Spinors (in SR and GR)

It is advantageous to use signature $(+,-,-,-)$

Approach influenced primarily by Penrose - the deepest and most extensive reference:

R. Penrose and W. Rindler: „Spinors & space-time

I. Two-spinor calculus and relativistic fields

II. Spinor and twistor methods in space-time geometry", Cambridge University Press 1984

See also F. A. E. Pirani, „Introduction to gravitational radiation theory," in Lectures on General Relativity, 1964, Vol. 1, Brandeis Summer Institute in Theoretical Physics, Prentice-Hall 1965 – this is more „pedagogical", explicit

We shall deal with 2-component spinors which form 2-valued representation of $L_+^\uparrow$ (proper ortochronous representation of Lorentz group – $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $\det\Lambda = +1$, $\Lambda^0{}_0 \geq 1$. (viz $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \Rightarrow \Lambda^0{}_0 \leq -1$, $\Lambda^0{}_0 \geq 1$)

There exists a mutually unique correspondence between Hermitian matrices of 2nd order and points in spacetime (Reminder: Herm. mat. $\overline{A_{mn}} = A_{nm} \Rightarrow A_{11} = \overline{A_{11}}$, $A_{22} = \overline{A_{22}}$ real

$\overline{A_{12}} = A_{21}$ ⋯ complex conjugate) $\Rightarrow$ <u>4 real numbers</u>

Any Hermitian matrix (of 2nd order) can be written in the form

$\begin{pmatrix} p+q, & r-is \\ r+is, & p-q \end{pmatrix}$ where p, q, r, s are 4 real numbers
these may be identified as t, x, y, z

we associate Herm. m. with a spacetime point in M_4:

$$\boxed{\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \longleftrightarrow (t, x, y, z)}$$

also call „position vector" x^μ

Importantly,
$$\underline{\det\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} = t^2 - x^2 - y^2 - z^2 = \eta_{\mu\nu} x^\mu x^\nu}$$

Every transformation from $L_+^\uparrow$ (so not translations in ST) preserves this form:

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2$$

$(x'^\mu = \Lambda^\mu{}_\nu x^\nu + \cancel{a})$

To each Lorentz transformation from $L_+^\uparrow$ there corresponds the following transformation of a Hermit. matrix

$$(1)\quad \begin{pmatrix} t'+z', & x'-iy' \\ x'+iy', & t'-z' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}}_{\equiv U} \begin{pmatrix} t+z, & x-iy \\ x+iy, & t-z \end{pmatrix} \underbrace{\begin{pmatrix} \bar\alpha, & \bar\gamma \\ \bar\beta, & \bar\delta \end{pmatrix}}_{=U^+},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ arbitrary complex numbers satisfying

$$(2)\qquad U = \underbrace{\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}}_{\equiv \det U} = \alpha\delta - \beta\gamma = 1 \ (+0i)$$

, U - is unimodular matrix

clearly, new t', x', y', z' are linear combinations of t, x, y, z

they are real and thanks to the unimodularity of U,

$$\det \begin{pmatrix} t'+z', & x'-iy' \\ x'+iy', & t'-z' \end{pmatrix} = t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2$$

so it is Lor. transf from $L_+^\uparrow$ - they are 6-parameter transformations - viz "$\eta\Lambda\Lambda = \eta$" ... 10 eqs, $16 - 10 = 6$ free par.

$\alpha, \beta, \gamma, \delta$ complex $\Rightarrow$ 8 parameters, condition (2) $\Rightarrow -2 \Rightarrow 6$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ form connected set containing δ_A^B - continuous changes $\Rightarrow$ no inversion

A given Lorentz transformation can be realized by just two unimodular matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $-\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$

$\Rightarrow$ 2-valued representation of $L_+^\uparrow$

matrices $\begin{pmatrix} t+z, & x-iy\\ x+iy, & t-z\end{pmatrix}$ transform by direkt product of $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\begin{pmatrix}\bar\alpha & \bar\beta\\ \bar\gamma & \bar\delta\end{pmatrix}$

The quantities which transform just by one $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = L^A{}_B$ $(A, B = 1, 2)$ are 2 component spinors $\xi^A = (\xi^1, \xi^2)$

The spin transformation (or transformation of spinors) is

$$\xi^{A'} = L^A{}_B\, \xi^B, \quad \det(L^A{}_B) = 1, \qquad (3)$$

both ξ^A and $L^A{}_B$ are complex

Each such transformation can be associated with Lor. trasf $\in L_+^\uparrow$ and each L.T can be associated with (3) and with $\xi^{A'} = -L^A{}_B\, \xi^B$ ~~2 valu~~ "ambiguity" two-valuedness

