Canonical decomposition of a symmetric spinor

Consider P spinor symmetric in all indices

$\underbrace{\phi_{AB\dots K}}_{P}$. Form the expression

$$\phi_{AB\dots K}\,\xi^A\xi^B\dots\xi^K$$

assume $\xi^A = (1, \xi)$ ... this can be easily achieved in general $\xi^A = (\xi^1, \xi^2)$, take ξ^A/ξ^1 this changes ξ^A to $(1, \xi)$ but the direction (null) determined by ξ does not change: $\xi^\mu = \sigma^\mu{}_{A\dot{x}}\,\xi^A\bar{\xi}^{\dot{x}} \rightarrow \xi^\mu = \underbrace{\frac{1}{|\xi^1\bar{\xi}^1|}}_{\equiv\lambda}\sigma^\mu{}_{A\dot{x}}\,\xi^A\bar{\xi}^{\dot{x}}$, so $\xi^\mu \to \lambda\xi^\mu$

Then $\phi_{AB\dots K}\,\xi^A\dots\xi^K$ is the polynomial of maximum degree P in complex variable ξ

From the fundamental theorem of algebra we can write the polynomial in terms of the roots:

$$\phi_{\dots}\,\xi^{\cdot}\xi^{\cdot}\dots\xi^{\cdot} = F(\xi) = \alpha_0\,\underbrace{(\xi-\alpha)(\xi-\beta)\dots(\xi-\kappa)}_{P}$$

introduce $\alpha_A = (-\alpha\alpha_0, \alpha_0)$, $\beta_B = (-\beta, 1), \dots, \kappa_K = (-\kappa, 1)$

Then

$$\phi_{AB\dots K}\,\xi^A\xi^B\dots\xi^K = (\alpha_A\xi^A)(\beta_B\xi^B)\dots(\kappa_K\xi^K)$$

$$\Rightarrow \quad \left(\phi_{AB\ldots K} - \alpha_{(A}\beta_{B}\cdots\kappa_{K)}\right)\xi^{A}\cdots\xi^{K} = 0$$

non-symm. part drops out

ξ^A is arbitrary $\Rightarrow$

$$\phi_{AB\ldots K} = \alpha_{(A}\beta_{B}\cdots\kappa_{K)}$$

$\alpha_A, \beta_B \ldots$ are determined up to multiplicative constants (of course they have to satisfy $\prod_{m=1}^{p} \overset{m}{C} = 1$)

We know that each spinor α_A determines real null vector $k^\mu = \sigma^\mu{}_{A\dot{B}}\, \alpha^A \bar{\alpha}^{\dot{B}} \ldots$ α^A determines up to constant but each $\alpha^A, \beta^B, \ldots$ determines uniquely a null direction. These null directions need not be different – this enables one to classify $\phi_{AB\ldots K}$

The simplest case is ϕ_{AB} ... electromagnetic field

2 possibilities:
- [1,1] general field $\phi_{AB} = \alpha_{(A}\beta_{B)}$
- [2] null field $\phi_{AB} = \alpha_A \alpha_B$

Proposition: If $\phi_{AB} = \alpha_A \alpha_B$, $k^\mu = \sigma^\mu{}_{C\dot{Z}}\, \alpha^C \bar{\alpha}^{\dot{Z}}$,
then $F_{\alpha\beta} k^\beta = F^*_{\alpha\beta} k^\beta = 0$, $F_{\alpha\beta} F^{\alpha\beta} = 0$, $F_{\alpha\beta} F^{*\alpha\beta} = 0$

electromagnetic „null field", e.g. plane waves

Proof: $$F_{\alpha\beta} = \sigma_\alpha{}^{A\dot W}\sigma_\beta{}^{B\dot X}\Big(\varepsilon_{AB}\underbrace{\bar\phi_{\dot W\dot X}}_{=\bar\alpha_{\dot W}\bar\alpha_{\dot X}} + \underbrace{\phi_{AB}}_{\alpha_A\alpha_B}\varepsilon_{\dot W\dot X}\Big)\underbrace{\sigma^\beta{}_{C\dot Z}\,\alpha^C\bar\alpha^{\dot Z}}_{=k^\beta}$$

but $$\sigma_\beta{}^{B\dot X}\sigma^\beta{}_{C\dot Z} = \delta^B_C\,\delta^{\dot X}_{\dot Z}$$

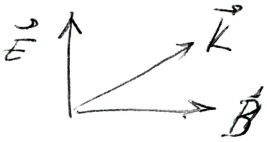
$$F_{\alpha\beta}k^\beta = \sigma_\alpha{}^{A\dot W}\big(\varepsilon_{AC}\bar\alpha_{\dot W}\bar\alpha_{\dot Z} + \alpha_A\alpha_C\varepsilon_{\dot W\dot Z}\big)\alpha^C\bar\alpha^{\dot Z}$$

