Group action on a set or a space

- if a certain abstract group is realised as transformations of a set or a space, we say that the group acts on such a set or a space, but in general, a group action is not necessarily a realization of the group (the same transformation can be assigned to more than one group element)

Def: (18) A group G acts on a set M if there exists a mapping

$$\varphi : G \times M \to M \quad \text{such that for all } g \in G \text{ and } m \in M$$

$$\varphi(g, m) = T(g)\, m \equiv g \cdot m \quad \text{, where } T(g) \text{ is some transformation of } M$$

and

$$\left.\begin{aligned} &g\cdot(h\cdot m) = (gh)\cdot m \quad \text{for all } g, h \in G \\ \text{and} \quad &e\cdot m = m \quad \text{for the identity } e \in G \end{aligned}\right\} \text{ and } m \in M.$$

- thus, the group action is a homomorphism from G to the group of all transformations of the set M (to two distinct elements $g, h \in G$, transformations $T(g)$ and $T(h)$ can be the same

- we will work mostly with a special case of the group action, a representation = linear action of G on linear (vector) spaces

Def: (19) An orbit of an element $m \in M$ under the action of G is a subset $G\cdot m \subset M$ consisting of elements $g \cdot m \in M$ for all $g \in G$

i.e. $\quad G\cdot m = \{ g\cdot m ,\ g \in G \}$

Def: (20) An isotropy group G_m for an element $m \in M$ is a set of $g \in G$ satisfying $\quad g\cdot m = m$.

- it is a subgroup $G_m \leq G$ as $e \in G_m$ (by definition)

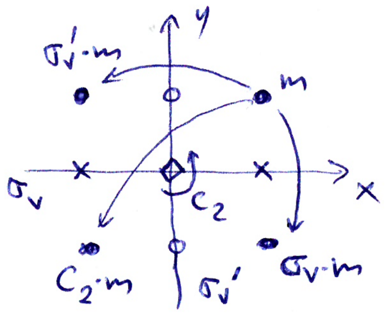
and if $g \in G_m$ then $m = e\cdot m = (g^{-1} g)\, m = g^{-1}(g\cdot m) = g^{-1}\cdot m$

and if $g\cdot m = m$ and $h \cdot m = m$ then $(g\cdot h)\, m = g(h\cdot m) = m$

thus $g^{-1} \in G_m$ and $(g\cdot h) \in G_m$ for any $g, h \in G_m$

Example: C_{2v} action in the plane (x,y)

- $C_{2v} = \{E, C_2, \sigma_v, \sigma_v'\}$ is the group of symmetry of H_2O



- a general point • has a 4-elem. orbit, $G_\bullet = \{E\}$
- a point on the x-axis × has a 2-el. orbit, $G_\times = \{E, \sigma_v\}$
- a point on the y-axis o has a 2-el. orbit, $G_o = \{E, \sigma_v'\}$
- an origin ◇ has a 1-el. orbit, $G_\diamond = C_{2v}$

Theorem: (12) For a finite group G, we have in general $\#G = \#G_m \cdot \#(G\cdot m)$ or the product of the numbers of elements of the isotropy group G_m and the orbit $G\cdot m$ for any point $m \in M$ is always the number of the group elements.

Proof: G_m is a subgroup of G and thus, according to Lagrange's theorem, $\#G_m$ divides $\#G$ and G can be decomposed to left cosets $g\cdot G_m$ satisfying $(gG_m)m = g\cdot m = n$, or all elements of one left coset give the same point n by acting on a point m.

But two different left cosets $g_1 G_m$ and $g_2 G_m$ $(g_1 \neq g_2)$ cannot give the same point n because then we would get $g_1 \cdot m = g_2 \cdot m = n \Rightarrow (g_1^{-1} g_2)\cdot m = m \Rightarrow g_1^{-1} g_2 \in G_m$

$\Rightarrow g_1(g_1^{-1} g_2) = g_2 \in g_1 G_m$ and also $g_2 \in g_2 \cdot G_m$ which is contradiction.

Thus we have as many points of the orbit as left cosets.

