Conjugate elements and conjugacy classes

Def. 6: Two elements g_1 and g_2 of G are <u>conjugate</u> if there exists an element $h \in G$ such that $g_1 = h g_2 h^{-1}$. ~~It is denoted~~ $g_1 \sim g_2$.

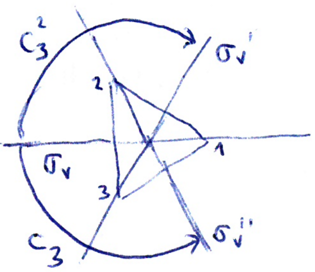
- it is an <u>equivalence relation</u>: 1) $g \sim g$ (through $e \in G$)
  2) $g_1 \sim g_2 \iff g_2 \sim g_1$ (through $h^{-1} \in G$)
  3) $g_1 \sim g_2$ and $g_2 \sim g_3 \Rightarrow g_1 \sim g_3$ (using $(h_1 h_2)^{-1} = h_2^{-1} h_1^{-1}$)

and thus, the group G decomposes into <u>conjugacy classes</u>, denoted C_g or (g) for an element g, containing all elements of G that are conjugate to g

- we will see later that these classes play an important role in the representation theory, particularly the number of all non-equivalent irreducible representations of any finite group is equal to the number of its conjugacy classes
- if G is abelian then $(g) = \{g\}$ for all $g \in G$ as $g_1 = h g_2 h^{-1} = g_2 h h^{-1} = g_2$

Example: for $C_{3v} = \{E, C_3, C_3^2, \sigma_v, \sigma_v', \sigma_v''\}$

we get 3 conjugacy classes:



$(E) = \{E\}$ as the identity forms always its own class ($g e g^{-1} = e$)

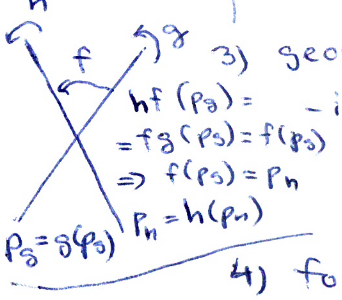
$(C_3) = \{C_3, C_3^2\}$ as $\sigma_v C_3 \sigma_v^{-1} = C_3^2 = \sigma_v' C_3 \sigma_v'^{-1} = \sigma_{v''} C_3 \sigma_{v''}^{-1}$ (and similarly for C_3^2)

$(\sigma_v) = \{\sigma_v, \sigma_v', \sigma_v''\}$ as $C_3 \sigma_v C_3^2 = \sigma_v'' = \sigma_v' \sigma_v \sigma_v'$ etc.

<u>notes</u>: 1) rotation C_3 is conjugate to $C_3^{-1} = C_3^2$ only using reflection σ_v, not using C_3 or $C_3^2 \Rightarrow$ the group C_3 has $(C_3) = \{C_3\}$, $(C_3^2) = \{C_3^2\}$

2) reflection σ_v is conjugate to σ_v' either using C_3, or using the third reflection σ_v''

From $f^{-1} h f = g$ we get



3) geometric interpretation of conjugate elements:
- in general in point groups, two operations are conjugate if there exists (in the given group) an operation that converts one symmetry element to another

4) for C_{3v}, we see that the number of elements of any class divides $\#G$; that is true for any finite group as we will prove later

Multiplication of conjugacy classes, class constants

- this concept will be important later for some proofs in the representation theory

Theorem: ⑧ Let us first consider a collection C of elements of a group G where each element can be included multiple times, thus C is not a subset of G, then

$$\underbrace{gCg^{-1}}_{\text{here we keep all elements, even when they are the same}} = C \text{ for all } g\in G \quad \underset{(\Longleftrightarrow)}{\text{iff}} \quad C = \sum_{(g_k)} a_k (g_k)$$

can be >1 if some class is in C more than once

sum over all different classes but for some a_k can be 0

Proof: $\Leftarrow$ since for all $g\in G$: $g(g_k)g^{-1}\in(g_k)$

and using RT (①) we have $g(g_k)g^{-1} = (g_k)$,

we get $gCg^{-1} = C$ as $C = \sum_{(g_k)} a_k(g_k)$

$\Rightarrow$ let assume that $gCg^{-1} = C$ and that $C = \sum_{(g_k)} a_k(g_k) + R$

where R is non-empty and not consisting of the whole classes

i.e. there exists $a\in R$ such that $(a)\not\subset R$,

but according to the asumption $gRg^{-1} = R$ for all $g\in G$

(as $gCg^{-1} = C$ and $g(g_k)g^{-1} = (g_k)$ for all (g_k) and g)

and thus $gag^{-1}\in R$ for all $g\in G$ which is contradiction.

