interwoven with Cartan's life as a scientist and teacher had been his family life, which was filled with an atmosphere of happiness and serenity. He had four children, three sons, Henri, Jean, and Louis, and a daughter, Hélène. Jean Cartan oriented himself towards music, and already appeared to be one of the most gifted composers of his generation, when he was cruelly taken by death. Louis Cartan was a physicist; arrested by the Germans at the beginning of the Resistance, he was murdered by them after a long period of detention. Henri Cartan followed in the footsteps of his father to become a leading mathematician.

Cartan's mathematical work can be roughly classified under three main headings: group theory, systems of differential equations, and geometry. These themes are, however, constantly interwoven with each other in his work. Almost everything Cartan did is more or less connected with the theory of Lie groups.

Sophus Lie introduced the groups of transformations which were named after him. The idea of considering the *abstract group* which underlies a given group of transformations came only later; it appears quite explicity in the first paper by Cartan. Whereas, for Lie, the problem of classification consisted in finding all possible transformation groups on a given number of variables (a far too difficult problem in the present stage of mathematics as soon as the number of variables is not very small), for Cartan the problem was to find all possible abstract structures of continuous groups. He solved the problem completely for "simple" groups (those having no proper normal subgroups). Once the structures of all simple groups were known, it became possible to look for all possible realizations of any one of these structures by transformations of a specific nature, and, in particular, for their realizations as groups of linear transformations. This is the problem of the determination of the representations of a given group; it was solved completely by Cartan for simple groups. The solution led in particular to the discovery, as early as 1913, of the *spinors*, which were to be rediscovered later in a special case by the physicists.

Cartan also investigated the infinite Lie groups, i.e., the groups of transformations whose operations depend *not* on a finite number of continuous parameters, but on arbitrary functions. In that case, one does not have the notion of the abstract underlying group. Cartan and Vessiot found, at about the same time and independently of each other, a substitute notion which consists in defining when two infinite Lie groups are to be considered as isomorphic. Cartan then proceeded to classify all possible types of non-isomorphic infinite Lie groups.

Cartan also paid much attention to the study of topological properties of groups consid[illegible] in the large. He showed how many of these topological problems may be reduced to purely algebraic questions; by so doing, he discovered the very remarkable fact that many properties of the group in the large may be read from the infinitesimal structure of the group, i.e., are already determined when some arbitrarily small piece of the group is given. His work along these lines resembles that of the paleontologist reconstructing the shape of a prehistoric animal from the peculiarities of some small bone.

The idea of studying the abstract structure of mathematical objects which hides itself beneath the analytical clothing was also the mainspring of Cartan's theory of differential systems. He insisted on having a theory of differential equations which is invariant under arbitrary changes in variables. Only in this way can the theory uncover the specific properties of the objects one studies by means of the differential equations they satisfy, in contradistinction to what depends only on the particular representation of these objects by numbers or sets of numbers. In order to achieve such an invariant theory, Cartan made a systematic use of the notion of the *exterior differential* of a differential form, a notion which he helped to create and which has the required property of being invariant with respect to any change of variables.

Raised in the French geometrical tradition, Cartan had a constant interest in differential geometry. He had the unusual combination of a vast knowledge of Lie groups, a theory of differential systems whose invariant character was particularly suited for geometrical investigations, and, most important of all, a remarkable geometrical intuition. As a result, he was able to see the geometrical content of very complicated calculations, and even to substitute geometrical arguments for some of the computations.

In the 1920's the general theory of relativity gave a new impulse to differential geometry. This gave rise to a feverish search for spaces with a suitable local structure. The most notable example of such a local structure is a Riemann metric. It can be generalized in various ways, by modifying the form of the integral which defines the arc length in Riemannian geometry (Finsler geometry), by studying only those properties pertaining to the geodesics or paths (geometry of paths of Eisenhart, Veblen, and T. Y. Thomas), by studying the properties of a family of Riemann metrics whose fundamental forms differ from each other by a common factor (conformal geometry), etc. While in all these directions the definition of a parallel displacement is considered to be the major concern, the approach of Cartan to these problems is most original and satisfactory. Again the notion of group plays the central role. Roughly speaking, a generalized space in the sense of Cartan is a space of tangent spaces such that two infinitely near tangent spaces are related by an infinitesimal transformation of a given Lie group. Such a structure is known as a connection.

Einstein–Cartan theory

Besides several books, Cartan published about 200 mathematical papers. His major specialties, in addition to geometry, were group theory and differential equations. Cartan's papers on group theory fall into two categories, distinguished from each other both by the nature of the question treated and by the time at which they were written.

The papers of the first cycle are purely algebraic in character; they are more concerned with what are now called Lie algebras than with group theory proper. The work of Cartan's second group-theoretic period is concerned with the groups themselves, and not with their Lie algebras, and in general with the global aspect of the group.

For an account of his algebraic discoveries we return to J. H. C. Whitehead once more.*

synovec A. N. Whitehead

In the years 1897, 1898 Cartan turned his attention from Lie algebras to linear associative algebras. In 1898 he proved the Wedderburn structure theorem [Chapter 28] for algebras over the real and complex fields. The methods which Wedderburn (1908) used in proving his theorem are more suitable than Cartan's to the problems of linear associative algebra. Indeed this paper of Wedderburn's is one of the outstanding contributions to the subject and it is reasonable to associate the theorem with his name. But the fundamental importance of Cartan's paper, which Wedderburn duly acknowledged, should not be forgotten.

During the years 1904 to 1909, there are his papers on infinite transformation groups, as defined by Lie. Such a group is "infinite" in the sense that its general transformation cannot be expressed in terms of a finite set of parameters. In general it is only defined as a local group, or pseudo-group, whose transformations operate on different open subsets of Cartesian space. Finally a (local) group, G, of this kind is defined as the totality of transformations which leave invariant a given set of differential equations. The transformations in G are themselves given by a set D, of differential equations. Since G is infinite the general solution of D is not expressed in terms of a finite set of parameters.

There have been very few, if any, new contributions to the general theory of infinite groups since these papers of Cartan. This is doubtless due to the difficulty of the subject and also to the appearance of temporary finality in Cartan's work. That is to say, there does not seem to have been much hope of greatly extending his theory with the methods which have been available during the last forty years. At the present time the obvious questions are those concerning global infinite groups, acting on n-dimensional manifolds. It may be that a theory of such groups will be constructed on the basis of Cartan's local theory. In this case it would not be surprising if the latter were eventually considered to be his greatest work.