Examples: Group actions of G on itself (i.e. $M = G$)

1) left or right multiplications by any $g \in G$ (left or right shifts)

$$L_g : G \to G \quad \text{or} \quad R_g : G \to G$$

given by $L_g : h \mapsto g\cdot h$ or $R_g : h \mapsto h\cdot g$

- according to RT, these are bijective mappings and the whole G is the only orbit and $G_h = \{e\}$ for all $h \in G$
  - such an action is called transitive

2) inner automorphism of G

- a group G acts on itself by conjugation

$$T(g)\cdot h = g h g^{-1} \text{ for all } g,h \in G$$

as we have $\quad T(e)\cdot h = e h e^{-1} = h$

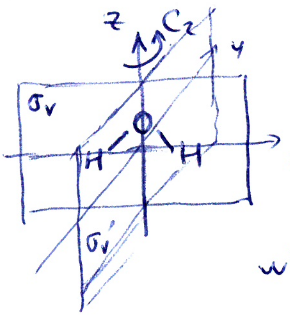
and $\quad T(g_1 g_2) h = (g_1 g_2) h (g_1 g_2)^{-1} = g_1 (g_2 h g_2^{-1}) g_1^{-1} = T(g_1) T(g_2)\cdot h$

- now, the orbits are all conjugacy classes and according to Theorem 12, $\#(g)$ divides $\#G$ for any class (g).

- for C_{3v} we get

$$\text{classes (orbits)} \begin{cases} (E) = \{E\} \\ (C_3) = \{C_3, C_3^2\} \\ (\sigma_v) = \{\sigma_v, \sigma_v', \sigma_v''\} \end{cases} \qquad \left.\begin{array}{l} G_E = C_{3v} \\ G_{C_3} = \{E, C_3, C_3^2\} \\ G_{\sigma_v} = \{E, \sigma_v\} \end{array}\right\} \text{isotropy groups (check this)}$$

Example: group action of $C_{2v} = \{E, C_2, \sigma_v, \sigma_v'\}$ on space $\mathbb{R}^3$



- in $\mathbb{R}^3$, the mapping φ is given by matrices 3x3 corresponding to particular transformations:
- in general, we have

$$D(g)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

where

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(C_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(\sigma_v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(\sigma_v') = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- these 4 matrices form a group with the standard matrix multiplication that is isomorphic to C_{2v}, we say that this "matrix representation" is faithful,

as this group is a subgroup of $O(3)$ = group of all matrices with $\det = \pm 1$

φ is also a homomorphism from C_{2v} to $O(3)$

- in this case, we can consider also C_{2v} action to x, y, or z separately

"matrices" 1x1

e.g. $D_x(E) = (1)$, $D_x(C_2) = (-1)$, $D_x(\sigma_v) = (1)$, and $D_x(\sigma_v') = (-1)$

is a homomorphism from C_{2v} to $G = (\{1,-1\}, \cdot)$ with $\mathrm{Ker}\, D_x = \{E, \sigma_v\}$

- similarly for D_y with $\mathrm{Ker}\, D_y = \{E, \sigma_v'\}$

but for z we get

$$D_z(E) = (1), D_z(C_2) = (1), D_z(\sigma_v) = (1), \text{ and } D_z(\sigma_v') = (1)$$

giving the trivial representation with $\mathrm{Ker}\, D_z = C_{2v}$

Representations of groups (and algebras)

- in general, it is an action of a certain mathematical structure on a vector (linear) space given by linear operators (mappings $V \to V$)
- consider an arbitrary linear space V (real, complex), then

  End(V) is a set of all linear operators on V (including zero operator)

  and

  Aut(V) is a set of all linear operators on V that have an inverse operator, i.e. a set of all linear transformations of V

- End(V) can be considered as
  1) associative algebra = a linear space with associative multiplication; $(f \cdot g)\vec{v} = f(g\,\vec{v})$

  or

  2) Lie algebra = a linear space with a commutator $[f,g]\vec{v} = (f\cdot g - g\cdot f)\vec{v}$ (see later)

  and Aut(V) forms a group of automorphisms of $V \sim GL(V)$ as $f^{-1} \in \mathrm{Aut}(V)$ for all $f \in \mathrm{Aut}(V)$

