One-parameter subgroups and exponential map

(LG 7)

- very important subgroups of Lie groups generated by elements of corresponding Lie algebras, we will see that they are integral curves of left-invariant vector fields and enable us to define the exponential map in general

Def: A one-parameter subgroup of a Lie group G is a curve $\gamma(t)$ satisfying $\gamma(t+s) = \gamma(t)\,\gamma(s)$, $\gamma(0) = e$, for all $t, s \in \mathbb{R}$

it is thus a homomorphism $\gamma : (\mathbb{R}, +) \xrightarrow{\text{into}} G$

- in general, it is not injective, as $\gamma(t)$ is defined for an arbitrary $t \in \mathbb{R}$ since $\gamma(t)\,\gamma(s)$ is always defined in the Lie group and, for compact Lie groups, $\gamma(t)$ for several or even infinite different values of t corresponds to the same $g = \gamma(t)$
- any element of the one-parameter subgroup can be written as $\gamma(t) = \gamma(\varepsilon)^n$ where $\varepsilon = \frac{t}{n}$ can be arbitrarily small giving $\gamma(\varepsilon)$ close to the identity e, thus, every $\gamma(t)$ for any $t \in \mathbb{R}$ can be written as a product of finite number of elements (for finite t) from some neighbourhood of the identity e

(note: remember
$$R^z(\varphi) = \lim_{n\to\infty} \underbrace{\left(1 - i\tfrac{\varphi}{n} L_z\right)}_{\text{small rotation around } z}{}^n = e^{-i\varphi L_z}$$
these are one-parameter subgroups of $SO(3)$ for various L)

- for compact Lie groups, every element $g \in G$ can be written as $\gamma(\varepsilon)^n$, i.e. g lies on a one-parameter subgroup,

but for a general, non-compact group, this is not true e.g. $SL(2,\mathbb{R})$ has elements that cannot be written as an exponential of some matrix from its Lie algebra (see the homework on Lie groups)

Theorem: Every one-parameter subgroup of a Lie group G is an integral curve of some leftinvariant vector field going through the identity, and vice versa, integral curves of left-invariant vector fields starting at e are one-parametric subgroups of G.

Proof: $\Rightarrow$ let $\gamma(t)$ be a one-param. subgroup of G and $\dot\gamma(0)$ its tangent vector at e, then it is enough to show that by push-forward $(L_{\gamma(t)})_*$ of $\dot\gamma(0)$ we get $\dot\gamma(t)$

– we get

$$\underset{\text{definition}}{\dot\gamma(t) =} \frac{d}{ds}\Big|_{s=0}\gamma(t+s) \overset{\text{assumption}}{=} \frac{d}{ds}\Big|_{s=0}\gamma(t)\gamma(s) \overset{\text{definition of left multiplication}}{=} \frac{d}{ds}\Big|_{s=0} L_{\gamma(t)}\gamma(s) =$$

$$\underset{\text{definition of push-forward}}{=} (L_{\gamma(t)})_* \frac{d\gamma(s)}{ds}\Big|_{s=0} = (L_{\gamma(t)})_*\,\dot\gamma(0)$$

and thus $\dot\gamma(t)$ is determined from $\dot\gamma(0)$ using $L_{\gamma(t)}$ and these form a left-inv. v. field as we wanted to show

$\Leftarrow$ let now $\gamma(t)$ be an integral curve of a left-inv. v. field V, then the curve $\Gamma(t) = \gamma(t+s)$ is also an integral curve of V starting at $\gamma(s)$ (because $\frac{d(t+s)}{dt} = 1$), but because of left-invariance of V, it is also an integral curve of the vector field $(L_{\gamma(s)})_* V$ which is also a curve given by left multiplication, that is

$$L_{\gamma(s)}\,\gamma(t) = \gamma(s)\gamma(t) \quad \text{and thus}$$

we have $\quad \gamma(t+s) = \gamma(s)\gamma(t)$

Example: One-parameter subgroups of $GL(n,\mathbb{R})$

– we know that left-inv. v. fields on $GL(n,\mathbb{R})$ are

$$V_C = x^i{}_k c^k{}_j \frac{\partial}{\partial x^i{}_j}$$ where C is an arbitrary real $n\times n$ matrix

