Lie group and its Lie algebra (geometric approach) (LG1)

Def. A topological group G is a topological space that is also a group with a binary operation ϕ and an inverse $i(g)$ that are continuous mappings in a topological sense

i.e. $\phi : G \times G \to G$ or $\phi : (g,h) \mapsto g \cdot h$ are both continuous

$i : G \to G$ or $i : g \to g^{-1}$

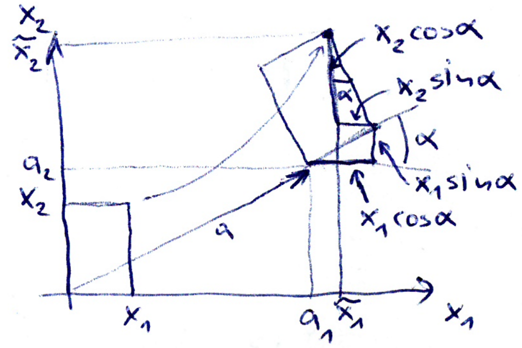
Def. A real Lie group G is a topological group that is also a real smooth manifold and the mappings ϕ and i are smooth mappings (or analytic mappings)

- a historical note: the fifth Hilbert problem from 1900 concerned the necessity of smoothness, i.e. the question whether it is not enough that it is a topological group in othe words, whether continuity implies smoothness
  - it was shown that under relatively general conditions it is true that continuity of φ and i together with the algebraic structure of a group really imply smoothness (or analyticity)

Example of a Lie group: special Euclidean group $E^+(2)$

= a group of all transformations of the Euclidean plane that preserve distances and orientation (only rotations and translations, but no reflections)

- it is given by transformations



$$\tilde{x}_1 = x_1 \cos\alpha - x_2 \sin\alpha + a_1$$
$$\tilde{x}_2 = x_1 \sin\alpha + x_2 \cos\alpha + a_2$$

or

$$\tilde{x} = \underbrace{R(\alpha)}_{\text{proper rotation by } \alpha} x + \underbrace{a}_{\text{translation by } a}$$

- composition of two such transformations gives again such transformation

$$\tilde{\tilde{x}} = R(\beta)\tilde{x} + b = \underbrace{R(\beta)R(\alpha)}_{\text{rotation by } \alpha+\beta} x + \underbrace{R(\beta)\,a + b}_{\text{translation by } c = R(\beta)\,a+b} = R(\gamma)\,x + c$$

(LG2)

- it is a 3-dimensional (or 3-parametric) Lie group,

the elements of which are characterized by 3 values (α, a_1, a_2) and the group operation $\phi : G \times G \to G$ is given by

$$\phi\left((\beta, b_1, b_2), (\alpha, a_1, a_2)\right) = (\gamma, c_1, c_2)$$

where $\gamma = \alpha + \beta$, $c = R(\beta)a + b$ or $c_1 = a_1\cos\beta - a_2\sin\beta + b_1$, $c_2 = a_1\sin\beta + a_2\cos\beta + b_2$

which are smooth (analytic) functions, and the inverse operation

$i : G \to G$ or $i((\alpha, a_1, a_2)) = (\beta, b_1, b_2)$

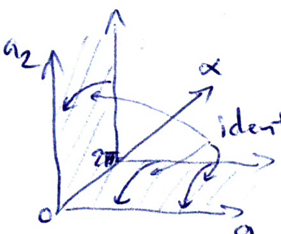
is given by $\beta = -\alpha$, $b = -R(-\alpha)a$ or $b_1 = -a_1\cos\alpha - a_2\sin\alpha$, $b_2 = a_1\sin\alpha - a_2\cos\alpha$

which are again smooth (analytic) functions

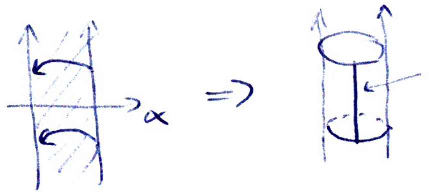
- note that $\tilde{\tilde{x}} = \underbrace{R(-\alpha)}_{R(\beta)} R(\alpha)x + \underbrace{R(-\alpha)}_{R(\beta)}a - R(-\alpha)a = x$

- because for $\alpha = 0$ and $\alpha = 2\pi$ we get the same result,
we cannot use one smooth parametrization (map) globally

- topologically, it is a 3-dim. smoot manifold
that is „a cylinder" in 4D constructed from the infinit layer in 3D and identification of boundary planes as for 3D cylinder from a slab in the plane