- thus on V, we can act by groups, associative or Lie algebras and such actions we call representations

Def: 21 A representation of a group G on V is a homomorphism

$$\varrho : G \to \mathrm{Aut}(V)$$

i.e. for all $g_1, g_2 \in G$: $\varrho(g_1 g_2) = \varrho(g_1)\varrho(g_2)$

and $\varrho(e) = E$ (identity on V)

A representation of a Lie algebra $\mathcal{L}$ on V is a homomorphism

$$\phi : \mathcal{L} \to \mathrm{End}(V)$$

i.e. for all $a, b \in \mathcal{L}$: $\phi(\alpha a + \beta b) = \alpha\phi(a) + \beta\phi(b)$ (for any α, β)

and $\phi([a,b]) = [\phi(a), \phi(b)]$

(in a similar way, we can define a repr. of an associative algebra, but we will not study them, although they are important even for mathematical theory of group repr.)

- dimension of the representation = dimension of V
- trivial representations - either all $g \in G$ are represented by E on V or all $a \in \mathcal{L}$ are represented by 0 on V

- faithful representation - if it is a monomorphism
  i.e. each $g \in G$ is represented by a different operator on V
- notation: we will use
  (ϱ, V) if it is necessary to stress both the mapping ϱ and space V
  $T(g)$ if the space is known from the context and we work with operators
  $D(g)$ for matrix representations and for $T(g)$ in a certain basis
  $U(g)$ for unitary operators
  (note that later, irreducible repr. will have special symbols depending on the particular group)
- for each group there are infinitely many representations on various vector spaces, there can be even multiple repres. on the same space (e.g. trivial and some non-trivial)
  but we will see that they are often equivalent (basically the same from a mathematical point of view as we can find a similarity relation between them)
  and moreover, they can be, in general, decomposed into less dimensional ones, called irreducible representations

Def. (22) Let V_1 and V_2 be two vector spaces and $\phi: V_1 \to V_2$ an isomorphism between them as vector spaces (i.e. there exists ϕ^{-1}, $\dim V_1 = \dim V_2$ and $\phi(\alpha\vec{v}+\beta\vec{w}) = \alpha\phi(\vec{v})+\beta\phi(\vec{w})$). Then the representation (ϱ_1, V_1) is equivalent to the repr. given by $(\phi \circ \varrho_1 \circ \phi^{-1} = \varrho_2, V_2)$.

Let (ϱ_1, V_1) and (ϱ_2, V_2) be two representations. We say that they are equivalent if there exists $\phi: V_1 \xrightarrow{\text{onto}} V_2$ such that
$$\varrho_2 = \phi \circ \varrho_1 \circ \phi^{-1} \quad (\text{again } \phi \text{ is an isomorphism})$$

Note: In mathematics, the notion of the intertwing operator is often used: $S: V_1 \to V_2$ satisfying $S \circ \varrho_1(g)\vec{v} = \varrho_2(g) \circ S\vec{v}$ for all $g \in G$ and $\vec{v} \in V_1$
and we can define that
(ϱ_1, V_1) and (ϱ_2, V_2) are equivalent if there exists an intertwining isomorphism between them.

$$\begin{array}{ccc} V_1 & \xrightarrow{S} & V_2 \\ {\scriptstyle \varrho_1}\downarrow & & \downarrow{\scriptstyle \varrho_2} \\ V_1 & \xrightarrow{S} & V_2 \end{array}$$

Examples: 1) $V=\mathbb{R}$, $G=(\{E,i\},\circ)$
identity, inversion

– there are two non-equivalent repr. of G on $\mathbb{R}$

a) (g,V) where $g(E)=1$, $g(i)=1$ (trivial repr.)

b) (σ,V) where $\sigma(E)=1$, $\sigma(i)=-1$ (antisymm. repr.)

as we cannot satisfy $1=g(i)=a(-1)a^{-1}=a\,\sigma(i)\,a^{-1}$
for any $a\in\mathbb{R}$