From 1916 onwards Cartan's papers, with one or two exceptions, were on differential geometry, including the theory of generalized spaces and differential geometry in the large. This work on differential geometry would, by itself, have been sufficient to establish Cartan among the leading mathematicians of this half-century. It is remarkable that he embarked on it when he was nearly fifty years old and maintained a steady output of first-class work throughout the subsequent thirty years.

As for Cartan's work on differential equations, probably the best authority is his own 1945 book, *Les systèmes différentiels extérieurs et leurs applications géométriques*, But we now return to the Chern-Chevalley biography for further details of Cartan's geometric work, which is the major theme of the present chapter.†

Einstein's theory of general relativity gave a new impetus to differential geometry. In their efforts to find an appropriate model of the universe, geometers have broadened

* Royal Society of London, *Obituary Notices of Fellows, loc. cit.*

† Shing-Shen Chern and C. Chevalley, *loc. cit.*

their horizon from the study of submanifolds in classical spaces (Euclidean, non-Euclidean, projective, conformal, etc.) to that of more general spaces intrinsically defined. The result is an extension of the work of Gauss and Riemann on Riemannian geometry to spaces with a "connection," which may be an affine connection, a Weyl connection, a projective connection, or a conformal connection. In these generalizations, sometimes called non-Riemannian geometry, an important tool is the absolute differential calculus of Ricci and Levi-Civita. The results achieved are of considerable geometric interest. For instance, in the theory of projective connections, developed independently by Cartan, Veblen, Eisenhart, and Thomas, it is shown that when the space has a system of paths defined by a system of differential equations of the second order, a generalized projective geometry can be defined in the space which reduces to ordinary projective geometry when the differential system is that of the straight lines. Numerous other examples can be cited. The problem at this stage is twofold: (1) to give a definition of "geometry" which will include most of the existing spaces of interest; (2) to develop analytic methods for the treatment of the new geometries, it being increasingly clear that the absolute differential calculus is inadequate.

For this purpose Cartan developed what seems to be the most comprehensive and satisfactory program and demonstrated its advantages in a decisive way. This contribution clearly illustrates his geometric insight and we consider it to be the most important among his works on differential geometry. It can be best explained by means of the modern notion of a "fiber bundle."

A *fiber bundle* is merely the generalization of a simpler concept considered earlier in this book (Chapter 7), namely, the idea of a *Cartesian product*. Let us recall that if there are two sets of numbers, $A = \{1, 2, 3\}$ and $B = \{7, 8\}$, then the Cartesian product symbolized by $A \times B$ and read as "*A cross B*" consists of all possible ordered number pairs formed from selecting the first number from A and the second from B. Thus, "A cross B," or $A \times B$, is the set of *six* number pairs, $\{(1, 7), (1, 8), (2, 7), (2, 8), (3, 7), (3, 8)\}$. Again, if A is the set of all real numbers between 0 and 2—that is, the interval [0, 2] on the X-axis in Figure 20.5—and B is the set of all real numbers in the interval [0, 1] on the Y-axis, then $A \times B$ is the shaded rectangle in Figure 20.5.

If A consists of all points of a circle and B of all points of a line segment, then $A \times B$ is the surface of a cylinder. To clarify this, suppose that the equation of the circle A is $x^2 + y^2 = 25$ (Figure 20.6), and that B is the interval [0, 1] on

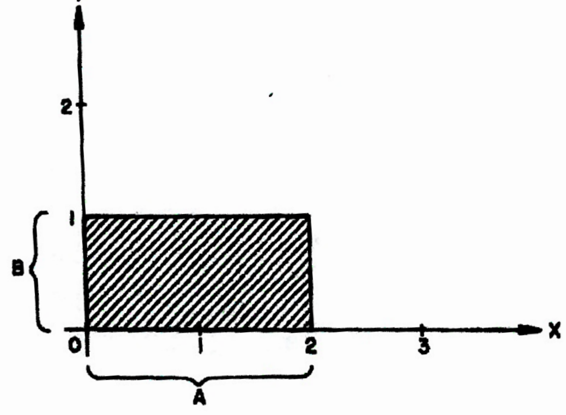


Figure 20.5 $A \times B$ for $A = [0, 2]$ and $B = [0, 1]$

Vaidya metric in differential forms - calculating the curvature tensor and field equations

The metric, using „−2" signature $(+---)$

$$\boxed{ds^2 = 2\,du\,dr + \left[1 - \frac{2m(u)}{r}\right]du^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)}$$

Spherically symmetric but non-stationary (u-dep.)
clearly, it cannot be vacuum metric (cf. Birkhoff thm.)
wish to choose the basic 1-forms $\theta^a = e^{(a)}_{\alpha}dx^{\alpha}$
such that g_{ab} in $ds^2 = g_{ab}\theta^a\theta^b$ are constants
(because then $\omega_{ab} = -\omega_{ba}$ for connection 1-forms)

writing

$$ds^2 = \underbrace{2du}_{=\theta^0}\underbrace{\left\{dr + \frac{1}{2}\left[1 - \frac{2m(u)}{r}\right]du\right\}}_{=\theta^1}\underbrace{-r^2 d\vartheta^2}_{=(\theta^2)^2}\underbrace{-r^2\sin^2\vartheta\, d\varphi^2}_{=(\theta^3)^2}$$

and choosing
so put

$$\theta^0 = du$$
$$\theta^1 = dr + \frac{1}{2}\left[1 - \frac{2m(u)}{r}\right]du$$
$$\theta^2 = r\,d\vartheta$$
$$\theta^3 = r\sin\vartheta\, d\varphi$$

$$\Rightarrow \quad ds^2 = g_{ab}\theta^a\theta^b = 2\theta^0\theta^1 - (\theta^2)^2 - (\theta^3)^2$$

so $g_{10} = g_{01} = 1$, $g_{22} = g_{33} = -1$
all other components $= 0$

It is easy to check that the inverse g^{ab} is the same, so $g^{ab} = g_{ab}$

The first equations of structure

$$(\mathrm{I}) \qquad \boxed{d\theta^a = -\omega^a{}_b \wedge \theta^b}$$

become (using $d(f\omega) = f d\omega + df \wedge \omega$)