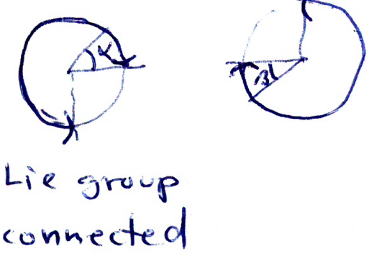


identification

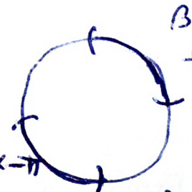


we glue $\alpha = 0$ and $\alpha = 2\pi$

- two maps are enough to cover the whole manifold
- the whole plane (a_1, a_2) and angles $\alpha \in (0, \frac{3}{2}\pi)$ **or** $\beta \in (0, \frac{3}{2}\pi)$ measured from $\alpha = \pi$



transition functions (maps) are



$\beta = \alpha + \pi$ for $\alpha \in (0, \frac{\pi}{2})$

$\beta = \alpha - \pi$ for $\alpha \in (\pi, \frac{3}{2}\pi)$

both smooth mappings

- it is a connected Lie group
but not simply connected
(a closed curve from $\alpha = 0$ to 2π cannot be contracted into a point smoothy)

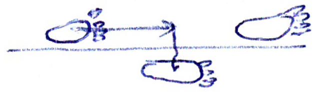
- $E^+(2)$ is a subgroup of the full Euclidean group

  $E(2) = ISO(2)$, a group of all isometries of a plane,

  - it has two components $E^+(2)$ and $E^-(2)$

    $E^-(2)$ contains all improper rotations (rotation multiplied by a reflection)

    but also for example glide reflection



shift and reflection

- $E(2)$ is a subgroup of more general group of affine transformations of a plane, denoted $GA(2,\mathbb{R})$

  i.e. $\tilde{x} = Ax + a$ where A is an arbitrary invertible real 2×2 matrix ($\det A \neq 0$)

  that can be written as

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \hdashline 1 \end{pmatrix} = \left(\begin{array}{c:c} A & a \\ \hdashline 0 & 1 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ \hdashline 1 \end{pmatrix}$$

  and thus it is also a subgroup of $GL(3,\mathbb{R}) =$

  $=$ real 3×3 matrices with $\det A \neq 0$

- we easily check the composition as $\tilde{\tilde{x}} = A_1 \tilde{x} + a_1 = A_1 A_2 x + A_1 a_2 + a_1$

  and

$$\left(\begin{array}{c:c} A_1 & a_1 \\ \hdashline 0 & 1 \end{array}\right) \left(\begin{array}{c:c} A_2 & a_2 \\ \hdashline 0 & 1 \end{array}\right) = \left(\begin{array}{c:c} A_1 A_2 & A_1 a_2 + a_1 \\ \hdashline 0 & 1 \end{array}\right)$$

- even though $E^+(2)$ is, as a topological space, a Cartesian product of $\mathbb{R}^2$ and $SO(2)$, $E^+(2)$ as a group is not a direct product of $T = (\mathbb{R}^2, +)$ (translations) and $SO(2)$ (rotations at the origin)

  because elements of these subgroups do not commute

$$\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & Ra \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & a \\ 0 & 1 \end{pmatrix}$$

  but it is a semidirect product

$$E^+(2) = T \rtimes SO(2)$$

  since T is a normal subgroup of $E^+(2)$

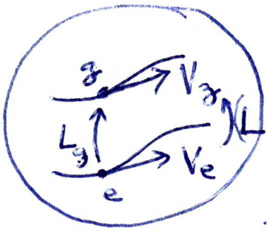
Left-invariant vectorfields on Lie groups

- on Lie groups, we have two diffeomorphisms
  - left and right multiplication by $g \in G$ (remember the rearrangement theorem)

i.e. $L_g : G \to G$ or $L_g : h \to gh$
$R_g : G \to G$ or $R_g : h \to hg$
by definition, these are smooth maps that are bijective and thus diffeomorphisms

<u>Def:</u> Let G be a Lie group. Then a vector field V is left- or right-invariant if it satisfies

$$(L_g)_* V = V \quad \text{or} \quad (R_g)_* V = V \quad \text{for all } g \in G$$

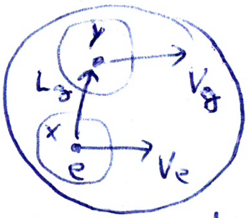


this denotes a push-forward map

- in coordinates we can write

components of V in the neighbourhood of g $\to$ $V^\nu(y(x)) = J^\nu{}_\mu(x)\, V^\mu(x)$ components of V in the neighbourhood of e

given by left multiplication



where $J^\nu{}_\mu(x) = \dfrac{\partial y^\nu(x)}{\partial x^\mu}$ is a Jacobian of change of coordinates corresponding to L_g