2) Equivalent matrix representations for different bases on V

– let (g,V) be a repr. of G on V ($\dim V=n$) and choose $\{e_i\}_{i=1}^n$ as a basis of V, then for all $x\in V$ we can write $x=\sum_{i=1}^n x_i e_i$

– action of G on V can be expressed as

$$\boxed{T(g)\,e_i=\sum_{k=1}^n D_{ki}(g)\,e_k}$$

[the result has to be a linear combination of the basis]

an operator representing some $g\in G$ on V

a matrix representing $g\in G$ for the basis e_i

and thus for a general vector x we get

$$x'=T(g)\,x=\sum_{i=1}^n x_i T(g)e_i=\sum_{i,k=1}^n x_i\,D(g)_{ki}\,e_k=\sum_{k=1}^n x'_k e_k$$

linearity of $T(g)$

or $$\boxed{x'_k=\sum_{i=1}^n D_{ki}(g)\,x_i}$$

[here we have a standard matrix times vector operation opposite to $T(g)e_i=\sum_k D_{ki}(g)e_k$]

– all matrices $D(g)$ form a matrix representation of G
(it is not then necessary to talk about V)

as we can easily check:

a) for $g=e$ we clearly have $D(e)=\mathbb{1}_{n\times n}$ (from $T(e)e_i=e_i$)

b) for any $g_1,g_2\in G$:

$$T(g_1)T(g_2)x=\sum_{i,k=1}^n x_i\,D(g_2)_{ki}\,T(g_1)\,e_k=\sum_{i,k,\ell} x_i\,D(g_2)_{ki}\,D(g_1)_{\ell k}\,e_\ell=$$

$$=\sum_{i,k,\ell} x_i\,D_{\ell k}(g_1)\,D_{ki}(g_2)\,e_\ell=\sum_{i,\ell} x_i\,D_{\ell i}(g_1g_2)\,e_\ell=T(g_1g_2)x$$

or $D(g_1g_2)=D(g_1)D(g_2)$ as it should be for a homomorphism from G to a matrix group

- now, choose a different basis on V, $\tilde{e}_i = \sum_{j=1}^{n} e_j A_{ji}$ and
A — transition matrix

for vector components we have

$$x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} \tilde{x}_i \tilde{e}_i = \sum_{i,j} x_j (A^{-1})_{ij} \tilde{e}_i \Rightarrow \tilde{x}_i = \sum_{j=1}^{n} (A^{-1})_{ij} x_j$$

- action of G in this new basis is

$$T(g)\tilde{e}_i = \sum_{\ell=1}^{n} \tilde{D}_{\ell i}(g)\tilde{e}_\ell = \quad (\tilde{D}(g) \text{ are different from } D(g) \text{ in general})$$

$$= \sum_{j=1}^{n} A_{ji} T(g) e_j = \sum_{j,k} A_{ji} D_{kj}(g) e_k = \sum_{j,k,\ell} A_{ji} D_{kj}(g) (A^{-1})_{\ell k} \tilde{e}_\ell$$

or

$$\tilde{D}_{\ell i}(g) = \sum_{k,j} (A^{-1})_{\ell k} D_{kj}(g) A_{ji} = \sum B_{\ell k} D_{kj}(g) (B^{-1})_{ji}$$

where $B = A^{-1}$

or in the matrix form $\tilde{D}(g) = B D(g) B^{-1}$

in other words, $\tilde{D}(g)$ and $D(g)$ are two equivalent matrix representations of G

- the same we get via vector components (in the matrix form)

$$\tilde{x} = Bx \quad , \quad x' = D(g)x$$

$$\tilde{x}' = \underline{\tilde{D}(g)}\tilde{x} = Bx' = B D(g) x = \underline{B D(g) B^{-1}}\tilde{x}$$

Reducible and irreducible representations

- when a group or a Lie algebra acts on the space V, it can happen that some non-trivial ($\neq \vec{0}, \neq V$) subspace W is preserved (unchanged), that is, for all $g \in G$: $g(g)\vec{w} \in W$ for all $\vec{w} \in W$.

Def: (23) A subspace $W \subset V$ is called invariant under the action of G if $G \cdot W \subset W$.