Together with $\xi^{A'} = L^A{}_B \xi^B$ consider also complex conjugate eq.

$$\overline{\xi^{A'}} = \bar{L}^A{}_B \bar{\xi}^B$$ ... another representation of $L_+^\uparrow$

to distinguish from those transforming by $L^A{}_B$, put "$\cdot$" above and "$-$" only above the letters, so write above eq. as

$$\bar{\xi}^{\dot{A}'} = \bar{L}^{\dot{A}}{}_{\dot{B}} \bar{\xi}^{\dot{B}}$$

and generally (when there is no use of corresponding spinor) just

$$\eta^{\dot{A}'} = \bar{L}^{\dot{A}}{}_{\dot{B}} \eta^{\dot{B}}$$

Let $l_A{}^B$ is inverse to $L^A{}_B$:

$$l_A{}^B L^A{}_C = \delta^B_C \Leftrightarrow L^C{}_B l_A{}^B = \delta^C_A$$

Two complex numbers ξ_A which transform as

$$\xi_A' = l_A{}^B \xi_B \qquad \text{("covariant spinor")}$$

form also 2-valued representation of $L_+^\uparrow$

Finally, the 4th representation is formed by

$$\xi_{\dot{A}}' = \bar{l}_{\dot{A}}{}^{\dot{B}} \xi_{\dot{B}}$$

$$\xi_A' \xi^{A'} = \xi_A \xi^A \text{ ...etc} \quad \text{are scalars}$$

<u>Note</u> The group of matrices L contains the same elements as e.g. the group $\bar{L}$.

$L_1 L_2 = L_3 \Rightarrow \bar{L}_1 \bar{L}_2 = \bar{L}_3$ but these 2 representations of $L_+^\uparrow$ are <u>not</u> equivalent since there exists no matrix S for which $SLS^{-1} = \bar{L}$

Proof by example:

the matrix $S = S^{-1} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ given <u>uniquely</u>

~~which~~ transforms $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ does $SLS^{-1} = \bar{L}$

but for a general matrix

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \neq \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$$

Spinors with arbitrary number of indices:

$$T'^{A\dot{X}}{}_B = L^A{}_C \bar{L}^{\dot{X}}{}_{\dot{Y}}\, l_B{}^D\, T^{C\dot{Y}}{}_D$$

<u>Theorem:</u> addition, symmetrization, multiplication by a number are covariant operations ... „evident"

<u>Note</u>: undotted and dotted indices must be written in a certain fixed order – but dotted and undotted indices can be arbitrarily interchanged, e.g. $T^{A\dot{W}}{}_B = T^{\dot{W}A}{}_B$ ...

Notation: P - spinor - if it has P indices of the same kind; PQ - spinor → P indices of the same kind and Q indices of the other kind

Def. PP spinor is called Hermitian if it has the same number of indices of both kinds and

$$\phi^{B_1 \dots B_p \dot{X}_1 \dots \dot{X}_p} = \overline{\phi^{X_1 \dots X_p \dot{B}_1 \dots \dot{B}_p}} \equiv \bar{\phi}^{B_1 \dots B_p \dot{X}_1 \dots \dot{X}_p}$$

in particular

$$\phi^{A\dot{B}} = \overline{\phi^{B\dot{A}}} \equiv \bar{\phi}^{\dot{B}A} \equiv \bar{\phi}^{A\dot{B}}$$

example the (Pauli) matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Def. of Levi-Civita symbol in 2 dim :

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon_{[AB]}\,, \quad \varepsilon_{12} = 1 \rightarrow [\varepsilon_{AB}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\varepsilon_{\dot{X}\dot{Y}} = \varepsilon^{\dot{X}\dot{Y}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

It is easy to see that $\varepsilon_{AB} \underbrace{\det(l_{\cdot}^{\cdot})}_{=1 \text{ (unimodular)}} = l_A{}^C l_B{}^D \varepsilon_{CD}$

$\Rightarrow$ ε_{AB} is covariant 2-spinor

Similarly $\varepsilon^{AB} = L^A{}_C L^B{}_D \varepsilon^{CD}$ ... contravariant 2-spinor