– for integral curves we have differential equations

$$\dot x^i{}_j(t) = (V_C)^i{}_j(x(t)) = x^i{}_k(t)\, c^k{}_j$$

with the initial condition $\quad x^i{}_j(0) = \delta^i{}_j$

- this set of ODEs (ordinary differential equations) has a solution

$$x^i{}_j(t) = (\exp(tc))^i{}_j = \delta^i{}_j + tc^i{}_j + \frac{t^2}{2!}(c^2)^i{}_j + \dots$$

or, in a matrix form,

$$\dot{x}(t) = x(t)\,C\ ,\quad x(0) = \mathbb{1} \quad \text{giving} \quad x(t) = e^{tC}$$

- we see that one-parameter subgroups of $GL(n,\mathbb{R})$ are given by the exponential of matrices C corresponding to left-invariant v. fields V_C, and we have 1-to-1 correspondence

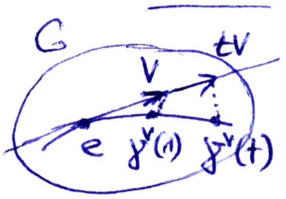
elements of LA of LG as matrices C $\longleftrightarrow$ left-invariant vector fields V_C $\longleftrightarrow$ one-parameter subgroups e^{tC}

$\Rightarrow$ motivation for the exponential map from LA to LG

Def.: Let $\gamma^V(t)$ be a one-parameter subgroup of G for a left-inv. v. field V from the Lie algebra $\mathfrak{g}$ of this group, i.e., $\dot{\gamma}^V(t) = V_{\gamma(t)}$, then we define an exponential map as $\exp : \mathfrak{g} \to G$

using $\exp : V \mapsto e^V \equiv \gamma^V(1)$

Theorem: A one-parametric subgroup $\gamma^V(t)$ of a left-invariant vector field V is given using the exponential map as $\gamma^V(t) = e^{tV}$.



Proof: for $tV \in \mathfrak{g}$ (LA of G) we have $e^{tV} = \gamma^{tV}(1)$, thus tV is a tangent vector of the curve $\gamma^{tV}(k)$ at $k=0$, but at the same time it is a tangent vector of $\gamma^V(tk)$ since

$$\left.\frac{d\gamma^V(tk)}{dk}\right|_{k=0} = \left.\frac{d\gamma^V(tk)}{d(tk)}\right|_{tk=0} \cdot \left.\frac{d(tk)}{dk}\right|_{k=0} = tV$$

and because $\gamma^{tV}(0) = e = \gamma^V(t\cdot 0) = \gamma^V(0)$, they are the same curves, thus $\gamma^{tV}(k) = \gamma^V(tk)$

and particularly $\gamma^{tV}(1) = \gamma^V(t) = e^{tV}$

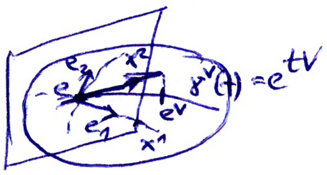
and $e^0 = \gamma^V(0) = e$ and $e^{-V} = \gamma^V(-1) = \gamma^V(1)^{-1} = (e^V)^{-1}$

Covering groups by the exponential map – selected results without proofs

Theorem: Every point $g \in G$ from a certain neighbourhood of e can be written as $g = e^V$ for a certain $V \in \mathcal{G} = T_e(G)$

(if g is close to e then V is close to 0 in $\mathcal{G}$)

- natural result as a neighbourhood $U(e) \subset G$ and also $U(0) \subset \mathcal{G}$ are diffeomorphic to $\mathbb{R}^n$, where $n = \dim G$ and $\dim \mathcal{G}$
- in this way, we can define „natural coordinates" close to e



when we assing coordinates $(x^1, \dots, x^n)$ to a point $g = e^V$ in the neighbourhood of e that are coefficients of V in the basis of $\mathcal{G}$, that is $V = \sum_{j=1}^{n} x^j e_j$

- in general, not every element $g \in G$ can be written as e^V for some $V \in \mathcal{G}$ (either the group G has more components, or for some non-compact groups it can happen that there are some unreachable elements even in the component connected to e, see homework on Lie groups for an example)
- but in general for connected groups (or its component connected to e) we have the following results

Theorem: Every element g of a Lie group G from its connected subgroup (with e) can be written as a finite product of exponentials from its Lie algebra, that is
$$g = \prod_{i=1}^{r} e^{V_i}, \quad V_i \in \mathcal{G}$$

Theorem: If a Lie group G is compact then every element g of its connected subgroup can be expressed as $g = e^V$ for a certain $V \in \mathcal{G}$

- more details and proofs can be found in some standard, but usually difficult to read, textbooks on theory of Lie groups like that by Pontryagin etc.