(0) $\quad d\theta^0 = d(du) = 0$

(1) $\quad d\theta^1 = \underbrace{d(dr)}_{=0} + d\Big[\underbrace{\tfrac{1}{2}\Big(1-\frac{2m}{r}\Big)}_{\text{scalar}} \underbrace{du}_{\text{1-form}}\Big] =$

notice:
$\frac{2dm}{r} \wedge du = \frac{2\dot{m}}{r} \underbrace{du \wedge du}_{=0}$

$$= -d\Big(\frac{m}{r}\Big)\wedge du = \frac{m}{r^2} dr \wedge du = \frac{m}{r^2}\theta^1 \wedge \theta^0$$

(2) $\quad d\theta^2 = dr \wedge d\vartheta = \frac{1}{r}\theta^1 \wedge \theta^2 - \frac{1}{2}\Big(1-\frac{2m}{r}\Big) du \wedge d\vartheta$

$$= \frac{1}{r}\theta^1 \wedge \theta^2 - \frac{1}{2}\Big(1-\frac{2m}{r}\Big)\frac{1}{r}\theta^0 \wedge \theta^2$$

$$= \frac{1}{r}\theta^1 \wedge \theta^2 - \frac{1}{2r}\Big(1-\frac{2m}{r}\Big)\theta^0 \wedge \theta^2$$

(3) $\quad d\theta^3 = d(r\sin\vartheta\, d\varphi) = \sin\vartheta\, dr \wedge d\varphi + r\cos\vartheta\, d\vartheta \wedge d\varphi =$

$$= \frac{1}{r}\Big[\theta^1 - \frac{1}{2}\Big(1-\frac{2m}{r}\Big)\theta^0\Big]\wedge\theta^3 + \frac{1}{r}\cot\vartheta\, \theta^2 \wedge \theta^3$$

Now by comparing these expressions (0)–(3) with the general form (I) of the 1st structure eqs. we shall guess the results for $\omega^a{}_b$.

But first show that $\omega^1{}_0 = 0$:

$$\omega^1{}_0 = g^{1s}\omega_{s0} = g^{10}\omega_{00} = 0 \quad \text{because}$$

from $0 = \underbrace{dg_{ab}}_{\text{consts}} = \omega_{ab} + \omega_{ba} \Rightarrow \omega_{00} = 0$

and also $\omega_{11} = \omega_{22} = \omega_{33} = 0$

Next regarding (1), i.e. $d\theta^1 = \frac{m}{r^2}\,\theta^1 \wedge \theta^0$,

and comparing with general

$$(\tilde{1}) \qquad d\theta^1 = -\omega^1{}_a \wedge \theta^a = -\omega^1{}_0 \wedge \theta^0 - \omega^1{}_1 \wedge \theta^1 - \omega^1{}_2 \wedge \theta^2 - \omega^1{}_3 \wedge \theta^3$$

but $\omega^1{}_0 = 0$, comparing: $\omega^1{}_1 = \frac{m}{r^2}\,\theta^0$

so that $-\omega^1{}_1 \wedge \theta^1 = -\frac{m}{r^2}\,\theta^0 \wedge \theta^1 = +\frac{m}{r^2}\,\theta^1 \wedge \theta^0$

and we, putting

$$\omega^1{}_2 = A\,\theta^2, \quad \omega^1{}_3 = B\,\theta^3,$$

coefficients A, B to be determined,

satisfy $(\tilde{1})$

Further, compare (2), i.e.

$$d\theta^2 = \frac{1}{r}\,\theta^1 \wedge \theta^2 - \frac{1}{2r}\left(1 - \frac{2m}{r}\right)\theta^0 \wedge \theta^2$$

with eq.

$$(\tilde{2}) \quad d\theta^2 = -\omega^2{}_a \wedge \theta^a = -\omega^2{}_0 \wedge \theta^0 - \omega^2{}_1 \wedge \theta^1 - \omega^2{}_2 \wedge \theta^2 - \omega^2{}_3 \wedge \theta^3$$

Guess: $\omega^2{}_1 = \frac{\theta^2}{r}$, $\omega^2{}_2 = g^{2s}\omega_{s2} = 0$, $\omega^2{}_3 = C\,\theta^3$

$\omega^2{}_0 = -\frac{1}{2r}\left(1 - \frac{2m}{r}\right)\theta^2$

Finally, compare (3), i.e.

$$d\theta^3 = \frac{1}{r}\left[\theta^1 - \frac{1}{2}\left(1 - \frac{2m}{r}\right)\theta^0\right] \wedge \theta^3 + \frac{1}{r}\cot\vartheta\,\theta^2 \wedge \theta^3$$

with

$$(\tilde{3}) \quad d\theta^3 = -\omega^3{}_a \wedge \theta^a = -\omega^3{}_0 \wedge \theta^0 - \omega^3{}_1 \wedge \theta^1 - \omega^3{}_2 \wedge \theta^2 - \underbrace{\omega^3{}_3}_{=0} \wedge \theta^3$$

and guess:

$$\omega^3{}_0 = -\frac{1}{2r}\left(1 - \frac{2m}{r}\right)\theta^3, \quad \omega^3{}_1 = \frac{1}{r}\,\theta^3, \quad \omega^3{}_2 = \frac{1}{r}\cot\vartheta\,\theta^3$$

The other ω_{ab} from $\omega_{ab} = -\omega_{ba}$:

For example, $\underbrace{\omega^1{}_2}_{} = g^{10}\omega_{02} = -\omega_{20} = -\omega^2{}_0 =$

$g^{1s}\omega_{s2}$ $\qquad = \frac{1}{2r}\left(1-\frac{2m}{r}\right)\theta^2$

similarly, $\omega^1{}_3 = -\omega^3{}_0 = \frac{1}{2r}\left(1-\frac{2m}{r}\right)\theta^3$

These results enable one to fix A, B (see, p.)