If this subspace does not contain any non-trivial subspace under G then it is called an irreducible invariant subspace.

Def: (24) Let (g, V) be a representation of G. If V contains a non-trivial G-invariant subspace then it is called a reducible space and (g, V) is a reducible representation. Otherwise, it is irreducible.

Def: (25) If W is an invariant subspace under the action of G in V and (g, V) is a representation on V then the restriction $(g|_W, W)$ is called a subrepresentation of (g, V).

– what do these terms mean for matrix representations?

- if we choose a basis in such a way that $e_1, \dots, e_r \in W$ ($\dim W = r$) and $e_{r+1}, \dots, e_d \in V$ but $\notin W$ ($\dim V = d$) then we set

$$D(g) = \begin{array}{c} \left(\begin{array}{c|c} D^W(g) & D^{WW'}(g) \\ \hline \underbrace{0}_{r} & \underbrace{D^{W'}(g)}_{d-r} \end{array}\right)\begin{array}{l}\}r \\ \}d-r\end{array}\end{array}$$

where W' is a linear span $W' = \mathcal{L}(\{e_{r+1}, \dots, e_d\})$

that is
$$T(g)\, e_i = \sum_{i=1}^{r} e_k \overset{W}{D}_{ki}(g) \quad \text{for } i = 1, \dots, r$$

and the matrices $D^W(g)$ form a subrepresentation of $D(g)$,

since
$$D(g_1) D(g_2) = \left(\begin{array}{c|c} D^W(g_1) D^W(g_2) & D^W(g_1) D^{WW'}(g_2) + D^{WW'}(g_1) D^{W'}(g_2) \\ \hline 0 & \underbrace{D^{W'}(g_1) D^{W'}(g_2)}_{} \end{array}\right)$$

these matrices also form a repr. of G but, in general, it is not a subrepr. of $D(g)$ if $D^{WW'}(g) \neq 0$

- vectors $e_1, \dots, e_d$ are called a representation basis (in this case for (g, V)) and $\{e_1, \dots, e_r\}$ is a basis of the subrepr. $(g|_W, W)$

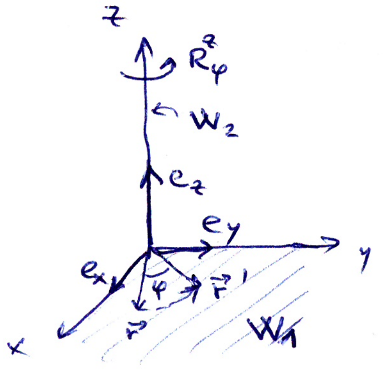
- in a general basis of V, $D(g)$ are not of a shape $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$, but full, but if $D(g)$ form a reducible repr., there exists a similarity transformation which transforms all $D(g)$ into this shape
- if one works just with matrices (not with V) then reducibility is defined by this property; that is a matrix representation $D(g)$ is reducible if it is equivalent to a repr. $D'(g)$ in which all matrices have the shape $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$, that is there exists A such that $D'(g) = A D(g) A^{-1}$ for all $g \in G$

Theorem: Every irreducible repr. of a finite group is finite-dimensional.

13 Proof: Let (g, V) is an IR of a finite group G and $x \in V$. Then $g(g)x$ for all $g \in G$ is a finite set of vectors in V and its linear span is a finite-dim. invariant subspace (thanks to the rearrangement theorem we cannot escape this linear span). But if (g, V) is irred. then this span must be the whole V.

Example: Let us consider an action of $SO(2)$ on $\mathbb{R}^3 = V$.

- $SO(2)$ is a group of all rotations around one fixed axis, for example the z-axis:



- we can see that vectors which are multiples of e_z do not change, but vectors $xe_x + ye_y$ are rotated in the plane (x,y)

thus subspaces
$$W_1 = \mathcal{L}(\{e_x, e_y\})$$
and $W_2 = \mathcal{L}(\{e_z\})$ are both invariant

- thus the corresponding representation of $SO(2)$ on $\mathbb{R}^3$ is reducible and if we restrict this repr. onto W_1 or W_2 we cannot find any other invariant subspaces, the subrepr. defined on W_1 or W_2 are irreducible.

