Induced homomorphism of Lie algebras

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- it is important relation between Lie algebras of two Lie groups when there exists a homomorphism $\phi : G \to H$ between them; because $\operatorname{Im}\phi$ forms a subgroup in H it is also important relation between the Lie algebras of a Lie group and its Lie subgroups

<u>Def:</u> A <u>Lie subgroup H</u> of a Lie group G is such a subgroup of G that is at the same time also a topological group with respect to topology induced from G onto H as its topological subspace

<u>Example:</u> $\{1,-1\} \subset SU(2)$ is not a Lie subgroup but one-parametric subgroups are

<u>Theorem:</u> Let $\phi : G \to H$ be a homomorphism between two Lie groups G and H and $\gamma^V(t) = e^{tV}$ is a one-parametric subgroup of G for a left-invariant vector field $V \in \mathfrak{g}$. Then $\phi(\gamma^V(t)) = \phi(e^{tV})$ is a one-parametric subgroup of H (trivially thanks to properties of homomorphism ϕ)

and thus $\phi(e^{tV}) = e^{tW}$ for some $W \in \mathfrak{h}$, $(=$ LA of $H)$

moreover W is given by push-forward $\phi_* : \mathfrak{g} \to \mathfrak{h}$ as $W = \phi_* V$ (it follows directly from the definition of the push-forward ϕ_*)

and $\phi_* : \mathfrak{g} \to \mathfrak{h}$ is homomorphism of Lie algebras

that is $\phi_*(\alpha V_1 + \beta V_2) = \alpha\, \phi_* V_1 + \beta\, \phi_* V_2$

and $\phi_*[V_1, V_2] = [\phi_* V_1, \phi_* V_2]$

which is called the <u>induced homomorphism</u> from the Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathfrak{h}$

- we have a commutating diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\phi"'] & H \end{tikzcd}$$

or $\phi(\exp(V)) = \exp(\phi_* V)$

- $\operatorname{Im}\phi_*$ is a subalgebra of the Lie algebra $\mathfrak{h}$

Subgroups of $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$

- we know that the Lie algebra $gl(n,\mathbb{R})$ (and similarly for $gl(n,\mathbb{C})$) is isomorphic to the LA of real (complex for $gl(n,\mathbb{C})$) $n\times n$ matrices
- it follows from the theorem about the induced homomorphism of LAs that every Lie subgroup of $GL(n,\mathbb{R})$, or $GL(n,\mathbb{C})$, has a Lie algebra that is a Lie subalgebra of $gl(n,\mathbb{R})$, or $gl(n,\mathbb{C})$ that is, it is isomorphic to some LA of $n\times n$ matrices

  (we can just take as the homomorphism ϕ in the theorem the 'identity' mapping from a subgroup $H\subset GL(n,\mathbb{R})$ as an abstract Lie group into $GL(n,\mathbb{R})$)
- in general, we can find matrices C that form the Lie algebra h of the subgroup $H\subset GL(n,\mathbb{R})$, or $GL(n,\mathbb{C})$ in such a way that we substitute the Taylor series

$$x(t) = e^{tC} = 1 + tC + \frac{t^2}{2!}C^2 + \dots$$

  into the condition(s) defining the subgroup of $GL(n,\mathbb{R})$ of interest, and we keep only terms of order t, the coefficient by t gives the condition defining the LA h
- examples of obtained conditions (see also the table on the next page)

Lie group condition		LA condition
$\det A = 1$	$\Rightarrow \det e^{tC} = e^{\mathrm{Tr}\, tC} = 1 \Rightarrow$	$\mathrm{Tr}\, C = 0$
$A^T A = 1$	$\Rightarrow (1+tC^T)(1+tC) = 1 + t(C+C^T) + O(t^2) = 1 \Rightarrow$	$C = -C^T$
$A^+ A = 1$	$\Rightarrow (1+tC^+)(1+tC) = 1 \Rightarrow$	$C = -C^+$
$A^T \eta A = \eta$ for $\eta = \begin{pmatrix} \underbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}_{r} & 0 \\ 0 & \underbrace{\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}}_{s} \end{pmatrix}$	$\Rightarrow (1+tC^T)\eta(1+tC) = \eta \Rightarrow$	$\eta C = -(\eta C)^T$
$A^+ \Omega A = \Omega$ for $\Omega = \begin{pmatrix} 0 & 1_{n\times n} \\ -1_{n\times n} & 0 \end{pmatrix}$	$\Rightarrow (1+tC^T)\Omega(1+tC) = \Omega \Rightarrow$	$\Omega C = (\Omega C)^T$

- the following table gives a basic summary of classical matrix groups

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	general linear	special linear	orthogonal	orthogonal	symplectic	complex →	complex →	unitary	special unitary
Lie group	$GL(n,\mathbb{R})$	$SL(n,\mathbb{R})$	$O(r,s)$ $SO(r,s)$	$O(n)$ $SO(n)$	$Sp(2n,\mathbb{R})$	$GL(n,\mathbb{C})$	$SL(n,\mathbb{C})$	$U(n)$	$SU(n)$
conditions on A	$\det A \neq 0$	$\det A = 1$	$A^T \eta A = \eta$ + $\det A = 1$	$A^T A = 1$ + $\det A = 1$	$A^T \Omega A = \Omega$ + $\det A = 1$	$\det A \neq 0$	$\det A = 1$	$A^\dagger A = 1$	$A^\dagger A = 1$ + $\det A = 1$
Lie algebra	$\mathfrak{gl}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{o}(r,s) \sim \mathfrak{so}(r,s)$	$\mathfrak{o}(n) \sim \mathfrak{so}(n)$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{u}(n)$	$\mathfrak{su}(n)$
conditions on C	—	$\mathrm{Tr}\, C = 0$	$(\eta C)^T = -\eta C$ + $\mathrm{Tr}\, C = 0$	$C^T = -C$ + $\mathrm{Tr}\, C = 0$	$\Omega C = (\Omega C)^T$ + $\mathrm{Tr}\, C = 0$	—	$\mathrm{Tr}\, C = 0$	$C^\dagger = -C$	$C^\dagger = -C$ + $\mathrm{Tr}\, C = 0$
dimension	n^2 (all elements independent)	$n^2 - 1$ just one condition	$\frac{n(n-1)}{2}$ number of independent elements = number of elements of an upper triangular matrix without the diagonal	$\frac{n(n-1)}{2}$ the same reasoning as for $\mathfrak{so}(r,s)$	$2n^2 + n$ number of elements of the upper triangular matrix $2n \times 2n$ with (the diagonal) $\frac{(2n+1)(2n)}{2} = 2n^2 + n$	$2n^2$ considered as real LG and LA	$2n^2 - 2$ as $GL(n,\mathbb{C})$ with two conditions (real and imaginary part of $\mathrm{Tr}\, C = 0$)	n^2 as for $\mathfrak{so}(n)$ but we can have imaginary parts on the diagonal $2\frac{n(n-1)}{2} + n = n^2$	$n^2 - 1$ as for $\mathfrak{u}(n)$ with one additional condition from $\mathrm{Tr}\, C = 0$ (only imaginary numbers)
other properties	not connected non-compact	connected non-compact	not connected only $SO(n)$ connected non-compact	compact	connected non-compact	connected non-compact	simply connected non-compact	connected compact	simply connected compact