$\omega^1{}_2 = A\,\theta^2$ comparing with above $\Rightarrow A = \frac{1}{2r}\left(1-\frac{2m}{r}\right)$

and

$\omega^1{}_3 = B\,\theta^3$ —— " —— $\Rightarrow B = \frac{1}{2r}\left(1-\frac{2m}{r}\right)$

similarly, C is fixed by

$\omega^2{}_3 = C\,\theta^3 = -\omega^3{}_2 = -\frac{1}{r}\cot\vartheta\,\theta^3$

$\Rightarrow C = -\frac{1}{r}\cot\vartheta$

we also find

$\omega^0{}_1 = g^{0s}\omega_{s1} = g^{01}\underbrace{\omega_{11}}_{=0} = 0$

$\omega^0{}_2 = g^{0s}\omega_{s2} = g^{01}\omega_{12} = \omega_{12} = -\omega_{21} = -\omega^2{}_1 =$
$= -\frac{\theta^2}{r}$

$\omega^0{}_3 = g^{0s}\omega_{s3} = g^{01}\omega_{13} = \omega_{13} = -\omega_{31} = -\omega^3{}_1 =$
$= -\frac{1}{r}\theta^3$

$\omega^0{}_0 = g^{0s}\omega_{s0} = g^{01}\omega_{10} = -\omega_{01} = -g_{0s}\omega^s{}_1 = -g_{01}\omega^1{}_1$
$= -\omega^1{}_1$
$= -\frac{m}{r^2}\theta^0$

From the results for all $\omega^0{}_s$ we can see that $d\theta^0 = \omega^0{}_s \wedge \theta^s = 0$ as it should be since $\theta^0 = du$ and thus $d\theta^0 = d^2 u = 0$

In this way we obtain unique (correct) solutions $\omega^a{}_b$ of the 1st structure eqs.

$$d\theta^a = -\omega^a{}_b \wedge \theta^b$$

This was the most complicated part of the game. The remaining part, the calculation of the curvature forms is straightforward - 2nd Cartan's structure eqs.

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

For example,

$$\Omega^1{}_1 = d\omega^1{}_1 + \omega^1{}_1 \wedge \omega^1{}_1 + \omega^1{}_2 \wedge \omega^2{}_1 + \omega^1{}_3 \wedge \omega^3{}_1$$

$$= d\left[\left(\frac{m}{r^2}\right)\theta^0\right] + 0 + \frac{1}{2r^2}\left(1-\frac{2m}{r}\right)\left(\underbrace{\theta^2\wedge\theta^2}_{0} + \underbrace{\theta^3\wedge\theta^3}_{0}\right)$$

$$= -\left(\frac{2m}{r^3}\right) dr \wedge \theta^0 = -\left(\frac{2m}{r^3}\right)\theta^1 \wedge \theta^0$$

Comparing with $\Omega^a{}_b$ expressed in terms of tetrad (frame) components of $R^\alpha{}_{\beta\gamma\delta}$, i.e. $R^a{}_{bcd}$ we get $\Omega^1{}_1 = \frac{1}{2} R^1{}_{1cd}\,\theta^c \wedge \theta^d$

$$\Rightarrow R^1{}_{101} = -R^1{}_{110} = \frac{2m}{r^3}, \quad \text{other } R^1{}_{1cd} = 0$$

Va 6

Similarly we find

we have $\omega^1{}_2 = A\theta^2;\ \omega^1{}_3 = B\theta^3$
$\omega^1{}_1 = \frac{m}{r^2}\theta^0;\ \omega^2{}_3 = C\theta^3$

$$\Omega^1{}_2 = d\omega^1{}_2 + \underbrace{\omega^1{}_0}_{0} \wedge \omega^0{}_2 + \omega^1{}_1 \wedge \omega^1{}_2 + \omega^1{}_2 \wedge \underbrace{\omega^2{}_2}_{=0}$$

$$+ \omega^1{}_3 \wedge \omega^3{}_2 =$$

$$= \Big[\underbrace{\frac{1}{2r}\Big(1-\frac{2m}{r}\Big)}_{A}\Big]_{,r} dr \wedge \theta^2 + \Big[\underbrace{\frac{1}{2r}\Big(1-\frac{2m}{r}\Big)}_{A}\Big]_{,u} du \wedge \theta^2$$

$$+ \frac{1}{2r}\Big(1-\frac{2m}{r}\Big) d\theta^2 +$$

$$+ 0 + \underbrace{\frac{m}{r^2}\theta^0}_{\omega^1{}_1} \wedge \underbrace{\frac{1}{2r}\Big(1-\frac{2m}{r}\Big)\theta^2}_{\omega^1{}_2}$$

$$+ \underbrace{\frac{1}{2r}\Big(1-\frac{2m}{r}\Big)}_{B} \underbrace{\theta^3 \wedge \frac{1}{r}\cot\vartheta\ \theta^3}_{=0} = \dots$$

only term $\sim \dot{m}$

$$= \Big[\frac{1}{2r}\Big(1-\frac{2m}{r}\Big)\Big]_{,u} \theta^0 \wedge \theta^2 + \dots$$

$$= -\frac{\dot{m}}{r^2}\theta^0 \wedge \theta^2 + \dots = +\frac{\dot{m}}{r^2}\theta^2 \wedge \theta^0 + \dots$$

all terms

$$= \Big[-\frac{1}{2r^2}\Big(1-\frac{2m}{r}\Big) + \frac{1}{2r}\frac{2m}{r^2}\Big]\Big(\theta^1 - \frac{1}{2}\Big(1-\frac{2m}{r}\Big)\theta^0\Big) \wedge \theta^2$$

$$- \frac{\dot{m}}{r^2}\theta^0 \wedge \theta^2 + \frac{1}{2r}\Big(1-\frac{2m}{r}\Big)\Big[\frac{1}{r}\theta^1 \wedge \theta^2 - \frac{1}{2r}\Big(1-\frac{2m}{r}\Big)\theta^0 \wedge \theta^2\Big]$$

$$+ \frac{m}{r^2}\frac{1}{2r}\Big(1-\frac{2m}{r}\Big)\theta^0 \wedge \theta^2$$

Similarly, after calculations one gets

$$\Omega^1{}_2 = \left(\frac{\dot m}{r^2}\right)\theta^2 \wedge \theta^0 + \left(\frac{m}{r^3}\right)\theta^1 \wedge \theta^2$$

$$\Omega^1{}_3 = \left(\frac{\dot m}{r^2}\right)\theta^3 \wedge \theta^0 + \left(\frac{m}{r^3}\right)\theta^1 \wedge \theta^3$$