- left- and right-invariant vector fields are completely determined by a tangent vector at identity e (or at any other point) using $V_g = (L_g)_* V_e$

  and they are smooth (as $g \cdot h$ is smooth)

- thus they form „only" n-dimensional vector space isomorphic to $T_e G$ and denoted as $\mathcal{L}(G)$ (for left-inv. vector fields)

- since for any diffeomorphism ϕ it can be shown $\phi_* [V, W] = [\phi_* V, \phi_* W]$, where $[V,W]_p(f) = V_p(Wf) - W_p(Vf)$

  we get $(L_g)_* [V,W] = [(L_g)_* V, (L_g)_* W] = [V,W]$ for all $V, W \in \mathcal{L}(G)$

and we see that also $[V,W] \in \mathcal{L}(G)$ and thus $\mathcal{L}(G)$ forms a Lie algebra with the commutator $[V,W]$

Def: The Lie algebra of the Lie group G is the vector space $\mathcal{L}(G)$ of all left-invariant vector fields on G with the commutator given by the Lie bracket of two vector fields $[V,W]f = V(Wf) - W(Vf)$

- to use the left-invariant vector fields was our choice we could use also right-inv. vector fields but they are in general different from left-inv. v. fields

- sometimes, you can find the definition of the Lie algebra of LG as the tangent space to the corresponding manifold at the identity e that is $T_e G$, but it is necessary to define a commutator to make it a Lie algebra and that can be in general done only through left-invariant vector fields generated from the elements of $T_e G$ as

$$[V_e, W_e] = [V,W]_e$$

where V and W are left-invariant v.f. generated from V_e and W_e

Lie algebra (LA) as an abstract mathematical structure

Def: A real Lie algebra $\mathcal{L}$ of dimension n is an n-dimensional real vector space with a binary operation (additional to the standard sum of two vectors) called the commutator $[a,b]$ for all $a,b \in \mathcal{L}$ satisfying

1) $[a,b] \in \mathcal{L}$ for all $a,b \in \mathcal{L}$ (closure)
2) $[\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c]$ (linearity)
3) $[a,b] = -[b,a]$ (antisymmetry)
4) $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ (Jacobi identity)

for all $a,b,c \in \mathcal{L}$ and all $\alpha, \beta \in \mathbb{R}$

- consequence of 2) and 3) is $[a, \beta b + \gamma c] = \beta[a,b] + \gamma[a,c]$ (bilinearity)
- in a similar way, one can define complex LAs or LAs for any field K

Examples: 1) vector space of matrices $n \times n$ with the commutator $[A,B] = AB - BA$

2) vector space of linear operators on some Hilbert space with $[a,b]f = a(bf) - b(af)$

<u>Ado's theorem</u>: Every finite-dimensional abstract Lie algebra over the field K of characteristic zero is isomorphic to some Lie algebra of matrices over the field K.

- thus, for finite-dimensional Lie algebras, it is enough to work with matrices and their Lie algebras (as $gl(n,\mathbb{R})$ for the Lie group $GL(n,\mathbb{R})$ later)

<u>Structure constants</u>

- let $\{e_i\}_{i=1}^n$ be a basis of $\mathcal{L}$, then $[e_i, e_j]$ must be a linear combination of these basis vectors and we write

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^{k} e_k$$

where c_{ij}^k are <u>structure constants</u> of the Lie algebra $\mathcal{L}$ with respect to the basis e_i

- thus, c^k_{ij} depend on the basis and they transform a a tensor of the third order when the basis is changed

- every commutator $[a,b]$ in $\mathcal{L}$ can be expressed using components of the vectors a, b in the basis and structure constants

$$[a,b] = \sum_{k,\ell,m} a^k b^\ell c^m_{k\ell} e_m \quad \text{where } a = \sum_{k=1}^{n} a^k e_k \text{ and } b = \sum_{\ell=1}^{n} b^\ell e_\ell$$

- from 3) and 4) in the definition of the Lie algebra we get

$$c^k_{ij} = -c^k_{ji} \quad \text{and} \quad c^s_{pq} c^t_{rs} + c^s_{qr} c^t_{ps} + c^s_{rp} c^t_{qs} = 0$$

- if Lie algebra is associated with a Lie group then the structure constants of LA are also considered as the structure constants of the Lie group