Theorem: ⑨ Let us now consider a collection (result of multiplication of two classes

$$(g_k)(g_\ell) = \{ g_i g_j \mid g_i\in(g_k) \text{ and } g_j\in(g_\ell)\}$$

where we keep all repeating elements, then for any (g_k) and (g_ℓ)

$$(g_k)(g_\ell) = \sum_{(g_m)} c^m_{k\ell}(g_m) \quad \text{where } c^m_{k\ell}\geq 0 \text{ are } \underline{\text{class constants}}$$

Proof: Immediate from $g(g_k)(g_\ell)g^{-1} = g(g_k)g^{-1}g(g_\ell)g^{-1} = (g_k)(g_\ell)$ and the previous theorem.

Example: C_{3v} has classes $(E) = \{E\}$,

$(C_3) = \{C_3, C_3^2\}$

and $(\sigma_v) = \{\sigma_v, \sigma_v', \sigma_v''\}$

we set $(E)(g_k) = (g_k)$ for any (g_k)

$(C_3)(C_3) = \{C_3^2, E, E, C_3\} = 2(E) + 1(C_3)$

$(C_3)(\sigma_v) = 2(\sigma_v)$

$(\sigma_v)(C_3) = 2(\sigma_v)$

$(\sigma_v)(\sigma_v) = 3(E) + 3(C_3)$

these are class constants

Normal (invariant) subgroups and factor (quotient) group

Def: (7) A subgroup $H \leq G$ is normal (invariant) if $gHg^{-1} \subset H$ for all $g \in G$.

- we write $H \triangleleft G$ for a normal subgroup
- from RT (Theorem (1)), it follows that

$$gHg^{-1} = H \quad \text{or} \quad gH = Hg \quad \text{for all } g \in G$$

or left and right cosets are the same for normal subgroups and vice versa

- for a normal subgroup, it is possible to multiply left (or right) cosets among themselves resulting in a left (or right) coset again (notice that now we do not keep the same elements more than once)

as $$(g_1H)(g_2H) = g_1(\underbrace{g_2H}_{= Hg_1})H = (g_1g_2)H \quad \text{where } HH = H$$

Def: (8) A set $G/H = \{gH : g \in G\}$ = decomposition of G into left (or it could be right) cosets where each coset is contained only once

with binary operation $(g_1H)(g_2H) = (g_1g_2)H$ is called a factor (quotient) group with the identity $eH = H$ and the inverse $(gH)^{-1} = g^{-1}H$

Example: $H = C_3 = \{E, C_3, C_3^2\}$ is a normal subgroup of $G = C_{3v}$

as $C_3C_3C_3^{-1} = C_3$, $\sigma_v C_3 \sigma_v^{-1} = C_3^2$ etc.

- left cosets are H and $\sigma_v H = \{\sigma_v, \sigma_v'', \sigma_v'\}$

and we get the factor group $C_{3v}/C_3 = \{H, \sigma_v H\} \sim C_s = \{E, \sigma\}$

- but the subgroup $\{E, \sigma_v\}$ is not normal

as for example $\sigma_v' \sigma_v \sigma_v' = C_3 \sigma_v' = \sigma_v'' \notin \{E, \sigma_v\}$

Theorem: (10) $H \triangleleft G$ iff H consists of whole conjugacy classes of G

Proof: It follows immediately from the definition of the normal subgroup and from Theorem (8)

$$\left(gCg^{-1} = C \iff C = \sum_{(g_k)} a_k(g_k) \right)$$

Direct product of groups

Def. 9: Let G_1 and G_2 be two groups and $g_1, h_1 \in G_1$ and $g_2, h_2 \in G_2$.

A set of all ordered pairs (g_1, g_2) with binary operation

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2)$$

forms a group called a direct product of G_1 and G_2, denoted as $G_1 \otimes G_2$ or $G_1 \times G_2$.

- identity is (e_1, e_2), inverse (g_1^{-1}, g_2^{-1})
- order for finite groups is $\#(G_1 \times G_2) = \#G_1 \cdot \#G_2$
- a subset $\{(g_1, e_2) : g_1 \in G_1\}$ forms a normal subgroup $(\sim G_1)$ and $\{(e_1, g_2) : g_2 \in G_2\}$ forms a normal subgroup $(\sim G_2)$
  - elements of these subgroups commute (obviously), these subgroups have only one common element (e_1, e_2) and any $(g_1, g_2) \in G_1 \times G_2$ can be expressed a unique product of two elements, each from one of these subgroups

$$(g_1, g_2) = (g_1, e_2)(e_1, g_2)$$

- all these properties motivate the following definition

Def. 10: We say that a certain group G is a direct product of its two subgroups G_1 and G_2, if it is isomorphic to $G_1 \times G_2$.