$$\Omega^2{}_1 = \left(\frac{m}{r^3}\right)\theta^2 \wedge \theta^0$$

$$\Omega^3{}_1 = \left(\frac{m}{r^3}\right)\theta^3 \wedge \theta^0$$

$$\Omega^2{}_3 = -\left(\frac{2m}{r^3}\right)\theta^2 \wedge \theta^3$$

To dermine other $R^{\cdot}{}_{\cdots}$ use antisymmetry

$$\Omega_{ab} = -\Omega_{ba}$$

By contracting tetrad components of Riemann $\Rightarrow$ Ricci

Using $\Omega^1{}_0 = \Omega_{00} = 0$, $\Omega^2{}_0 = -\Omega^1{}_2$ we get

$$\boxed{R_{00} = R^2{}_{002} + R^3{}_{003} = -R^1{}_{202} - R^1{}_{303} = \dots = \frac{2\dot m}{r^2}}$$

all other components $R_{\cdot\cdot} = 0$

The coordinate components of $R_{\alpha\beta}$ are

$$\boxed{R_{\alpha\beta} = R_{ab}\, e^{(a)}_\alpha e^{(b)}_\beta = R_{00}\, e^{(0)}_\alpha e^{(0)}_\beta = 2\left(\frac{\dot m}{r^2}\right)\partial_\alpha u\, \partial_\beta u}$$

i.e. $R_{uu} = 2\dot m / r^2$ other components $\Rightarrow 0$

Vaidya SpTime
(solution)

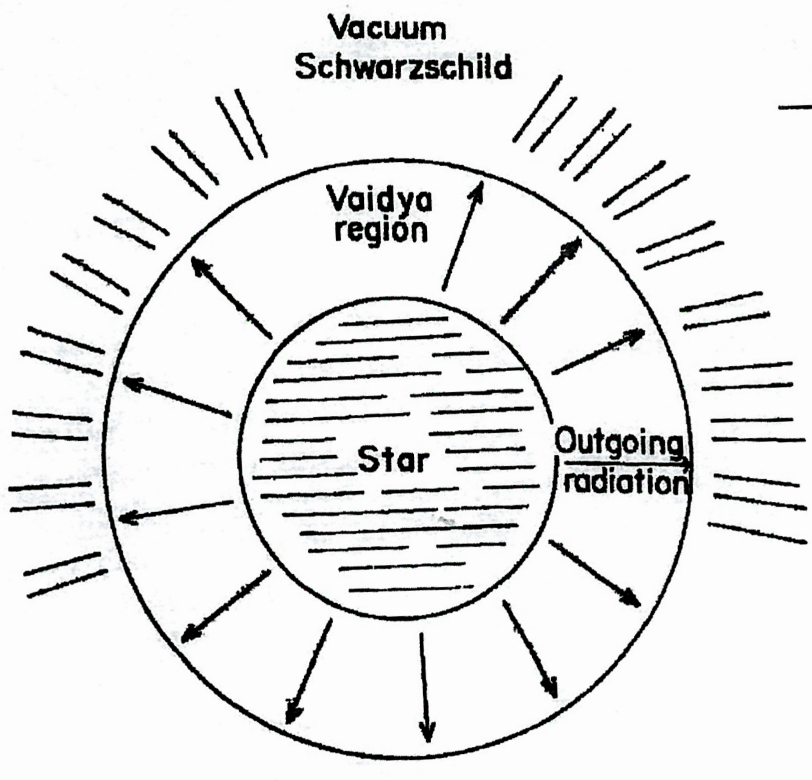


Fig. 19 A schematic diagram for the radiating star configuration. The interior of the star is matched with the Vaidya metric outside which describes the outwards flowing radiation. This is joined smoothly to the vacuum Schwarzschild geometry at the boundary of the radiation zone.

to the body having a charge. In the case of a normal star, the effect of radiation on the overall exterior metric could be considered negligible when effects such as rotation, magnetic fields and so on, are considered which cause perturbations from spherical symmetry. However, the radiation effects would be relevant during the later stages of gravitational collapse when the star would be throwing away considerable mass in the form of radiation or when abundant supply of neutrinos is radiated from a collapsing supernova core (see for example, Kahana, Baron and Cooperstein, 1984).

Such a non-static distribution as the radiating star would then be surrounded by an ever-expanding zone of radiation. One could treat this radiating system, together with its radiation, as forming an isolated system in an otherwise empty, asymptotically flat universe. Then, beyond the zone of pure radiation, the space-time is described by the empty Schwarzschild solution (Fig. 19).

One is thus looking here for a spherically symmetric solution to Einstein equations $G_{ij} = 8\pi T_{ij}$ with the geometrical optics type stress-energy tensor for the radiation with form

$$T_{ij} = \sigma k_i k_j, \tag{3.60}$$

where k_i is a null vector radially directed outwards. The metric is best given in the null coordinates (u, r, θ, ϕ):

$$\boxed{ds^2 = -\left(1 - \frac{2m(u)}{r}\right) du^2 - 2dudr + r^2 d\Omega^2,} \tag{3.61}$$

below we shall use signature $+---$!

with $m(u)$ being an arbitrary non-increasing function of the retarded time u. (In Section 6.4, where we shall be concerned with the application of Vaidya space-times to examine the cosmic censorship hypothesis, we will consider imploding radiation shells, rather than the outgoing case considered here. Then, the function m will be taken to be non-decreasing and the advanced null coordinate $t + r$ will be used.) The above gives the Vaidya metric in the radiation zone, which is to be matched by the interior metric of the radiating body at the boundary of the star and is matched by the Schwarzschild metric in the exterior.

In the form (3.60) for the energy-momentum tensor, σ is defined to be the energy density of radiation as measured locally by an observer with a four-velocity vector v^i. Thus, σ is the energy flux as well as energy density measured in this frame,

$$\boxed{\sigma \equiv T_{ij} v^i v^j,} \tag{3.62}$$

with $v^i v_i = -1$. Working out the connection coefficients from eqn (3.61), the Ricci tensor in null coordinates is given by

$$R_{ij} = -\frac{2}{r^2} \frac{dm(u)}{du} \delta^0{}_i \delta^0{}_j. \tag{3.63}$$

This implies that the Ricci scalar $R^i_i = R = 0$ and hence the Einstein equations give

$$T_{ij} = -\frac{1}{4\pi r^2} \frac{dm(u)}{du} \delta^0{}_i \delta^0{}_j, \tag{3.64}$$

which is the energy-momentum tensor of a radiating field in the geometric optics form. From eqns (3.62) and (3.64) we get

$$\sigma = -\frac{1}{4\pi r^2} \frac{dm(u)}{du}, \tag{3.65}$$

which is the expression for the energy density of radiation.