ε_{AB}, ε^{AB} play the role of the metric tensor in spinors — see below

With $\varepsilon_{..}$ one raises, lowers indices:

$$\xi_A = \xi^B \varepsilon_{BA} \iff \xi^B = \varepsilon^{BA} \xi_A$$

careful with sign ${}^{B}\searrow{}_{B}$ ${}^{A}\leftrightarrow{}_{A}$ — indices are lowered by the 1st index in ε raised by the 2nd

Easy to prove:

$$\varepsilon^{AC}\varepsilon_{BC} = \delta^A{}_B, \quad \varepsilon^{\dot W \dot X}\varepsilon_{\dot Y \dot X} = \delta^{\dot W}{}_{\dot Y} \quad \text{etc}$$

$\xi_A \eta^A$ is scalar for which

$$\boxed{\xi_A \eta^A = \xi^B \varepsilon_{BA}\varepsilon^{AC}\eta_C = -\xi^B \underbrace{\varepsilon_{AB}\varepsilon^{AC}}_{=\delta^C{}_B}\eta_C = -\xi^C\eta_C = -\xi^A\eta_A}$$

The relation between spinors & tensors

introduce (compare p. 2)

$$x^{A\dot B} = \frac{1}{\sqrt{2}}\begin{pmatrix} t+z, & x-iy \\ x+iy, & t-z \end{pmatrix} = \sigma_\mu{}^{A\dot B} x^\mu$$

$\sigma_i{}^{A\dot B}$ are $= \frac{1}{\sqrt{2}}$ (Pauli matrices), $\sigma_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$,

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \quad \sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$$

checking $x^{1\dot 2} = \underbrace{\sigma_1{}^{1\dot 2}}_{=1} x + \underbrace{\sigma_2{}^{1\dot 2}}_{-i} y = \frac{1}{\sqrt{2}}(x - iy)$ ✓

Easy to see:

$$(*) \quad \sigma_\mu{}^{B\dot X}\,\sigma^\mu{}_{C\dot Y} = \delta^B_C\,\delta^{\dot X}_{\dot Y}$$

$$\sigma^\mu{}_{C\dot Y}\,\sigma_\nu{}^{C\dot Y} = \delta^\mu_\nu$$

($B, C, \dot X, \dot Y$ indices in „σ" raised/lowered by ε.., μ,ν by $\eta^{\mu\nu}$)

Proposition

$\sigma_\mu{}^{A\dot B}$ enable one to connect tensors and spinors

To any tensor corresponds uniquely spinor
For example consider $T_\varkappa{}^{\mu\nu}$, then form

$$T_{C\dot Y}{}^{A\dot W B\dot X} = \sigma_\mu{}^{A\dot W}\,\sigma_\nu{}^{B\dot X}\,\sigma^\varkappa{}_{C\dot Y}\,T_\varkappa{}^{\mu\nu}$$

as consequence of (*) we get conversely

$$T_\varkappa{}^{\mu\nu} = \sigma^\mu{}_{A\dot W}\,\sigma^\nu{}_{B\dot X}\,\sigma_\varkappa{}^{C\dot Y}\,T_{C\dot Y}{}^{A\dot W B\dot X}$$

one often just writes: $T_\varkappa{}^{\mu\nu} \leftrightarrow T_{C\dot Y}{}^{A\dot W B\dot X}$

Expressing the metric $\eta_{\mu\nu}$:

the relation $\det\begin{vmatrix} t+z, & x-iy \\ x+iy, & t-z \end{vmatrix} = t^2 - x^2 - y^2 - z^2 = \eta_{\mu\nu}x^\mu x^\nu$

$$\Rightarrow \underbrace{2\,|x^{A\dot B}|} = \eta_{\mu\nu}x^\mu x^\nu$$

$$= \varepsilon_{AC}\varepsilon_{\dot B\dot D}\,x^{A\dot B}x^{C\dot D} = \varepsilon_{AC}\varepsilon_{\dot B\dot D}\,\sigma_\mu{}^{A\dot B}\sigma_\nu{}^{C\dot D}x^\mu x^\nu = \eta_{\mu\nu}x^\mu x^\nu$$

arbitrary

$$\Rightarrow \varepsilon_{AC}\,\varepsilon_{\dot B\dot D}\,\sigma_\mu{}^{A\dot B}\sigma_\nu{}^{C\dot D} = \eta_{\mu\nu}, \quad \text{so } \varepsilon_{AC}\,\varepsilon_{\dot B\dot D} \leftrightarrow \eta_{\mu\nu}$$