- the above properties are sufficient conditions, i.e.
  if $G_1 \leq G$ and $G_2 \leq G$ with 1) only one common element $e \in G$
  and 2) all elements $g_1 \in G_1$ commute with all $g_2 \in G_2$
  and 3) every $g \in G$ can be written as $g = g_1 g_2$
  then $G \sim G_1 \times G_2$ and G is a direct product of G_1 and G_2
- uniqueness of decomposition $g = g_1 g_2$ follows from $G_1 \cap G_2 = \{e\}$
  Proof: assume $g = g_1 g_2 = h_1 h_2$, then $\underset{\Uparrow\, G_1}{h_1^{-1} g_1} = \underset{\Uparrow\, G_2}{h_2 g_2^{-1}} = e$
  and thus $h_1 = g_1$, $h_2 = g_2$ (contradiction)
- commutativity condition is equivalent to G_1 and G_2 are normal subgroups
  Proof: $\Rightarrow$ let $g = h_1 h_2$, then $g \overset{G_1}{\check{g}_1} g^{-1} = h_1 h_2 g_1 h_2^{-1} h_1^{-1} = h_1 g_1 h_1^{-1} \in G_1$
  and similarly for G_2
  $\Leftarrow$ consider a commutator $\underbrace{g_1 g_2 g_1^{-1}}_{\text{according to assumptions} \in G_2} \underset{\in G_1}{g_2^{-1}} \in G_1$ and also $\in G_2$
  thus $= e$
  $\Rightarrow g_1 g_2 = g_2 g_1$

Semidirect product

Def: (11) A group G is a semidirect product of its two subgroups G_1 and G_2, i.e. $G = G_1 ⓢ G_2$ or $G = G_1 \rtimes G_2$, if

1) $G_1 \cap G_2 = \{e\}$
2) every $g \in G$ can be written as $g = g_1 g_2$, $g_1 \in G_1$ and $g_2 \in G_2$
3) G_1 is a normal subgroup of G (but not G_2)

- again 1) implies uniqueness of $g = g_1 g_2$

Examples: 1) C_{3v} is a semidirect product of $G_1 = \{E, C_3, C_3^2\}$ (normal) and $G_2 = \{E, \sigma_v\}$ (not normal), or

$$C_{3v} = \{E, C_3, C_3^2\} \rtimes \{E, \sigma_v\} = C_3 \rtimes C_s$$

as each $g \in C_3$ can be written as $g_1 \cdot g_2$:

$$E = E \cdot E\,,\ C_3 = C_3 \cdot E\,,\ C_3^2 = C_3^2 \cdot E\,,\ \sigma_v = E \cdot \sigma_v\,,\ \sigma_v' = C_3 \sigma_v\,,\ \sigma_v'' = C_3^2 \sigma_v$$

- it is not a direct product: $\{E, \sigma_v\}$ is not normal and thus C_3 and C_3^2 do not commute with σ_v
- the direct product $\{E, C_3, C_3^2\} \times \{E, \sigma_v\}$ is isomorphic to the 6-element cyclic group with the generator $g^6 = (C_3, \sigma_v)^6 = (E, E)$

2) Euclidean group E^3

of all linear transformations of $\mathbb{R}^3$ preserving distances and angles with elements $\{R, \vec{t}\}$

$$\{\underset{\text{proper or improper rotation}}{R},\ \underset{\text{translation}}{\vec{t}}\}$$

and composition $\{R_1, \vec{t}_1\} \circ \{R_2, \vec{t}_2\} = \{\overbrace{R_1 R_2}^{\text{rotation}}, \overbrace{R_1 \vec{t}_2 + \vec{t}_1}^{\text{translation}}\}$

as $\vec{r}\,' = \{R_2, \vec{t}_2\}\vec{r} = R_2 \vec{r} + \vec{t}_2$ and $\vec{r}\,'' = \{R_1, \vec{t}_1\}\vec{r}\,' = R_1 R_2 \vec{r} + R_1 \vec{t}_2 + \vec{t}_1$

- identity $e = \{1, \vec{0}\}$, inverse $\{R, \vec{t}\}^{-1} = \{R^{-1}, -R^{-1}\vec{t}\}$
- all translations $G_1 = \{1, \vec{a}\}$ form a normal subgroup as $\{R, \vec{t}\} \circ \{1, \vec{a}\} \circ \{R, \vec{t}\}^{-1} = \{1, R\vec{a}\} \in G_1$
- and because $G_2 = \{R, \vec{0}\}$ is not normal but $\{R, \vec{t}\} = \{1, \vec{t}\} \circ \{R, \vec{0}\}$, it follows that $E^3 = G_1 \rtimes G_2$
- similarly the Poincaré group is a semidirect product

Some other notions

Def: (12) The center of a group G is a subgroup $Z(G)$ of all elements of G that commute with all $g \in G$, thus it is abelian.