$=0$

$=0$

Similarly for $F^*_{\alpha\beta}$

Indeed, for the "null" electromagnetic field (like plane wave)



$F_{\alpha\beta}k^\beta = F^*_{\alpha\beta}k^\beta = 0$, i.e. $\vec E\vec k = \vec B\vec k = 0$

and $F_{\alpha\beta}F^{\alpha\beta} = 0$, $\vec E^2 - \vec B^2 = 0$

$k^\alpha(k^0;\vec k)$

$F_{\alpha\beta}F^{*\alpha\beta} = 0$, $\vec E\cdot\vec B = 0$

Classification of the Weyl tensor (Riemann in vacuum)

$$R_{\alpha\beta\gamma\delta} \leftrightarrow A_{ABCD}\,\varepsilon_{\dot W\dot X}\varepsilon_{\dot Y\dot Z} + \varepsilon_{AB}\varepsilon_{CD}\bar A_{\dot W\dot X\dot Y\dot Z}$$

Possibilities	Petrov type		eqs. with k^α
[1111]	I	$A_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$	$k_{[\alpha}R_{\beta]\gamma\delta[\mu}k_{\nu]}k^\gamma k^\delta$
[211]	II	$A_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$	$=0$
[22]	D	$A_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)}$	
[31]	III	—"— $= \alpha_{(A}\alpha_B\alpha_C\beta_{D)}$	
[4]	N	$A_{ABCD} = \alpha_A\alpha_B\alpha_C\alpha_D$	$R_{\alpha\beta\gamma\delta}k^\delta = 0$
	O	$A_{\dots} = 0$	$R_{\dots} = 0$

$$\begin{tikzcd}
 & \mathrm{I}_{[1111]} \arrow[dl] \arrow[dr] & & \\
\mathrm{II}_{[211]} \arrow[rr] \arrow[d] \arrow[dr] & & D_{[22]} \arrow[dl] \arrow[d] & \\
\mathrm{III}_{[31]} \arrow[r] & N_{[4]} \arrow[r] & O &
\end{tikzcd}$$

Penrose diagram ... increasing special algebraic type

With „null type" – $N_{[4]}$ when $A_{ABCD} = \alpha_A \alpha_B \alpha_C \alpha_D$

it is easy to prove (cp. electromagnetic case) that

if $k^\alpha = \sigma^\alpha{}_{A\dot{B}}\, \alpha^A \bar{\alpha}^{\dot{B}} \Rightarrow R_{\alpha\beta\gamma\delta} k^\delta = 0, \; {}^*R_{\alpha\beta\gamma\delta} k^\delta = 0$

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 0, \quad R_{\alpha\beta\gamma\delta}\, {}^*R^{\alpha\beta\gamma\delta} = 0$$

Precisely these properties has exact plane wave in GR – see „skripta" Bičák & Rud., GR and GW, §4.2

– remark on the peeling-off theorem

waves from an insular system

Covariant derivative of spinors

Until now we considered spinors at special relativity (globally) or at a given event in GR (say in local in. frame)
The covariant derivative of spinors can be introduced axiomatically by requiring (for any spinor $S_{\cdots}^{\cdots}$ and $T_{\cdots}^{\cdots}$)

- $\nabla_\alpha \left(S_{\cdots}^{\cdots} + T_{\cdots}^{\cdots} \right) = \nabla_\alpha S_{\cdots}^{\cdots} + \nabla_\alpha T_{\cdots}^{\cdots}$ here S, T of the same rank

- Leibniz: $\nabla_\alpha \left(S_{\cdots}^{\cdots} T_{\cdots}^{\cdots} \right) = \left(\nabla_\alpha S_{\cdots}^{\cdots} \right) T_{\cdots}^{\cdots} + S_{\cdots}^{\cdots} \left(\nabla_\alpha T_{\cdots}^{\cdots} \right)$

- for scalars $\nabla_\alpha \varphi = \partial_\alpha \varphi$

- reality $\overline{\nabla_\alpha S_{\cdots}^{\cdots}} = \nabla_\alpha \overline{S_{\cdots}^{\cdots}}$