Note: if we are in arbitrary coordinates in a curved spacetime of signature $(+---)$ we can locally at given point work in the tangent vector space and write $\varepsilon_{AC}\,\varepsilon_{\dot B\dot D} \longleftrightarrow g_{\mu\nu}$ — $\sigma_\mu{}^{A\dot B}$ also change when we go from local inertial frame $\mathring{x}^\mu$ to general x^μ and transform $\sigma_\mu{}^{A\dot B}$ as covariant vector. So, by giving $\varepsilon_{..}$, $\sigma_{.}{}^{..}$, i.e. spinor structure, then from $g_{\mu\nu} \Leftrightarrow \varepsilon_{AB}\,\varepsilon_{\dot C\dot D}$ follows the signature in this sense the spinor structure is more „fundamental" than the Riemannian since the signature follows from it.

Spinor algebra

Since spinor indices $A, B, \ldots$ have only 2 values, $A = 1, 2$ ... it is easy to see that

$$\varepsilon_{A[B}\,\varepsilon_{CD]\,cycl.} = 0$$

i.e.

$$\varepsilon_{AB}\,\varepsilon_{CD} + \varepsilon_{AC}\,\varepsilon_{DB} + \varepsilon_{AD}\,\varepsilon_{BC} = 0 \quad / \xi^{CD}$$

$$\Rightarrow \quad \varepsilon_{AB}\,\xi_C{}^C - \xi_{AB} + \xi_{BA} = 0$$

$$\Rightarrow \quad \xi_{AB} = \xi_{(AB)} + \tfrac{1}{2}\,\varepsilon_{AB}\,\xi_C{}^C \qquad (+)$$

viz.

$$\xi_{AB} = \underbrace{\xi_{(AB)}}_{\frac{1}{2}(\xi_{AB}+\xi_{BA})} + \underbrace{\xi_{[AB]}}_{\frac{1}{2}(\xi_{AB}-\xi_{BA})}$$

Let in particular $\xi_{AB} = \varphi_A \psi_B$

$(+) \Rightarrow \quad \varphi_A \psi_B - \psi_A \varphi_B = \varepsilon_{AB} \varphi_C \psi^C \qquad (++)$

if $\varphi_A \sim \psi_A \quad \Rightarrow \quad \varphi_C \psi^C = 0$

if, on the other hand, $\varphi_C \psi^C = 0 \Rightarrow \varphi_A \psi_B = \psi_A \varphi_B \Rightarrow$

$\Rightarrow \varphi_A = k \psi_A$

$(+) \Rightarrow$ „only symmetric spinors matter" (R. Penrose)

Theorem: Each finite dim. irreducible representation of $L_+^\uparrow$ is equivalent to some spinor representation in which spinors are symmetric in all indices (undotted, dotted)

Spinors which are not symmetric in all indices transform by reducible representation which in spinor algebra is reflected by the fact that antisym. spinors can be replaced by products of ε_{AB} and spinors of lower rank - one writes $\underbrace{\phi_{AB\ldots K}}_{p} \underset{p}{\cong} \phi_{(AB\ldots K)}$

Lemma

Any couple of spinors κ_A, μ_A for which $\kappa_A \mu^A = 1$ can serve as the basis in the spinor space:

from $(++) \Rightarrow \varepsilon_{AB} = \kappa_A \mu_B - \mu_A \kappa_B \Rightarrow \delta_A^B = \kappa_A \mu^B - \mu_A \kappa^B \quad | \times \xi^A$

$\Rightarrow \xi^B = (\xi^A \kappa_A) \mu^B + (-\xi^A \mu_A) \kappa^B$

components of linear combination

Spinor equivalents of some vectors and tensors

to any ξ^μ there corresponds

$$\xi^{A\dot{X}} = \sigma_\mu{}^{A\dot{X}} \xi^\mu \qquad \xi^\mu = \sigma^\mu{}_{A\dot{X}} \xi^{A\dot{X}}$$

one has $\xi^\mu \xi_\mu = 2 \det|\xi^{A\dot{X}}|$

$\Rightarrow$ if ξ^μ is null vector (in general complex)

$\Leftrightarrow$ $\det|\xi^{A\dot{X}}| = 0$, so the matrix rank is 1

$$\Rightarrow \xi^{A\dot{X}} = \xi^A \bar{\eta}^{\dot{X}}$$

if ξ^μ is real, $\xi^{A\dot{X}} = \xi^{\dot{X}A} = \bar{\xi}^{A\dot{X}} \Rightarrow$

$$\xi^A \bar{\eta}^{\dot{X}} = \eta^A \bar{\xi}^{\dot{X}} \,/\times \eta_A, \quad \eta_A \eta^A = 0$$

$$\Rightarrow \eta_A \xi^A = 0 \Rightarrow \xi^A = \lambda \eta^A, \ \lambda \text{ real}$$