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$$

- check that it is really a subgroup and that it is normal
- if G is abelian then $Z(G) = G$

Def: (13) A group G is simple if its only normal subgroups are the trivial ($= \{e\}$ or G) subgroups.

Def: (14) A group G is semisimple if it has no non-trivial abelian normal subgroup.

- these characteristics of groups play an important role for classification of groups and also for representations of some Lie groups

Homomorphism and other morphisms

- important for relations among groups
- a special homomorphism will be a representation of a group on a linear space, it is a homomorphism from a group to the group of all linear transformations on the linear space

Def: (15) A homomorphism from a group G to a group H is a mapping $\varphi: G \to H$ satisfying $\varphi(a \cdot b) = \varphi(a)\varphi(b)$ for $a, b \in G$

- that is, φ preserves the group operation
- for $b = e_G$ we get $\varphi(a e_G) = \varphi(a) = \varphi(a)\varphi(e_G) \Rightarrow \varphi(e_G) = e_H$
  - similarly $\varphi(a^{-1}) = \varphi(a)^{-1}$

Def: (16) A bijective (one-to-one) homomorphism is an isomorphism.

A injective (one-to-one, but not onto) homom. is a monomorphism.

$Im_\varphi G = H$ = A surjective (onto) homomorphism is an epimorphism.

If $G = H$ then a homomorphism is an endomorphism.

an isomorphism is an automorphism

- an automorphism of a group G given by $\overset{\varphi(a)}{\phi_a}(g) = a g a^{-1}$
for all $a, g \in G$ is called an <u>inner automorphism</u> (see later)

- analogous terminology is used for other mathematical structures, for linear spaces $\varphi(\alpha\vec{v}+\beta\vec{w}) = \alpha\varphi(\vec{v})+\beta\varphi(\vec{w})$
or for Lie algebras $\varphi([\vec{v},\vec{w}]) = [\varphi(\vec{v}),\varphi(\vec{w})]$

<u>Def:</u> 17 Let $\varphi: G \to H$ be a homomorphism. Then

<u>$\mathrm{Ker}\,\varphi$</u> $= \{ g \in G : \varphi(g) = e_H \}$ is a kernel of homomorphism

and <u>$\mathrm{Im}\,\varphi$</u> $= \{ h \in H : h = \varphi(g) \text{ for some } g \in G \}$ is ~~its~~ image.

<u>Theorem:</u> 11 $\mathrm{Ker}\,\varphi$ is a normal subgroup of G and
$\mathrm{Im}\,\varphi$ is a subgroup of H.

<u>Proof:</u> 1) Let $g \in \mathrm{Ker}\,\varphi$ and $a \in G$. Then
$$\varphi(a g a^{-1}) = \varphi(a)\varphi(g)\varphi(a^{-1}) = \varphi(a)\, e_H\, \varphi(a^{-1}) = e_H$$
or $a g a^{-1} \in \mathrm{Ker}\,\varphi$ for all $a \in G$ $\Rightarrow$ $\mathrm{Ker}\,\varphi$ is normal
(that it is a subgroup, one can show straightforwardly)

2) To show that $\mathrm{Im}\,\varphi$ is a subgroup of H we use the relation $\varphi(a \cdot b) = \varphi(a)\varphi(b)$ that induces closure and associativity, moreover, $e_H \in \mathrm{Im}\,\varphi$ as $\varphi(e_G) = e_H$
and $\varphi(a)^{-1} \in \mathrm{Im}\,\varphi$ as $\varphi(a)^{-1} = \varphi(a^{-1})$ for all $a \in G$

<u>Example:</u> <u>natural projection</u> (or quotient map) is a fundamental homomorphism from a group G onto its factor group G/H given by $\pi : G \to G/H$
or $\pi : g \mapsto gH$ for all $g \in G$

- clearly it is not one-to-one, but it is surjective (onto) with the kernel $\mathrm{Ker}\,\pi = H$

<u>Proof:</u> for all $g, h \in G$: $\pi(gh) = ghH = (gH)(hH) = \pi(g)\pi(h)$
and for $h \in H$: $\pi(h) = hH = eH$ = identity in G/H
and if $g \notin H$: $\pi(g) = gH \neq H$