- in matrix form

$$R_\varphi^z = \left(\begin{array}{cc|c} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$R_\varphi^z \downarrow W_1 = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

this is a real irreducible repr. of $SO(2)$ (2-dim.)

$$R_\varphi^z \downarrow W_2 = (1)$$

this is a trivial repr.

- we will learn later that <u>all complex IR of an abelian group</u> ($SO(2)$ is clearly abelian) <u>are necessarily one-dimensional</u> but here we have a real repr.

- if we would consider an action of $SO(2)$ on some complex space, for example on a subspace of the Hilbert space for an electron in a central field formed as a linear span of $\{p_x, p_y, p_z\}$ orbitals we would find one-dim. invariant subspaces $\{p_z\}, \{p_x + ip_y\}, \{p_x - ip_y\}$

- in our example, if we allow complex combinations $z = c^x e_x + c^y e_y$ then $z_1 = e_x + ie_y$ and $z_2 = e_x - ie_y$ generate 1-dim. invariant subspaces and $W_1^{\mathbb{C}}$ („complexification" of W_1) is reducible

since $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = e^{\mp i\varphi}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ and $R_\varphi^z \downarrow W_1^{\mathbb{C}}$ can be diagonalized using

$$\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}}_{A} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{A^{-1} = A^\dagger} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

these are special cases of general IR of $SO(2)$ given by $e^{\pm in\varphi}$, $n = 0, 1, 2, \dots$

Complete reducibility

- in the example, we observed that the repr. of $SO(2)$ on $\mathbb{R}^3$
  decomposed into two subrepresentations $R^2_\varphi \downarrow W_1$ and $R^2_\varphi \downarrow W_2$
  where $W_1 \perp W_2$ and R^2_φ was block-diagonal $\left(\begin{array}{c|c} \ast & 0 \\ \hline 0 & \ast \end{array}\right)$
  that is the submatrix $D^{WW'}(g)$ from the previous exposition of reducibility
  is here a zero-matrix

Def: (26) A representation (g, V) of a group G (or a Lie algebra) is completely reducible (also decomposable) if there exists a decomposion $V = \sum_{j \in J} \oplus V_j$ where V_j are irreducible invariant subspaces under G and J is a general set of indices.
(for finite-dim. spaces, it's finite, but in general can be some ordered set of multiindices)

- in a finite-dim. space V, we can choose
  a basis $\underbrace{e_{11}, \ldots, e_{1d_1}}_{\text{basis of } V_1}, \underbrace{e_{21}, \ldots, e_{2d_2}}_{V_2}, \ldots, \underbrace{e_{p1}, \ldots e_{pd_p}}_{V_p}$
  and we get a matrix representation of the form

$$D(g) = \begin{pmatrix} D^1(g) & 0 & 0 \ldots & 0 \\ 0 & D^2(g) & 0 \ldots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & D^p(g) \end{pmatrix}$$

or $D(g) = D^1(g) \oplus D^2(g) \oplus \ldots \oplus D^p(g)$ for all $g \in G$ where $D^1(g)$ etc. are irred. matrix repr. of G

- in applications in physic, we usually write such decomposition as $\Gamma = \Gamma^1 \oplus \Gamma^2 \oplus \ldots \oplus \Gamma^p$

- for matrix representations we can again define complete reducibility in such a way that there exists a matrix A such that

$$D'(g) = A D(g) A^{-1} \text{ is block-diagonal for all } g \in G$$

and blocks on the diagonal are irreducible.