$|\lambda|^{1/2}$ insert to η

Proposition

If ξ^μ is a real null vector, $\xi_\mu \xi^\mu = 0$,

$\Leftrightarrow$ $\xi^{A\dot{X}} = \pm \xi^A \bar{\xi}^{\dot{X}}$, or $\xi^\mu \Leftrightarrow \pm \xi^A \bar{\xi}^{\dot{X}}$

It can be seen that given relation between $\xi^{A\dot{X}}$ and ξ^A implies time orentation of the manifold

if $\xi^{A\dot{X}} = + \xi^A \bar{\xi}^{\dot{X}} \Rightarrow \xi^{1\dot{1}} = \xi^1 \bar{\xi}^{\dot{1}} = \xi^1 \overline{\xi^1} \geq 0$

but $\xi^\mu = \sigma^\mu{}_{A\dot{X}} \xi^{A\dot{X}} \Rightarrow \xi^{1\dot{1}} = \xi^0 + \xi^3 > 0 \Rightarrow \xi^0 > 0$

$$\xi^{1\dot{1}} < 0 \Rightarrow \xi^0 < 0$$

since for null vector $|\xi^0| \geq |\xi^3|$ ....

Spinor equivalent of a real antisym. tensor

$F_{\mu\nu} = F_{[\mu\nu]}$ real $\quad F_{\mu\nu} \leftrightarrow F_{A\dot W B\dot X}$

spinor equivalent must have the same symmetry

$$\boxed{F_{A\dot W B\dot X}} = \sigma^\mu{}_{A\dot W}\,\sigma^\nu{}_{B\dot X}\,F_{\mu\nu} = -\sigma^\mu{}_{B\dot X}\,\sigma^\nu{}_{A\dot W}\,F_{\mu\nu} =$$

$= -F_{\nu\mu}$ interchange $\nu \leftrightarrow \mu$

$$= \boxed{-F_{B\dot X A\dot W}}$$

$$\Rightarrow F_{A\dot W B\dot X} = \frac{1}{2}\left(F_{A\dot W B\dot X} - F_{B\dot X A\dot W}\right) =$$

$$= \frac{1}{2}\Big(\underbrace{F_{A\dot W B\dot X} - F_{B\dot W A\dot X}}_{\text{antisym.}} + \underbrace{F_{B\dot W A\dot X} - F_{B\dot X A\dot W}}_{\text{antisym.}}\Big)$$

$$= \frac{1}{2}\left(\varepsilon_{AB}\,F_{H\dot W}{}^{H}{}_{\dot X} + \varepsilon_{\dot W\dot X}\,F_{B\dot P A}{}^{\dot P}\right) \qquad (\text{recall } \xi_{AB} - \xi_{BA} = \varepsilon_{AB}\,\xi_C{}^C)$$

denote $\phi_{AB} \equiv \frac{1}{2} F_{B\dot P A}{}^{\dot P}$

then $\phi_{AB} = -\frac{1}{2} F_{A}{}^{\dot P}{}_{B\dot P} = \frac{1}{2} F_{A\dot P B}{}^{\dot P} = \phi_{BA}$ $\quad \phi$ is symmetric

Similarly show $F_{H\dot W}{}^{H}{}_{\dot X} = F_{H\dot X}{}^{H}{}_{\dot W}$

But $F_{\mu\nu}$ real $\Rightarrow F_{A\dot W B\dot X} = F_{AB\dot W\dot X} \overset{\text{is Hermitian}}{=} \overline{F_{\dot W\dot X \dot A\dot B}} = \bar F_{\dot W\dot X AB} = \bar F_{A\dot W B\dot X}$

$$\Rightarrow \frac{1}{2}F_{H\dot W}{}^{H}{}_{\dot X} = \frac{1}{2}\bar F_{H\dot W}{}^{H}{}_{\dot X} = \frac{1}{2}\bar F_{\dot W H\dot X}{}^{H} = \frac{1}{2}\overline{F_{W\dot H X}{}^{\dot H}} = \bar\phi_{WX} = \bar\phi_{\dot W\dot X}$$

