# Problems on Lie groups

**Deadline**: recommended by Monday **6.1.2025**

## Homomorphism from $SL(2,\mathbb{C})$ to $L^0$

Let $\phi$ be the homomorphism from $SL(2,\mathbb{C})$ to $L^0$ constructed during lectures (see notes on the website).

1. (5 points) Show that the kernel of this homomorphism is $\operatorname{Ker}\phi = \{\mathbf{1}, -\mathbf{1}\}$.
   *Note:* It is not enough to show that the matrices $\mathbf{1}$ and $-\mathbf{1}$ belong to $\operatorname{Ker}\phi$, but that there is no other matrix $A \in SL(2,\mathbb{C})$ that maps to the identity in $L^0$.

2. (5 points) Find the matrix $A \in SL(2,\mathbb{C})$ that maps to the transformation of the 4-vector $(x_0, x_1, x_2, x_3)^T$ which is a rotation about the $x_1$ axis by an angle $\alpha$.

## Left-invariant vector fields of the group $GA(1,\mathbb{R})$ and its one-parameter subgroups

The group $GA(1,\mathbb{R})$ is the group of all linear transformations

$$x' = ax + b, \qquad a \neq 0,\ b \in \mathbb{R}\,.$$

Therefore, it is a two-parameter Lie group, each element of which is given by a pair of parameters $g = (a, b)$ with the group operation given by the composition of two such transformations.

1. (5 points) Determine the left-invariant vector fields of this group and the structure constants in a suitable basis of the corresponding Lie algebra.

2. (5 points) Determine the one-parameter subgroup of this group corresponding to the general element of its Lie algebra, i.e. for a general linear combination of basis elements of this Lie algebra.
   *Note:* This subproblem can also be solved in matrix form by rewriting the transformations using $2 \times 2$ matrices and finding its matrix Lie algebra.

## One-parameter subgroups of the Lie group $SL(2,\mathbb{R})$ which is not covered by the exponential mapping

The group $SL(2,\mathbb{R})$ is the group of all real matrices $A_{2\times 2}$ for which $\det A = 1$, or

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1\,.$$

1. (5 points) Determine the one-parameter subgroups of this group in the matrix form. *Hint:* Write the general form of the matrix $C$ from the corresponding Lie algebra $sl(2,\mathbb{R})$ of this group and calculate $\exp Ct$ directly, e.g. using Taylor expansion, depending on the sign of $\det C$.

2. (5 points) Based on the result of the previous subproblem, show that the exponential map does not cover the entire group $SL(2,\mathbb{R})$, although this group is connected. *Hint:* Consider, for example, the traces of the resulting matrices.