- „commutativity" with ε_{AB} and $\sigma_\alpha{}^{B\dot{X}}$:

$$\left. \begin{aligned} \nabla_\alpha \varepsilon_{AB} &= 0 \\ \nabla_\alpha \sigma_\beta{}^{B\dot{X}} &= 0 \end{aligned} \right\} \Rightarrow \text{ since } g_{\alpha\beta} = \sigma_\alpha{}^{A\dot{X}} \sigma_\beta{}^{B\dot{Y}} \varepsilon_{AB} \varepsilon_{\dot{X}\dot{Y}} \Rightarrow \nabla_\gamma g_{\alpha\beta} = 0$$

Expressed in spinor form

$$\nabla_\alpha = \sigma_\alpha{}^{A\dot{X}} \nabla_{A\dot{X}} \iff \nabla_{A\dot{X}} = \sigma^\alpha{}_{A\dot{X}} \nabla_\alpha$$

in particular, in flat spacetime $\nabla_{A\dot{W}} = \dfrac{\partial}{\partial x^{A\dot{W}}}$, $x^{A\dot{W}} = \sigma_\mu{}^{A\dot{W}} x^\mu$

Proposition:

Field equations (in vacuum) for a general field of spin s and with zero rest mass have the form

$$\nabla^{A\dot{X}} \underbrace{\phi_{AB\ldots K}}_{2s} = 0$$

(I) For $s = \frac{1}{2}$, (Weyl) neutrino

$$\nabla^{A\dot{X}} \phi_A = \sigma^{\mu A\dot{X}} \nabla_\mu \phi_A = 0$$

$$\Rightarrow \sigma^{0A\dot{X}} \partial_t \phi_A + \vec{\sigma}^{A\dot{X}} \vec{\nabla} \phi_A = 0$$

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $\frac{1}{\sqrt{2}}$ (Pauli matrices)

$$\Rightarrow \quad \boxed{\partial_t \phi = - \vec{\sigma} \vec{\nabla} \phi} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Standard form in QFT (see e.g. S. Schweber, An Introduction to Relativistic Quantum Field Theory, New York, Harper & Row 1961):

$$i\hbar \partial_t \phi = - c i \hbar \vec{\sigma} \vec{\nabla} \phi$$

put $\hbar = 1$, $c = 1$, multiply by i^{-1}

II Maxwell equations $s=1$

$$\boxed{\nabla^{A\dot{X}} \phi_{AB} = 0} \quad (*)$$

Let $F^{\mu\nu} = \sigma^{\mu A\dot{W}} \sigma^{\nu B\dot{X}} \left(\varepsilon_{AB} \bar{\phi}_{\dot{W}\dot{X}} + \varepsilon_{\dot{W}\dot{X}} \phi_{AB} \right)$

$$\Rightarrow F^{\mu\nu}{}_{;\nu} = \left[\left(\varepsilon_{AB} \bar{\phi}_{\dot{W}\dot{X}} + \varepsilon_{\dot{W}\dot{X}} \phi_{AB} \right) \sigma^{\mu A\dot{W}} \sigma^{\nu B\dot{X}} \right]_{;\nu}$$

$$= \underbrace{\sigma_{\nu C\dot{D}} \nabla^{C\dot{D}}}_{\nabla_\nu} \left[\sigma^{\mu A\dot{W}} \sigma^{\nu B\dot{X}} \left(\varepsilon_{AB} \bar{\phi}_{\dot{W}\dot{X}} + \varepsilon_{\dot{W}\dot{X}} \phi_{AB} \right) \right]$$

$$= \varepsilon_{AB} \delta_C^B \delta_{\dot{D}}^{\dot{X}} \nabla^{C\dot{D}} \bar{\phi}_{\dot{W}\dot{X}} + \varepsilon_{\dot{W}\dot{X}} \delta_C^B \delta_{\dot{D}}^{\dot{X}} \nabla^{C\dot{D}} \phi_{AB}$$

$$= \varepsilon_{AB} \underbrace{\nabla^{B\dot{X}} \bar{\phi}_{\dot{W}\dot{X}}}_{\substack{=0 \text{ from } (*) \\ \text{if we complex} \\ \text{conjugate it}}} + \varepsilon_{\dot{W}\dot{X}} \underbrace{\nabla^{B\dot{X}} \phi_{AB}}_{= \nabla^{B\dot{X}} \phi_{BA} = 0 \text{ from } (*)}$$

Note:
In case of curvature (gravity) one can prove - by using the spinor equivalent of Riemann tensor Bianchi identities in the form $\nabla^{d} \overset{*}{R}_{abcd} = 0$

where $R^*_{abcd} = \frac{1}{2} \varepsilon_{cd}{}^{gh} R_{abgh}$

Geometrical interpretation of 1-spinors

1-spinor ξ^A defines a real null vector

$$\xi^\mu = \sigma^\mu{}_{A\dot{X}}\, \xi^A \bar{\xi}^{\dot{X}}$$

when we change the phase $\xi^A \to \xi^A e^{i\theta}$, θ real then $\bar{\xi}^{\dot{A}} \to \bar{\xi}^{\dot{A}} e^{-i\theta}$, so the spinor changes but the null vector not. So we cannot associate ξ^A uniquely with ξ^μ. Better use "square $\xi^A \xi^B$".