$\Rightarrow$ Finally $\boxed{F_{\mu\nu} = \sigma_\mu{}^{A\dot W}\,\sigma_\nu{}^{B\dot X}\left(\varepsilon_{AB}\,\bar\phi_{\dot W\dot X} + \varepsilon_{\dot W\dot X}\,\phi_{AB}\right)}$

where $\phi_{AB} = \frac{1}{2} F_{B\dot P A}{}^{\dot P}$, briefly: $\underline{F_{\mu\nu} \leftrightarrow \varepsilon_{AB}\,\bar\phi_{\dot W\dot X} + \varepsilon_{\dot W\dot X}\,\phi_{AB}}$

Real antisymmetric tensor has 6 independent components (e.g. $\vec{E}$ and $\vec{B}$). It is determined by a symmetric spinor ϕ_{AB} which has 3 independent complex components $\phi_{11}, \phi_{12}=\phi_{21}, \phi_{22}$, i.e. by 6 real quant.

Very useful is to have the spinor equivalent of the Riemann tensor, Levi-Civita tensor etc.

Consider first the spinor equivalent of the tensor $B_{\alpha\beta\gamma\delta}$, for which there is antisym. in first and second pair of indices, i.e. $B_{[\alpha\beta][\gamma\delta]}$ and $B_{\alpha\beta\gamma\delta}=B_{\gamma\delta\alpha\beta}$

Then $$B_{\alpha\beta\gamma\delta} \leftrightarrow B_{A\dot{W}B\dot{X}C\dot{Y}D\dot{Z}}$$

We shall continue without detailed calculations when we aply preceding procedure for $F_{\mu\nu}$ on each pair of indices we expect the terms of the type

$$(\varepsilon_{AB}\bar{\phi}_{\dot{W}\dot{X}}+\phi_{AB}\varepsilon_{\dot{W}\dot{X}})(\varepsilon_{CD}\bar{\psi}_{\dot{Y}\dot{Z}}+\psi_{CD}\varepsilon_{\dot{Y}\dot{Z}})$$

we find

$$B_{A\dot{W}B\dot{X}C\dot{Y}D\dot{Z}} = B_{ABCD}\,\varepsilon_{\dot{W}\dot{X}}\varepsilon_{\dot{Y}\dot{Z}} + \bar{B}_{\dot{W}\dot{X}\dot{Y}\dot{Z}}\,\varepsilon_{AB}\varepsilon_{CD} + {}$$
$$+ C_{AB\dot{Y}\dot{Z}}\,\varepsilon_{CD}\varepsilon_{\dot{W}\dot{X}} + C_{CD\dot{W}\dot{X}}\,\varepsilon_{AB}\varepsilon_{\dot{Y}\dot{Z}},$$

similarly $B_{ABCD}=\frac{1}{4}B_{A\dot{P}B}{}^{\dot{P}}{}_{C\dot{Q}D}{}^{\dot{Q}}$ and $\left.\begin{array}{l} B_{ABCD}= \\ =B_{(AB)(CD)} \end{array}\right. = B_{CDAB}$

$$\boxed{C_{..}=\frac{1}{4}B_{A\dot{P}B}{}^{\dot{P}}{}_{G\dot{Y}}{}^{G}{}_{\dot{Z}} = C_{(AB)(\dot{Y}\dot{Z})} = \bar{C}_{AB\dot{Y}\dot{Z}}}$$

($\dot{\sim}$ $C \leftrightarrow \phi\bar{\psi}$ is indeed Hermitian)

Now, as for any spinor, B_{ABCD} can be written in terms of a completely symmetrical part and of $\varepsilon_{..}$ and spinors arising from contractions. One finds

$$B_{ABCD} = B_{(ABCD)} + \frac{1}{6}\left(\varepsilon_{BC}\varepsilon_{AD} + \varepsilon_{AD}\varepsilon_{AC}\right)B$$

where $B = B_{RS}{}^{RS}$

Proof: we know $B_{ABCD} = B_{CDAB}$

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cycl.

$$B_{ABCD} = \frac{1}{3}\left(B_{ABCD} + B_{ACDB} + B_{ADBC}\right)$$

$$+ \frac{1}{3} B_{ABCD} - \frac{1}{3} B_{ACBD} \quad - \quad \text{is antisym. in } {}^{BC} \Rightarrow = \frac{1}{3}\epsilon_{BC} B_{AH}{}^{H}{}_{D}$$

$$+ \underbrace{\frac{1}{3} B_{ABCD}}_{=\frac{1}{3}B_{ABDC}} - \frac{1}{3} B_{ADBC} \quad = \quad - \text{"} - \text{ in } {}^{BD} = \frac{1}{3}\epsilon_{BD} B_{AH}{}^{H}{}_{C}$$