- are all reducible repr. completely reducible?
  - in general no: Example: let us consider $G = (\mathbb{R}, +)$
    and its 2-dim. repr. $D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$
    - it is really repr. as $D(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$D(x)\,D(y) = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = D(x+y)$$

    - all vectors $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ form an irred. invariant subspace as $D(x)\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, but vectors $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$, orthogonal to $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, do not form an invariant subspace since

$$D(x)\begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} \beta x \\ \beta \end{pmatrix}$$

    - thus $D(x)$ is reducible, but not completely reducible as there is no matrix S such that

$$D'(x) = S\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}S^{-1} \quad \text{where } D'(x) = \begin{pmatrix} f(x) & 0 \\ 0 & g(x) \end{pmatrix}$$

where $f(x)$ and $g(x)$ are some functions since then we would have

$$\det D(x) = \det D'(x) \Rightarrow 1 = f(x)\,g(x)$$
$$\mathrm{Tr}\, D(x) = \mathrm{Tr}\, D'(x) \Rightarrow 2 = f(x) + g(x)$$

and $f(x) = g(x) = 1$ which is not possible because $D(x)$ is not equivalent to the trivial repr. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

as $\quad S\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for any S.

- but it turns out that reducibility implies complete reducibility for important classes of groups used in physics and also for an important type of repr. used in physics

Def. (27) A representation of a group G on some vector space $\mathcal{H}$ with a scalar product $\langle\psi|\varphi\rangle$ (we will use here a standard notation for Hilbert spaces in physics) is called unitary if all $g \in G$ are represented by a unitary operator $U(g)$ satisfying

$$U(g)\,U(g)^\dagger = U(g)^\dagger U(g) = \mathbb{1} \qquad \text{for all } g \in G$$

and $\langle U(g)\psi | U(g)\varphi\rangle = \langle\psi|\varphi\rangle$ and $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$.

- for unitary matrix representations, we have unitary matrices: $D(g)^{-1} = D(g)^{\dagger}$

[note: in ∞-dim. Hilbert spaces we need also boundness of $U(g)$. i.e. $\|U(g)\,\psi\| \leq M\|\psi\|$ for all $|\psi\rangle \in \mathcal{H}$ and some $M < +\infty$ but we will not talk about ∞-dim representations much]

Theorem 14: Every finite-dimensional unitary reducible repr. of G on $\mathcal{H}$ is completely reducible.

Proof: Let $\mathcal{H}_1 \subset \mathcal{H}$ be a non-trivial invariant subspace under G.

- Take any $|\psi\rangle \in \mathcal{H}_1^{\perp}$ (orthogonal complement to $\mathcal{H}_1$: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}$)

then $\langle U(g)\psi|\varphi\rangle = \langle U(g)\psi|U(g)U(g)^{-1}\varphi\rangle \underset{\text{unitarity}}{=} \langle\psi|\underbrace{U(g^{-1})\varphi}_{\in\mathcal{H}_1}\rangle = 0$ for all $|\varphi\rangle \in \mathcal{H}_1$

which is invariant

as $\langle\psi|\varphi\rangle = 0$ for all $|\varphi\rangle \in \mathcal{H}_1$.

- thus, also $\mathcal{H}_1^{\perp}$ is invariant, because $U(g)|\psi\rangle \in \mathcal{H}_1^{\perp}$ for all $g \in G$
- if $\mathcal{H}_1$ and $\mathcal{H}_1^{\perp}$ are reducible, continue in decomposition (to stop, we need finite-dimen.)

- in our example with $G = (\mathbb{R}, +)$ and $D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ we had a non-unitary repr. that cannot be transformed to a unitary repr. by a similarity transformation
- when are non-unit. repr. equivalent to some unitary repr. and thus completely reducible?
  - as we have seen not always but for certain classes of groups always

Theorem 15: (Maschke's theorem)
Every finite-dimensional reducible representation of a finite (or a compact Lie) group is completely reducible.

Proof: Basic idea: Let $T(g)$ be a red. repr. on $\mathcal{H}$ which is not unitary with respect to $\langle\cdot|\cdot\rangle$ (scalar product on $\mathcal{H}$).

- then construct a new scalar product that i equivalent with $\langle\cdot|\cdot\rangle$

using $\langle\psi|\varphi\rangle' = \underbrace{\frac{1}{\#G}\langle T(g)\psi|T(g)\varphi\rangle}_{\text{group averaging}}$ with respect to which $T(g)$ is unitary (check) and thus completely reducible

note: for Lie groups we need finite integrals $\Rightarrow$ only works for compact Lie groups.