Penrose: ξ^A can be used to define simple bivector (i.e. antisym. 2nd rank tensor of the form $a^\mu b^\nu - a^\nu b^\mu$) and then the phase can be interpreted.

Introduce as usually antisym. tensor and corresponding spinor

$$F_{\mu\nu} = \sigma_\mu{}^{A\dot{X}} \sigma_\nu{}^{B\dot{Y}} \left(\varepsilon_{\dot{X}\dot{Y}} \phi_{AB} + \varepsilon_{AB} \bar{\phi}_{\dot{X}\dot{Y}} \right) \qquad (*)$$

and take

$$\phi_{AB} = \xi_A \xi_B \to \bar{\phi}_{\dot{X}\dot{Y}} = \bar{\xi}_{\dot{X}} \bar{\xi}_{\dot{Y}} \qquad (**)$$

introduce further an auxiliary spinor η_A which forms, with ξ_A, the basis, i.e.

$$\xi_A \eta^A = 1$$

$$\Leftrightarrow \quad \varepsilon_{AB} = \xi_A \eta_B - \eta_A \xi_B \qquad (***)$$

viz. in general $\psi_{AB} - \psi_{BA} = \varepsilon_{AB} \psi_D{}^D$

if $\psi_{AB} = \xi_A \eta_B$ $\quad \Rightarrow \xi_A \eta_B - \xi_B \eta_A = \varepsilon_{AB} \underbrace{\xi_D \eta^D}_{=1}$

Substitute from (***) and (**) into $F_{\mu\nu}$ in (*):

$$F_{\mu\nu} = \sigma_\mu{}^{A\dot{X}}\, \sigma_\nu{}^{B\dot{Y}} \Big[\big(\bar{\xi}_{\dot{X}} \bar{\eta}_{\dot{Y}} - \bar{\eta}_{\dot{X}} \bar{\xi}_{\dot{Y}} \big) \xi_A \xi_B$$

$$+ \big(\xi_A \eta_B - \eta_A \xi_B \big) \bar{\xi}_{\dot{X}} \bar{\xi}_{\dot{Y}} \Big]$$

$$= \sigma_\mu{}^{A\dot{X}} \xi_A \bar{\xi}_{\dot{X}}\, \sigma_\nu{}^{B\dot{Y}} \xi_B \bar{\eta}_{\dot{Y}} - \sigma_\mu{}^{A\dot{X}} \xi_A \bar{\eta}_{\dot{X}}\, \sigma_\nu{}^{B\dot{Y}} \xi_B \bar{\xi}_{\dot{Y}}$$

$$+ \sigma_\mu{}^{A\dot{X}} \xi_A \bar{\xi}_{\dot{X}}\, \sigma_\nu{}^{B\dot{Y}} \bar{\xi}_{\dot{Y}} \eta_B - \sigma_\mu{}^{A\dot{X}} \eta_A \bar{\xi}_{\dot{X}}\, \sigma_\nu{}^{B\dot{Y}} \xi_B \bar{\xi}_{\dot{Y}}$$

$$= \sigma_\mu{}^{A\dot{X}} \xi_A \bar{\xi}_{\dot{X}} \Big[\sigma_\nu{}^{B\dot{Y}} \big(\xi_B \bar{\eta}_{\dot{Y}} + \bar{\xi}_{\dot{Y}} \eta_B \big) \Big]$$

$$- \sigma_\nu{}^{B\dot{Y}} \xi_B \bar{\xi}_{\dot{Y}} \Big[\sigma_\mu{}^{A\dot{X}} \big(\xi_A \bar{\eta}_{\dot{X}} + \eta_A \bar{\xi}_{\dot{X}} \big) \Big]$$

$\Rightarrow \quad F_{\mu\nu} = \xi_\mu w_\nu - \xi_\nu w_\mu$ <u>is</u> simple bivector

where $\quad w_\mu = \sigma_\mu{}^{A\dot{X}} \big(\xi_A \bar{\eta}_{\dot{X}} + \eta_A \bar{\xi}_{\dot{X}} \big)$