$= B_{ABCD}$

$$= \frac{1}{3}\left(B_{ABCD} + B_{ACBD} + B_{ADBC}\right) + \frac{1}{3}\varepsilon_{BC} B_{AH}{}^{H}{}_{D} + \frac{1}{3}\varepsilon_{BD} B_{AH}{}^{H}{}_{C}$$

i.e. $B_{AH}{}^{H}{}_{D} = -B_{A}{}^{H}{}_{HD} = -B_{HDA}{}^{H} = -B_{DH}{}^{H}{}_{A}$

i.e. $B_{AH}{}^{H}{}_{D}$ is antisym. in A, D

$$B_{AH}{}^{H}{}_{D} = \frac{1}{2}\epsilon_{AD} B_{RS}{}^{RS} = B$$

Finally, denoting $B_{(ABCD)} \equiv A_{ABCD}$ we have

$$A_{ABCD} = \frac{1}{4} \varepsilon_{\dot W \dot X} \varepsilon_{\dot Y \dot Z} B_{(A}{}^{\dot W}{}_{B}{}^{\dot X}{}_{C}{}^{\dot Y}{}_{D)}{}^{\dot Z}$$ and

$$(+)\begin{cases} B_{A\dot W B\dot X C\dot Y D\dot Z} = A_{ABCD}\, \varepsilon_{\dot W\dot X} \varepsilon_{\dot Y\dot Z} + \varepsilon_{AB}\varepsilon_{CD} \bar A_{\dot W\dot X\dot Y\dot Z} + \\ + \frac{1}{6}\left(\varepsilon_{AD}\varepsilon_{BC} + \varepsilon_{AC}\varepsilon_{BD}\right) \varepsilon_{\dot W\dot X}\varepsilon_{\dot Y\dot Z} B + \\ + \frac{1}{6}\left(\varepsilon_{\dot W\dot Z}\varepsilon_{\dot X\dot Y} + \varepsilon_{\dot W\dot Y}\varepsilon_{\dot X\dot Z}\right)\varepsilon_{AB}\varepsilon_{CD}\bar B + \\ + C_{AB\dot Y\dot Z}\,\varepsilon_{CD}\varepsilon_{\dot W\dot X} + C_{CD\dot W\dot X}\,\varepsilon_{AB}\varepsilon_{\dot Y\dot Z} \end{cases}$$

Each of these terms can be written interms of $B_{A\dot W B\dot X C\dot Y D\dot Z}$

Next, assume further symmetries of the tensor $B_{\alpha\beta\gamma\delta}$

1. $B_{\alpha(\beta\gamma)\delta} = 0$, i.e $B_{\alpha\beta\gamma\delta} = -B_{\alpha\gamma\beta\delta}$

$\Rightarrow$ it is easy to see that then $B_{\alpha\beta\gamma\delta}$ is antisymmetric in all indices $\Rightarrow$ $B_{\cdots}$ has only one independent component and must be a multiple of Levi-Civita tensor

Then from (+) interchange $B \leftrightarrow C$, $\dot X \leftrightarrow \dot Y$, add an symmetrize in $(ABCD)$ $\Rightarrow$ $A_{ABCD} = 0$, then symmetrize in (ABC) $\Rightarrow$ $C_{AB\dot Y\dot Z} = 0$, B must be purely imaginary: $B = i\beta$, β real

Then the spinor equivalent of $B_{\alpha\beta\gamma\delta}$ is

$$B_{A\dot W B\dot X C\dot Y D\dot Z} = \frac{1}{3} i\beta\left(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot W\dot X}\varepsilon_{\dot Y\dot Z} - \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot W\dot Y}\varepsilon_{\dot X\dot Z}\right)$$

and to be $B_{\cdots}$ directly equivalent to Levi-Civita tensor (symm) it must be $\beta = -3$, so $\varepsilon_{\alpha\delta}{}^{\beta\gamma} \leftrightarrow \varepsilon_{A\dot W D\dot Z}{}^{B\dot X C\dot Y} =$

$$= i\left(\delta_A^B \delta_D^C \delta_{\dot W}^{\dot Y} \delta_{\dot Z}^{\dot X} - \delta_A^C \delta_D^B \delta_{\dot W}^{\dot X} \delta_{\dot Z}^{\dot Y}\right)$$