In the case when $m(u) = \text{const.}$ the relationship of the null coordinates in eqn (3.61) with the Schwarzschild coordinates (t, r, θ, ϕ) is not difficult to see. In such a case, one can use the transformation given by Finkelstein (1958) to diagonalize eqn (3.61)

$$\boxed{u = T - r - 2m \log(r - 2m),} \tag{3.66}$$

this gives the Schwarzschild metric in (T, r, θ, φ) coordinates

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Professor P C Vaidya (* 23.5.1918) – 12.3.2010

1 message

Prahalad Chunnilal

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To: psj <psj@tifr.res.in>

Wed, Mar 17, 2010 at 3:45 PM

Veteran Gandhian Mathematician P.C.Vaidya passes away
<http://DeshGujarat.Com/2010/03/12/veteran-gandhian-mathematician-p-c-vaidya-passes-away/
<http://DeshGujarat.Com/2010/03/12/veteran-gandhian-mathematician-p-c-vaidya-passes-away/>>

*Veteran Gandhian Mathematician P.C.Vaidya passes away
*By Japan K Pathak
Ahmedabad, DeshGujarat, 12 March, 2010

Veteran Mathematician and Ex-Vice Chancellor of Gujarat University professor Shri Prahalad Chunnilal Vaidya(P.C.Vaidya) -- famous for his Vaidya Metric internationally -- passed away today morning. He was in his early 90s. Internationally renown for his several researches and theories in the field of mathematics, and truly Gandhian PC Vaidya was one of the last educationists of Gandhian generation in Gujarat.

When he was a Vice Chancellor of Gujarat University, he used to ride on a bicycle. Till he was healthy, he used to visit Gujarat University's department that he represented in early days of his career on big black bicycle wearing Gandhi topi. For last several years, Vaidya Saheb was almost limited to his house in Ahmedabad's Shardanagar area due to poor health.

Einstein's theory of gravity is described by a set of rather complicated equations which use the mathematics of Riemannian geometry. It is very difficult to solve these equations, and particularly so to find solutions which describe physically interesting solution. But in 1942 Prahalad Chunnilal Vaidya(PC Vaidya) did pioneering work which led to just such a solution. The Vaidya Metric provided a solution to Einstein's equations which describes the gravitational field of a star which emits a great deal of radiation(search Vaidya Metric in Google and you would be amazed to know what amount of research has been made on the bases of Vaidya Sir's theory).

Being a freedom fighter, Shri Vaidya joined Ahimsak Vyayam Sangh in 1930s. When Vaidya heard Professor V.V.Narlikar(
father of Jayant
Narlikar)'s lecture in 1937 in Mumbai, he wrote a postcard requesting that he wanted to work with him. Shri Narlikar replied positively and invited Vaidya to join him. Thus Shri Vaidya went to Banaras with his wife and 6-month-old child. Vaidya then shifted to Mumbai to work with Shri Homi Bhabha at Tata Institute of Fundamental Research. He then became professor of Mathematics in Vallabh Vidyanagar, then in Visnagar and finally in Ahmedabad. Here he became Vice Chancellor of Gujarat University. Vaidya was instrumental in starting the Gujarat Mathematical Society. He had friendly relation with Shri Vikram Sarabhai in 1960s. A meeting with Sarabhai led to formation of the Vikram Sarabhai Community Science Centre in Ahmedabad where first mathematical laboratory was set up. Shri Vaidya always believed that if could be difficult to teach mathematics but it is certainly not difficult to learn mathematics. He believed that mathematics is something that is in our culture.

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Global Aspects in Gravitation and Cosmology

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Tata Institute of Fundamental Research
Bombay, India

CLARENDON PRESS · OXFORD
1993

such as the Vaidya–Papapetrou radiation collapse, Tolman–Bondi inhomogeneous dust space-times, general self-similar collapse of a perfect fluid, and also certain non-self-similar situations. The implications of this analysis towards the formulation and proof of cosmic censorship are discussed as and when relevant and also in the next chapter, where possible future directions are indicated.

6.5 Censorship violation in radiation collapse

We begin our study of gravitational collapse scenarios, other than the special case of completely homogeneous dust cloud collapse, by investigating the collapse of inflowing radiation. This is the situation in which a thick shell of radiation collapses at the center of symmetry in an otherwise empty universe which is asymptotically flat far away. The situation discussed in this section could be relevant to the gravitational collapse of a massive star because in the very final stages of the collapse the collapsing matter would be largely radiation dominated. Our main purpose here is to examine the final fate of such a collapse with special reference to the occurrence of naked singularities and the cosmic censorship hypothesis.

When should one regard a naked singularity forming in a gravitational collapse as a serious situation which must guide the formulation and proof of the censorship hypothesis, or must be regarded as an important counter-example? The following could be imposed as a minimum set of conditions for this purpose. Firstly, the naked singularity has to be visible at least for a finite period of time to any far away observer. If only a single null geodesic escaped, it would provide only an instantaneous exposure to the observer by means of a single wave front. In order to yield any observable consequences, a necessary condition is that, a family of future directed non-spacelike geodesics should terminate at the naked singularity in past. Next, this singularity must not be gravitationally weak, creating a mere space-time pathology, but must be a strong curvature singularity in a suitable sense as characterized in Section 5.4. This would ensure that the space-time does not admit any continuous extension through the singularity in any meaningful manner, and hence such a singularity cannot be avoided. The physical effects should then be important near such a strong curvature singularity. Finally, the form of matter should be reasonable in that it must satisfy a suitable energy condition ensuring the positivity of energy, and collapses gravitationally from an initial spacelike hypersurface with a well-defined non-singular initial data. We study here the phenomena of gravitational collapse of a spherical shell of radiation in this context and examine the nature and structure of the resulting naked singularity.