ζ is <u>real</u>

and $\boxed{w_\mu \xi^\mu = 0}$ $\quad \to \quad w_\mu$ is spacelike (since ξ^μ is null)

w_μ is <u>not</u> determined uniquely because η_A is not determined uniquely:

the only condition is that $\xi_A \eta^A = 1$; and we change $\eta^A \to \tilde{\eta}^A = \eta^A + \lambda \xi^A$ condition $\tilde{\eta}^A \xi_A = 1$

is satisfied. How does w^μ change?

$$\tilde{w}_\mu = w_\mu + \sigma_\mu^{B\dot{X}}\left(\xi_B \bar{\lambda} \bar{\xi}_{\dot{X}} + \lambda \xi_B \bar{\xi}_{\dot{X}}\right) =$$

$$= w_\mu + \underbrace{(\bar{\lambda}+\lambda)\left(\sigma_\mu^{B\dot{X}} \xi_B \bar{\xi}_{\dot{X}}\right)}$$

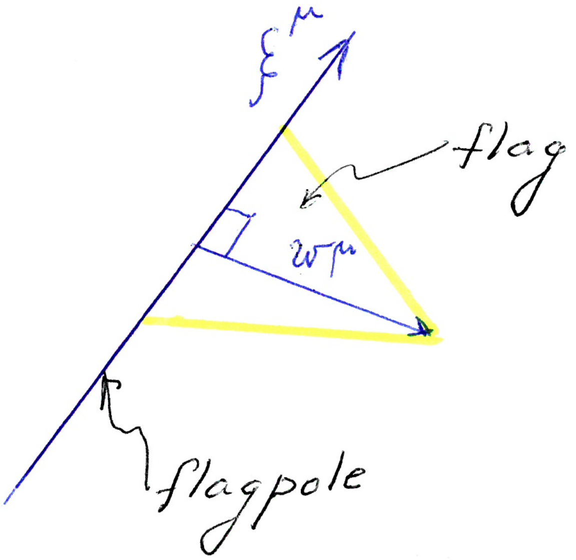
$$= \underbrace{(\bar{\lambda}+\lambda)}_{= k \text{ which is real}} \xi_\mu$$

$$w_\mu \to \tilde{w}_\mu = w_\mu + k \xi_\mu$$

However, $F_{\mu\nu}$ does not change (it is seen that $F_{\mu\nu}$ is independent of change of η, and of w

$\Rightarrow$ Spinor ξ^A determines a 2-space given by ξ_μ and arbitrary w_μ which is orthogonal to ξ_μ

$\Rightarrow$ flag



flagpole determined by the direction of ξ^μ

the flag determined by the bivector $F_{\mu\nu}$, i.e. by the phase ξ^A

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When the phase is changed

$$\xi^A \to \xi^A e^{i\theta},$$

in order to preserve to condition on the basis, it must be that

$$\eta^A \to \eta^A e^{-i\theta}$$

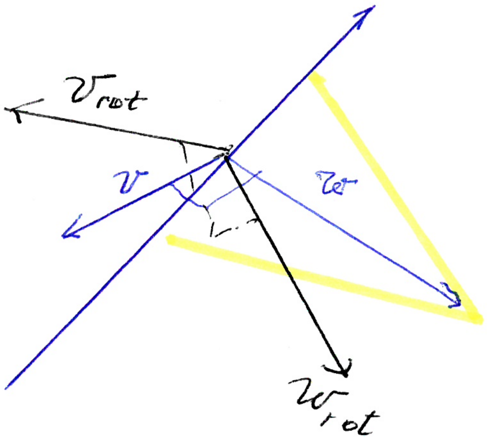
$$\Rightarrow w_\mu \to \sigma_\mu{}^{B\dot{x}}\left(\xi_B \bar{\eta}_{\dot{x}}\, e^{2i\theta} + \eta_B \bar{\xi}_{\dot{x}}\, e^{-2i\theta}\right)$$

Define $v_\mu = i\,\sigma_\mu{}^{B\dot{x}}\left(\xi_B \bar{\eta}_{\dot{x}} - \eta_B \bar{\xi}_{\dot{x}}\right)$

is real

and $v \perp w$, $v \perp \xi$, when $\xi^A \to \xi^A e^{i\theta}$, $\eta^A \to \eta^A e^{-i\theta}$

we find $w_\mu \to w_\mu \cos 2\theta + v_\mu \sin 2\theta \Rightarrow$



when $\xi^A \to \xi^A e^{i\theta}$

the flag rotates around the flagpole by 2θ

when $\theta = \pi$, $\xi^A \to -\xi^A$ but the flag and the flagpole do not change!

related to:

" spin transformations give 2-valued representation of the Lorentz transformations! "