Dual $F^*_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\delta}{}^{\beta\gamma}F_{\beta\gamma}$ and its spinor equivalent

we know that $F_{\alpha\beta} \leftrightarrow \varepsilon_{AB}\bar{\phi}_{\dot{W}\dot{X}} + \varepsilon_{\dot{W}\dot{X}}\phi_{AB}$

then

$$F^*_{\alpha\delta} = \frac{1}{2}\varepsilon_{\alpha\delta}{}^{\beta\gamma}F_{\beta\gamma} \leftrightarrow \frac{1}{2}i\left(\delta_A{}^B\delta_D{}^C\delta_{\dot{W}}{}^{\dot{Y}}\delta_{\dot{Z}}{}^{\dot{X}} - \delta_A{}^C\delta_D{}^B\delta_{\dot{W}}{}^{\dot{X}}\delta_{\dot{Z}}{}^{\dot{Y}}\right)$$

$$\times\left(\varepsilon_{BC}\bar{\phi}_{\dot{X}\dot{Y}} + \varepsilon_{\dot{X}\dot{Y}}\phi_{BC}\right) =$$

$$= \frac{1}{2}i\left(\varepsilon_{AD}\bar{\phi}_{\dot{Z}\dot{W}} + \varepsilon_{\dot{Z}\dot{W}}\phi_{AD} - \varepsilon_{DA}\bar{\phi}_{\dot{W}\dot{Z}} - \varepsilon_{\dot{W}\dot{Z}}\phi_{DA}\right)$$

$$= i\left(\varepsilon_{AD}\bar{\phi}_{\dot{W}\dot{Z}} - \varepsilon_{\dot{W}\dot{Z}}\phi_{AD}\right) \quad \text{(remember } \phi_{\dot{W}\dot{Z}} = \phi_{\dot{Z}\dot{W}}\text{)}$$

Then, since $e^{i\theta} = \cos\theta + i\sin\theta$, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, $\sin\theta = \dots$

„the dual rotation"

$F_{\alpha\beta} \to F_{\alpha\beta}\cos\theta + F^*_{\alpha\beta}\sin\theta$ in spinors $\phi_{AB} \to \phi_{AB}e^{-i\theta}$

(2.) Assume $B_{\alpha[\beta\gamma\delta]\text{cycl.}} = 0 \Rightarrow$ all symmetries of the Riemann tensor $\to B = \bar{B}$, so to a general Riemann tensor corresponds spinor $B_{A\dot{W}B\dot{X}C\dot{Y}D\dot{Z}}$ given by (+) with $B = \bar{B}$

(3.) Next, assume further that $B_{\alpha\beta}{}^{\beta}{}_{\delta} = 0$

($R_{\alpha\beta}{}^{\beta}{}_{\delta} = R_{\alpha\delta}$ in vacuum Ricci $= 0$)

Then in (+) contract in corresponding spinor pairs, so multiply by $\varepsilon^{BC}\varepsilon^{\dot{X}\dot{Y}} \Rightarrow$

$$\underline{A_{ABCD}}\,\varepsilon_V{}^{BC} = 0 \Rightarrow 2\,C_{AD\dot{W}\dot{Z}} - \frac{1}{2}\varepsilon_{AD}\varepsilon_{\dot{W}\dot{Z}}(B + \bar{B}) = 0$$

is sym.

we symmetrize $\Rightarrow C_{\dots} = 0$ and also $\varepsilon_{AD}\varepsilon_{\dot{W}\dot{Z}}(B + \bar{B}) = 0$

but B must be real (see (2.) above) $\Rightarrow B = 0$

⇒ Proposition

The spinor corresponding to the Riemann tensor in vacuum is symmetric in all 4 indices:

$$R_{\alpha\beta\gamma\delta} \leftrightarrow A_{ABCD}\, \varepsilon_{\dot{W}\dot{X}}\, \varepsilon_{\dot{Y}\dot{Z}} + \varepsilon_{AB}\, \varepsilon_{CD}\, \bar{A}_{\dot{W}\dot{X}\dot{Y}\dot{Z}}$$

Remark: if we are not in vacuum we can form the Weyl tensor $C_{\alpha\beta\gamma\delta}$ which has the same properties (symmetries) as the vacuum Riemann tensor. It is defined as follows:

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\,\delta^{[\alpha}_{[\gamma}\left(R^{\beta]}_{\delta]} - \frac{1}{6} R\, \delta^{\beta]}_{\delta]}\right)$$

So in general spacetime the Weyl tensor $C^{\alpha\beta}{}_{\gamma\delta} \leftrightarrow A_{ABCD}$

next applications