The imploding radiation is described by the Vaidya space-time, given in (v, r, θ, ϕ) coordinates as

note sign at $-+++$ | V ... "advanced time"

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{6.12}$$

flat if $v = t + 1$

$m = C$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The radiation collapses at the origin of coordinates, $v = 0, r = 0$. Throughout the discussion here the null coordinate v denotes the advanced time and $m(v)$ is an arbitrary but non-negative increasing function of v. The stress-energy tensor for the radial flux of radiation is

not lik

$u = t -$

we us

$$T_{ij} = \rho k_i k_j = \frac{1}{4\pi r^2} \frac{dm}{dv} k_i k_j, \tag{6.13}$$

$i, j = 0, 1, 2, 3 = v, r, \theta, \phi$

so far

with

we had

$$k_i = -\delta^v{}_i, \quad k_i k^i = 0,$$

$m(u)$

which represents the radially inflowing radiation along the world lines $v =$ const. Note that

m decr

$$\frac{dm}{dv} \geq 0,$$

implies that the weak energy condition is satisfied. The situation is that of a radially injected radiation flow into an initially flat and empty region, which is focused into a central singularity of growing mass by a distant source (see Fig. 47). The source is turned off at a finite time T when the field settles to the Schwarzschild space-time. The Minkowski space-time for $v < 0, m(v) = 0$ here is joined to a Schwarzschild space-time for $v > T$ with mass $m_0 = m(T)$ by way of the Vaidya metric (6.12). Assuming $m(v)$ to be a linear function, the central singularity $v = 0, r = 0$ was studied by Papapetrou (1985) and Kuroda (1984), who showed that it will be naked and persistent when the collapse is sufficiently slow. The radial null geodesics of this space-time were examined by Papapetrou which clarified the null structure of the space-time for a linear mass function. The particle creation effects associated with such a shell-focusing singularity were studied by Hiscock, Williams and Eardley (1982). They considered the situation when the space-time admits marginally naked singularity, where the Cauchy horizon coincides with the event horizon allowing only an isolated null trajectory to escape to infinity; and studied the particle creation by such a naked singularity. The other cases when the mass function is not linear are discussed by Lake (1986), Rajagopal and Lake (1987) and others. We discuss this situation in detail in Section 6.6.

Our purpose now is to examine the structure and curvature strength of naked singularity arising in the radiation collapse. We would first like to specify here all the families of future directed non-spacelike geodesics

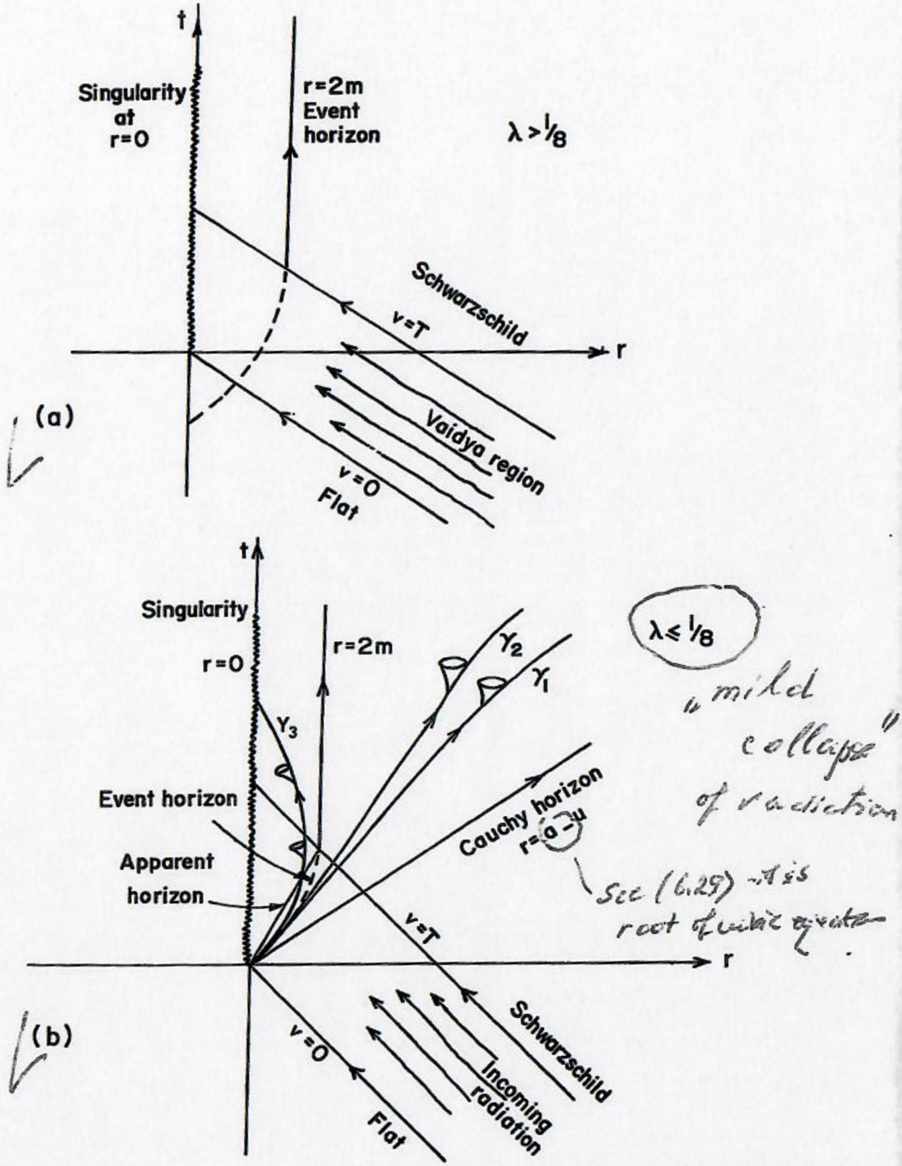

Fig. 48 (a) The event horizon completely covers the singularity at $r = 0$ when $\lambda > 1/8$. (b) For $\lambda \leq 1/8$, a naked singularity forms at the origin. Families of trajectories such as γ_1 and γ_2 escape away to infinity from the singularity with a definite tangent. The non-spacelike curve γ_3, which is emitted after the event horizon, crosses the apparent horizon and falls back to the singularity. The Cauchy horizon is the first ray coming out from the singularity.

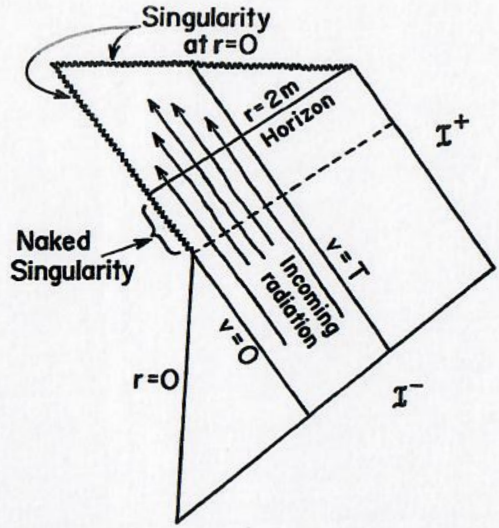

Fig. 49 A Penrose diagram for the naked singularity forming in the radiation collapse.

$$\frac{dr}{dX} = \frac{r[1 + Ak - (1 + \lambda X^2 - X)Q(X)]}{(2 + \lambda X^2 - X)XQ(X)}. \tag{6.30}$$

Integration of the above yields the equation of geodesic curves in (v, r) plane to give $r = r(X)$ for $\lambda \leq 1/8$:

$$\mathrm{D}r = \frac{1}{X-4} \exp\left(\frac{-4}{X-4} + \int \frac{-L^2 + (A/C)r^2}{Q(Q+1+Ak)} dX\right) \quad \text{for } \lambda = 1/8, \tag{6.31}$$

$$\mathrm{D}r = \frac{(X - a_+)^{a_-/(a_+ - a_-)}}{(X - a_-)^{a_+/(a_+ - a_-)}} \exp\left(\int \frac{-L^2 + (A/C)r^2}{Q(Q+1+Ak)} dX\right) \quad \text{for } \lambda < 1/8. \tag{6.32}$$

where D is a constant which labels different geodesics in the (v, r) plane. For example, $D = 0$, or $D = \infty$ implies $X = v/r =$ const. and gives rise to geodesics which are rectilinear (straight line) in the (v, r) plane. The behaviour of singular geodesics near the singularity is immediate from eqns (6.31) and (6.32). For families of non-rectilinear geodesics (that is, $\mathrm{D} \neq 0, \infty$), terminating at $r = 0$, the allowed values of X are obtained by simply letting $r \to 0$ in eqn (6.32) and finding the corresponding values of X. It follows that either $X = a_+$, or $X = c$, where c is the double and

$$\lim_{k\to 0} k^2\psi = 4\lambda.$$

For all other geodesics ($\ell \neq 0$), we have again

$$\psi \simeq (4\lambda/k^2) + (2\ell^2 D/k^{(2-n)}),$$

and again

$$\lim_{k\to 0} k^2\psi = 4\lambda.$$

Next, for $X = c$ value, the behaviour of the curvature scalar ψ for the rectilinear non-radial trajectories is given by

$$\psi = \lambda c^2/4k^2,$$

and the limit above for $k^2\psi$ works out to be $\lambda c^2/4$. For non-radial trajectories in this class, given by $\ell = \ell_{crt}$, we have $\psi \simeq \lambda c^2/4k^2$ which gives

$$\lim_{k\to 0} k^2\psi = \frac{\lambda c^2}{4}.$$

Finally, for the rectilinear radial geodesic with $X = a_-$, one has

$$\psi = \frac{4\lambda}{k^2},$$

and

$$\lim_{k\to 0} k^2\psi = 4\lambda.$$

Thus the strong curvature condition is again satisfied.

The important point to note is that the limiting value of $k^2\psi$ for all non-spacelike geodesics that terminate at the singularity is always positive, even though its actual value is path dependent. Away from the singularity the growth of $k^2\psi$ for non-radial geodesics shows deviation. This is due to the contribution of non-radial terms in eqn (6.43). In region I of the space-time non-radial terms add to the growth while in region II they check the growth. This results in a somewhat unusual peak in the value of $k^2\psi$ in region III and it can be seen that the peak increases as the value of ℓ gets nearer to ℓ_{crt}. For ℓ further than ℓ_{crt} the growth aligns itself more and more as $1/k^2$ and for $\ell = 0$, or $\ell = \infty$, becomes as $1/k^2$ (see Fig. 50).

Having seen that the Vaidya–Papapetrou naked singularity is a strong curvature singularity, one would expect it to be a scalar polynomial singularity as well. This is another notion matching well with our concept of singular behaviour familiar from 'big-bang' cosmologies (divergence of the

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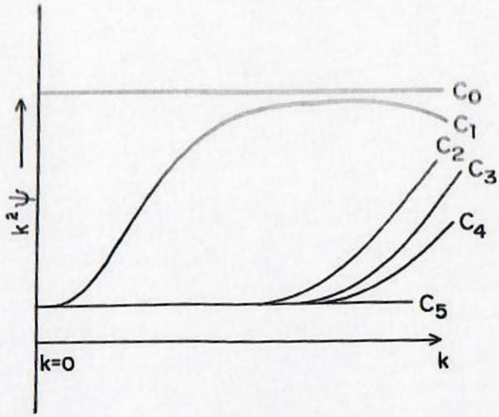

Fig. 50 The curvature growth along different non-spacelike trajectories meeting the naked singularity in the past for the radiation collapse models.

Ricci scalar) and the Schwarzschild models (divergence of the Kretschmann scalar $R^{ijkl}R_{ijkl}$). In fact, all known cases of strong curvature singularities in Einstein's equations are scalar polynomial singularities as well. Any such singularity will be an end point of at least one curve on which a scalar polynomial in the metric and the Riemann tensor takes unboundedly large values.

We therefore examine now the behaviour of the Kretschmann scalar near the naked singularity forming in the Vaidya–Papapetrou models. The Kretschmann scalar is given by

$$\alpha = R^{ijkl}R_{ijkl} = \frac{48M(v)^2}{r^6} = \frac{12\lambda^2 X^2}{r^6}, \tag{6.48}$$

along the families of the non-spacelike geodesics joining the singularity. It is verified that this scalar always diverges, thus establishing the scalar polynomial singularity as expected. However, an interesting directional behaviour is revealed by the singularity as far as the scalar α is concerned, which was not the case for the scalar ψ examined above. Using the relation $r = r(X)$ given by eqns (6.32), (6.36), and (6.40), and $X = X(k)$ from eqn (6.46), calculations are straightforward and the behaviour of the Kretschman scalar α along different families near the singularity can be described as follows.

Again, we discuss the behaviour of α near the naked singularity in terms of the limiting value of X. For the class given by $X = a_+$,

$$\alpha = \frac{12\lambda^2 a_+^2}{C_1^4 k^{8/a_+}}.$$